CSE 548: Analysis of Algorithms

Lecture 3
(Divide-and-Conquer Algorithms: Matrix Multiplication)

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Iterative Matrix Multiplication

$$z_{ij} = \sum_{k=1}^{n} x_{ik} y_{kj}$$

$$\begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} \times \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix}$$

```
Iter-MM ( Z, X, Y )          { X, Y, Z are n × n matrices,
                               where n is a positive integer }
for i ← 1 to n do
    for j ← 1 to n do
        Z[ i ][ j ] ← 0
        for k ← 1 to n do
            Z[ i ][ j ] ← Z[ i ][ j ] + X[ i ][ k ] · Y[ k ][ j ]
```

Recursive (Divide & Conquer) Matrix Multiplication

$$\underbrace{\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}}_{Z} = \underbrace{\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}}_{X} \times \underbrace{\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}}_{Y}$$

$$= \begin{bmatrix} X_{11}\,Y_{11} + X_{12}\,Y_{21} & X_{11}\,Y_{12} + X_{12}\,Y_{22} \\ X_{21}\,Y_{11} + X_{22}\,Y_{21} & X_{21}\,Y_{12} + X_{22}\,Y_{22} \end{bmatrix}$$

Recursive (Divide & Conquer) Matrix Multiplication

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$$

$$= \begin{bmatrix} X_{11}Y_{11} + X_{12}Y_{21} & X_{11}Y_{12} + X_{12}Y_{22} \\ X_{21}Y_{11} + X_{22}Y_{21} & X_{21}Y_{12} + X_{22}Y_{22} \end{bmatrix}$$

```
Rec-MM ( X, Y )    { X and Y are n × n matrices,
                     where n = 2^k for integer k ≥ 0 }

Let Z be a new n × n matrix
if n = 1 then
    Z ← X · Y
else
    Z11 ← Rec-MM ( X11, Y11 ) + Rec-MM ( X12, Y21 )
    Z12 ← Rec-MM ( X11, Y12 ) + Rec-MM ( X12, Y22 )
    Z21 ← Rec-MM ( X21, Y11 ) + Rec-MM ( X22, Y21 )
    Z22 ← Rec-MM ( X21, Y12 ) + Rec-MM ( X22, Y22 )
endif
return Z
```

\# recursive matrix products: 8

\# matrix sums: 4

$$T(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ 8T\left(\frac{n}{2}\right) + \Theta(n^2), & otherwise. \end{cases}$$

$$= \Theta(n^3)$$

Strassen's Algorithms for Matrix Multiplication (MM)

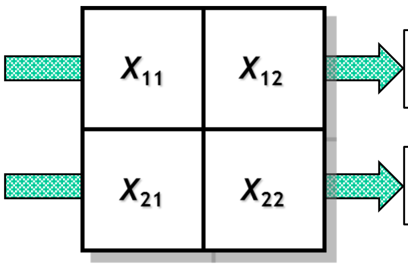
In 1968 Volker Strassen came up with a recursive MM algorithm that runs asymptotically faster than the classical $\Theta(n^3)$ algorithm.

In each level of recursion the algorithm uses:

7 recursive matrix multiplications (instead to 8), and

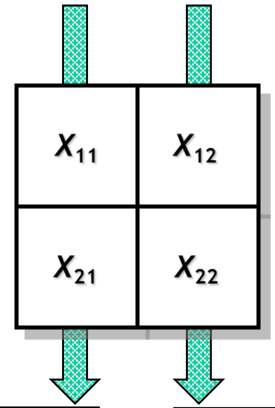
18 matrix additions (instead of 4).

Strassen's MM: 10 Matrix Additions/Subtractions



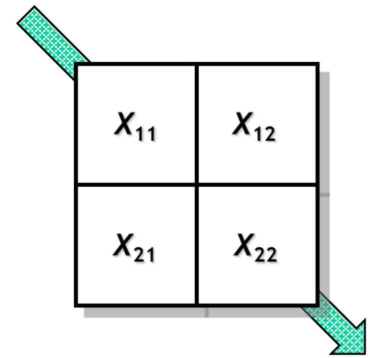
$X_{r1} = X_{11} + X_{12}$

$X_{r2} = X_{21} + X_{22}$

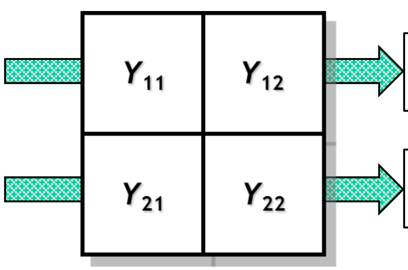


$X_{c1} = X_{11} - X_{21}$

$X_{c2} = X_{12} - X_{22}$

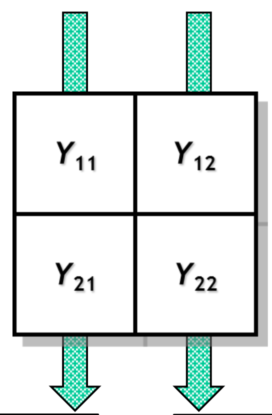


$X_{d1} = X_{11} + X_{22}$



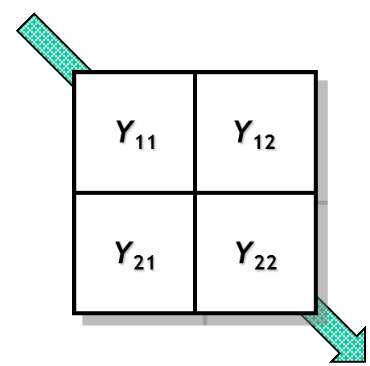
$Y_{r1} = Y_{11} + Y_{12}$

$Y_{r2} = Y_{21} + Y_{22}$



$Y_{c1} = Y_{11} - Y_{21}$

$Y_{c2} = Y_{12} - Y_{22}$



$Y_{d1} = Y_{11} + Y_{22}$

Strassen's MM: 7 Matrix Products

X-cell • *Y*-col

	X_{11}	X_{12}	X_{21}	X_{22}
Y_{11}				
Y_{21}				
Y_{12}	+			
Y_{22}	−			

$$P_{11} = X_{11} \cdot Y_{c2}$$

	X_{11}	X_{12}	X_{21}	X_{22}
Y_{11}				+
Y_{21}				−
Y_{12}				
Y_{22}				

$$P_{22} = X_{22} \cdot Y_{c1}$$

X- row • *Y*-cell

	X_{11}	X_{12}	X_{21}	X_{22}
Y_{11}				
Y_{21}				
Y_{12}				
Y_{22}	+	+		

$$P_{r1} = X_{r1} \cdot Y_{22}$$

	X_{11}	X_{12}	X_{21}	X_{22}
Y_{11}			+	+
Y_{21}				
Y_{12}				
Y_{22}				

$$P_{r2} = X_{r2} \cdot Y_{11}$$

X-col • *Y*-row

	X_{11}	X_{12}	X_{21}	X_{22}
Y_{11}	+		−	
Y_{21}				
Y_{12}	+		−	
Y_{22}				

$$P_{c1} = X_{c1} \cdot Y_{r1}$$

	X_{11}	X_{12}	X_{21}	X_{22}
Y_{11}				
Y_{21}		+		−
Y_{12}				
Y_{22}		+		−

$$P_{c2} = X_{c2} \cdot Y_{r2}$$

X-diag • *Y*-diag

	X_{11}	X_{12}	X_{21}	X_{22}
Y_{11}	+			+
Y_{21}				
Y_{12}				
Y_{22}	+			+

$$P_{d1} = X_{d1} \cdot Y_{d1}$$

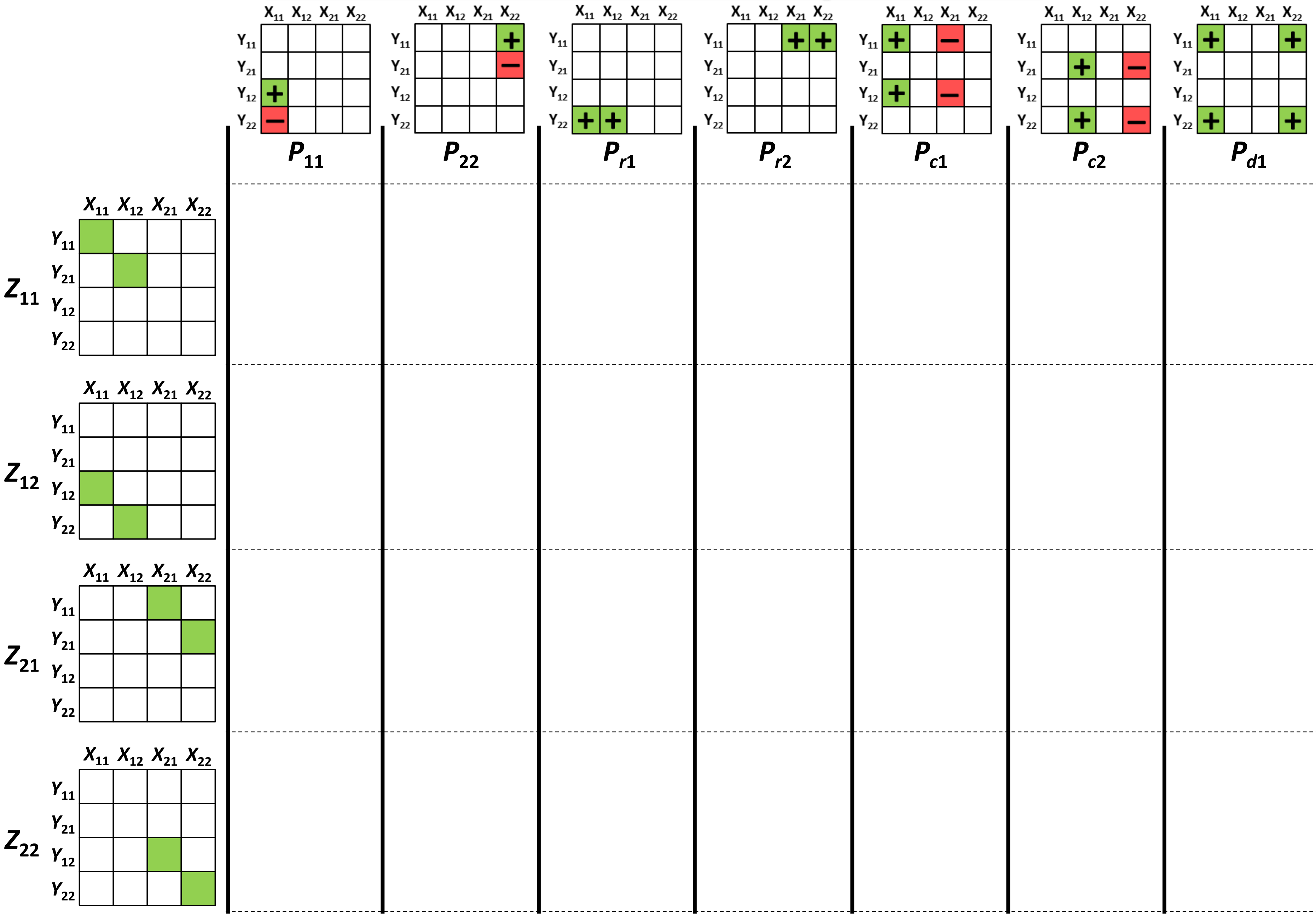
Strassen's MM: 8 More Matrix Additions/Subtractions

$$
\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \times \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}
$$

$$
= \begin{bmatrix} X_{11}Y_{11} + X_{12}Y_{21} & X_{11}Y_{12} + X_{12}Y_{22} \\ X_{21}Y_{11} + X_{22}Y_{21} & X_{21}Y_{12} + X_{22}Y_{22} \end{bmatrix}
$$

$$
= \begin{bmatrix} -P_{r1} - P_{22} + P_{d1} + P_{c2} & +P_{r1} + P_{11} \\ +P_{r2} - P_{22} & -P_{r2} + P_{11} + P_{d1} - P_{c1} \end{bmatrix}
$$

Strassen’s Matrix Multiplication



Strassen's Matrix Multiplication

P_{11}

	X_{11}	X_{12}	X_{21}	X_{22}
Y_{11}				
Y_{21}				
Y_{12}	+			
Y_{22}	−			

P_{22}

	X_{11}	X_{12}	X_{21}	X_{22}
Y_{11}				+
Y_{21}				−
Y_{12}				
Y_{22}				

P_{r1}

	X_{11}	X_{12}	X_{21}	X_{22}
Y_{11}				
Y_{21}				
Y_{12}				
Y_{22}	+	+		

P_{r2}

	X_{11}	X_{12}	X_{21}	X_{22}
Y_{11}			+	+
Y_{21}				
Y_{12}				
Y_{22}				

P_{c1}

	X_{11}	X_{12}	X_{21}	X_{22}
Y_{11}	+		−	
Y_{21}				
Y_{12}	+		−	
Y_{22}				

P_{c2}

	X_{11}	X_{12}	X_{21}	X_{22}
Y_{11}				
Y_{21}		+		−
Y_{12}				
Y_{22}		+		−

P_{d1}

	X_{11}	X_{12}	X_{21}	X_{22}
Y_{11}	+			+
Y_{21}				
Y_{12}				
Y_{22}	+			+

Z_{11}

	X_{11}	X_{12}	X_{21}	X_{22}
Y_{11}				
Y_{21}				
Y_{12}				
Y_{22}				

Z_{12}

	X_{11}	X_{12}	X_{21}	X_{22}
Y_{11}				
Y_{21}				
Y_{12}				
Y_{22}				

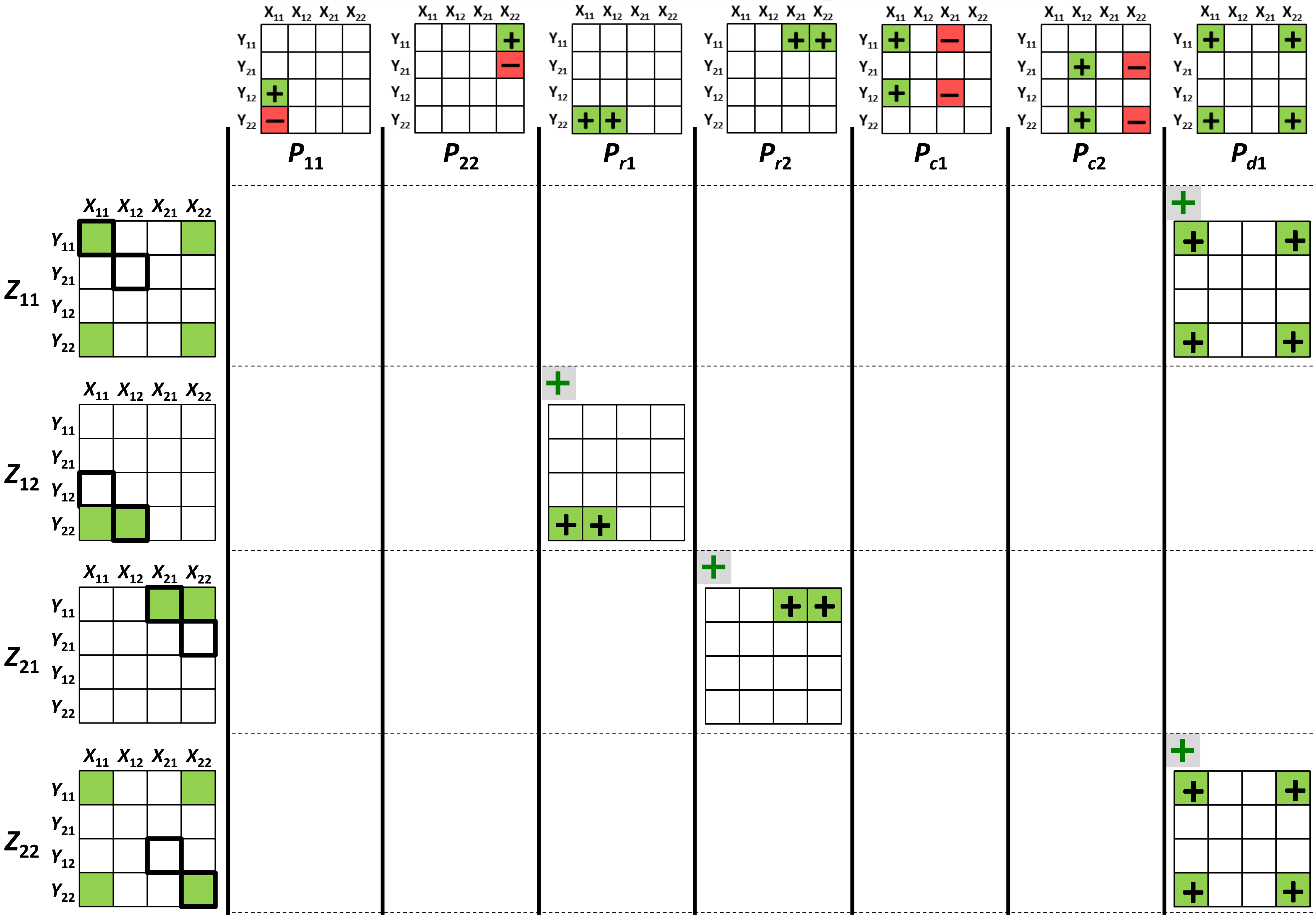
Z_{21}

	X_{11}	X_{12}	X_{21}	X_{22}
Y_{11}				
Y_{21}				
Y_{12}				
Y_{22}				

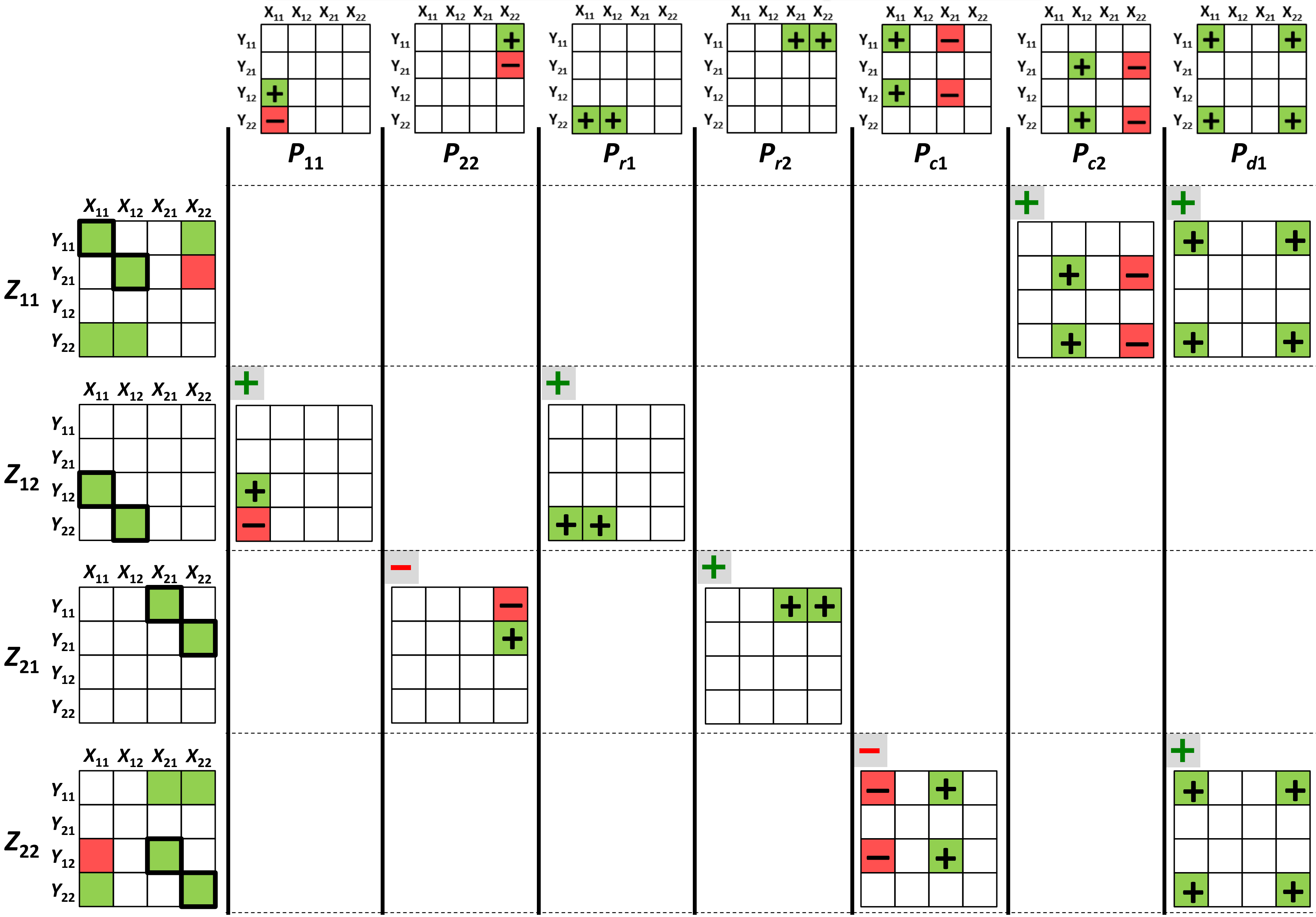
Z_{22}

	X_{11}	X_{12}	X_{21}	X_{22}
Y_{11}				
Y_{21}				
Y_{12}				
Y_{22}				

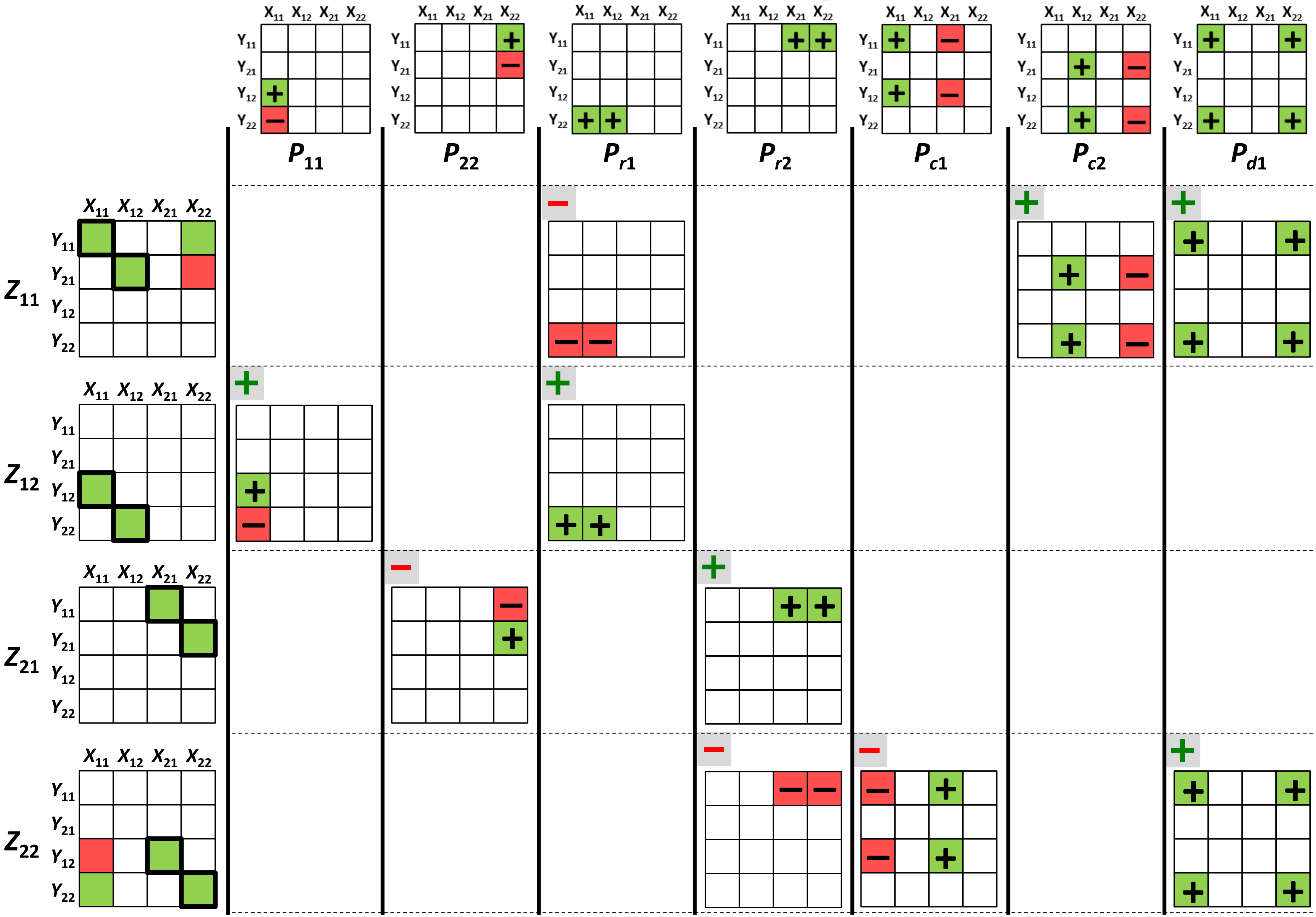
Strassen's Matrix Multiplication



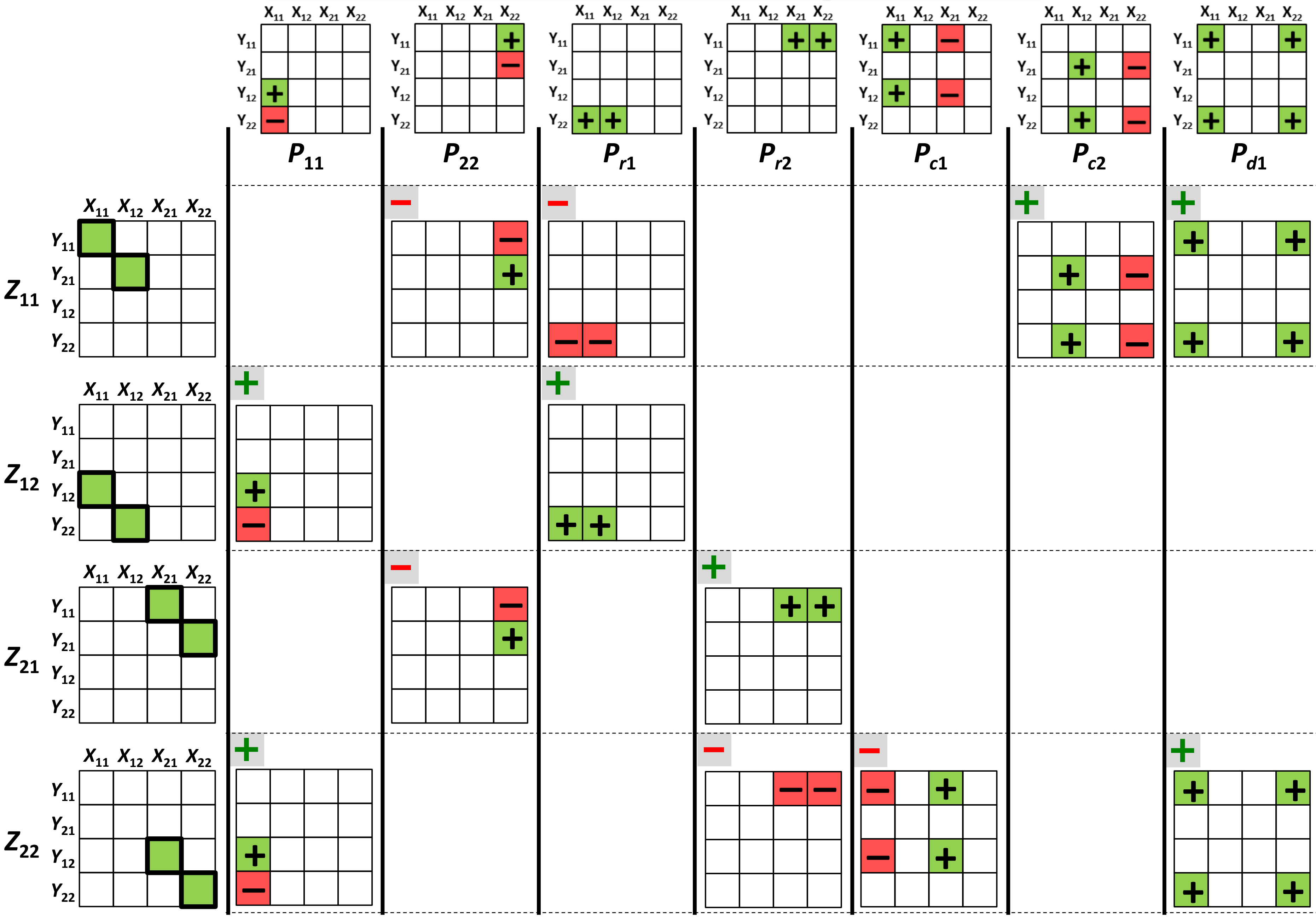
Strassen's Matrix Multiplication



Strassen's Matrix Multiplication



Strassen's Matrix Multiplication



Strassen's Matrix Multiplication

$$\begin{array}{|c|c|}\hline Z_{11} & Z_{12} \\ \hline Z_{21} & Z_{22} \\ \hline \end{array} = \begin{array}{|c|c|}\hline X_{11} & X_{12} \\ \hline X_{21} & X_{22} \\ \hline \end{array} \times \begin{array}{|c|c|}\hline Y_{11} & Y_{12} \\ \hline Y_{21} & Y_{22} \\ \hline \end{array} = \begin{array}{|c|c|}\hline X_{11}Y_{11} + X_{12}Y_{21} & X_{11}Y_{12} + X_{12}Y_{22} \\ \hline X_{21}Y_{11} + X_{22}Y_{21} & X_{21}Y_{12} + X_{22}Y_{22} \\ \hline \end{array}$$

$$= \begin{array}{|c|c|}\hline -P_{r1} - P_{22} + P_{d1} + P_{c2} & +P_{r1} + P_{11} \\ \hline +P_{r2} - P_{22} & -P_{r2} + P_{11} + P_{d1} - P_{c1} \\ \hline \end{array}$$

Sums:

$$\begin{aligned} X_{r1} &= X_{11} + X_{12} & \quad Y_{r1} &= Y_{11} + Y_{12} \\ X_{r2} &= X_{21} + X_{22} & \quad Y_{r2} &= Y_{21} + Y_{22} \\ X_{c1} &= X_{11} - X_{21} & \quad Y_{c1} &= Y_{11} - Y_{21} \\ X_{c2} &= X_{12} - X_{22} & \quad Y_{c2} &= Y_{12} - Y_{22} \\ X_{d1} &= X_{11} + X_{22} & \quad Y_{d1} &= Y_{11} + Y_{22} \end{aligned}$$

Products:

$$\begin{aligned} P_{11} &= X_{11} \cdot Y_{c2} & \quad P_{c1} &= X_{c1} \cdot Y_{r1} \\ P_{22} &= X_{22} \cdot Y_{c1} & \quad P_{c2} &= X_{c2} \cdot Y_{r2} \\ P_{r1} &= X_{r1} \cdot Y_{22} & \quad P_{d1} &= X_{d1} \cdot Y_{d1} \\ P_{r2} &= X_{r2} \cdot Y_{11} \end{aligned}$$

Running Time:

$$T(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ 7T\left(\frac{n}{2}\right) + \Theta(n^2), & otherwise. \end{cases}$$

$$= \Theta\left(n^{\log_2 7}\right) = O\left(n^{2.81}\right)$$

Deriving Strassen's Algorithm

Use the *Feynman Algorithm*:

Step 1: write down the problem

Step 2: think real hard

Step 3: write down the solution

Deriving Strassen's Algorithm

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}}_{X}\underbrace{\begin{bmatrix} e \\ g \\ f \\ h \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} p \\ r \\ q \\ s \end{bmatrix}}_{Z}$$

We will try to minimize the number of multiplications needed to evaluate Z using special matrix products that are easy to compute.

Type	Product	#Mults
$(\cdot)$	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e \\ g \end{bmatrix} = \begin{bmatrix} ae + bg \\ ce + dg \end{bmatrix}$	4
(A)	$\begin{bmatrix} a & a \\ a & a \end{bmatrix}\begin{bmatrix} e \\ g \end{bmatrix} = \begin{bmatrix} a(e+g) \\ a(e+g) \end{bmatrix}$	1
(B)	$\begin{bmatrix} a & a \\ -a & -a \end{bmatrix}\begin{bmatrix} e \\ g \end{bmatrix} = \begin{bmatrix} a(e+g) \\ -a(e+g) \end{bmatrix}$	1
(C)	$\begin{bmatrix} a & 0 \\ a-b & b \end{bmatrix}\begin{bmatrix} e \\ g \end{bmatrix} = \begin{bmatrix} ae \\ ae + b(g-e) \end{bmatrix}$	2
(D)	$\begin{bmatrix} a & b-a \\ 0 & b \end{bmatrix}\begin{bmatrix} e \\ g \end{bmatrix} = \begin{bmatrix} a(e-g) + bf \\ bf \end{bmatrix}$	2

Deriving Strassen's Algorithm

$$\begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix} = \underbrace{\begin{bmatrix} b & b & 0 & 0 \\ b & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{Type\ A\ (\ 1\ Mult\ )} + \underbrace{\begin{bmatrix} a-b & 0 & 0 & 0 \\ c-b & d-b & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}}_{\Delta_1}$$

$$\Delta_1 = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c & c \\ 0 & 0 & c & c \end{bmatrix}}_{Type\ A\ (\ 1\ Mult\ )} + \underbrace{\begin{bmatrix} a-b & 0 & 0 & 0 \\ c-b & d-b & 0 & 0 \\ 0 & 0 & a-c & b-c \\ 0 & 0 & 0 & d-c \end{bmatrix}}_{\Delta_2}$$

$$\Delta_2 = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ c-b & 0 & 0 & c-b \\ -(c-b) & 0 & 0 & -(c-b) \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{Type\ B\ (\ 1\ Mult\ )} + \underbrace{\begin{bmatrix} a-b & 0 & 0 & 0 \\ 0 & d-b & 0 & b-c \\ c-b & 0 & a-c & 0 \\ 0 & 0 & 0 & d-c \end{bmatrix}}_{\Delta_3}$$

$$\Delta_3 = \underbrace{\begin{bmatrix} a-b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (a-b)-(a-c) & 0 & a-c & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{Type\ C\ (\ 2\ Mult\ )} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & d-b & 0 & (d-c)-(d-b) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d-c \end{bmatrix}}_{Type\ D\ (\ 2\ Mult\ )}$$

Algorithms for Multiplying Two *n*×*n* Matrices

A recursive algorithm based on multiplying two $m \times m$ matrices using k multiplications will yield an $\mathrm{O}\left(n^{\log_m k}\right)$ algorithm.

To beat Strassen's algorithm: $\log_m k < \log_2 7 \Rightarrow k < m^{\log_2 7}$.

So, for a 3×3 matrix, we must have: $k < 3^{\log_2 7} < 22$.

But the best known algorithm uses 23 multiplications!

Inventor	Year	Complexity
Classical	—	$\Theta\left(n^3\right)$
Volker Strassen	1968	$\Theta\left(n^{2.807}\right)$
Victor Pan (multiply two 70 × 70 matrices using 143,640 multiplications)	1978	$\Theta\left(n^{2.795}\right)$
Don Coppersmith & Shmuel Winograd (arithmetic progressions)	1990	$\Theta\left(n^{2.3737}\right)$
Andrew Stothers	2010	$\Theta\left(n^{2.3736}\right)$
Virginia Williams	2011	$\Theta\left(n^{2.3727}\right)$

Lower bound: $\Omega\left(n^2\right)$ (why?)