CSE 548: Analysis of Algorithms

Lecture 2
(Divide-and-Conquer Algorithms: Integer Multiplication)

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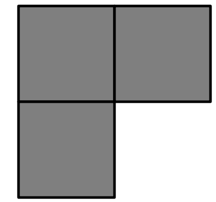
Fall 2015

Tromino Cover

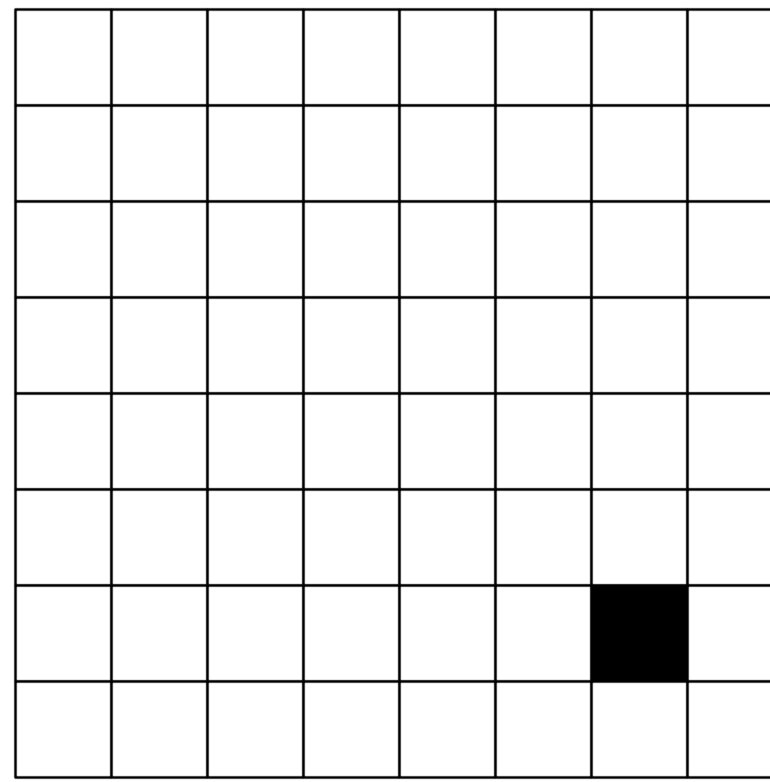
A *right tromino* is an L-shaped tile formed by three adjacent squares.



Puzzle: You are given a $2^n \times 2^n$ board with one missing square.

– you must cover all squares except the missing one exactly using right trominoes

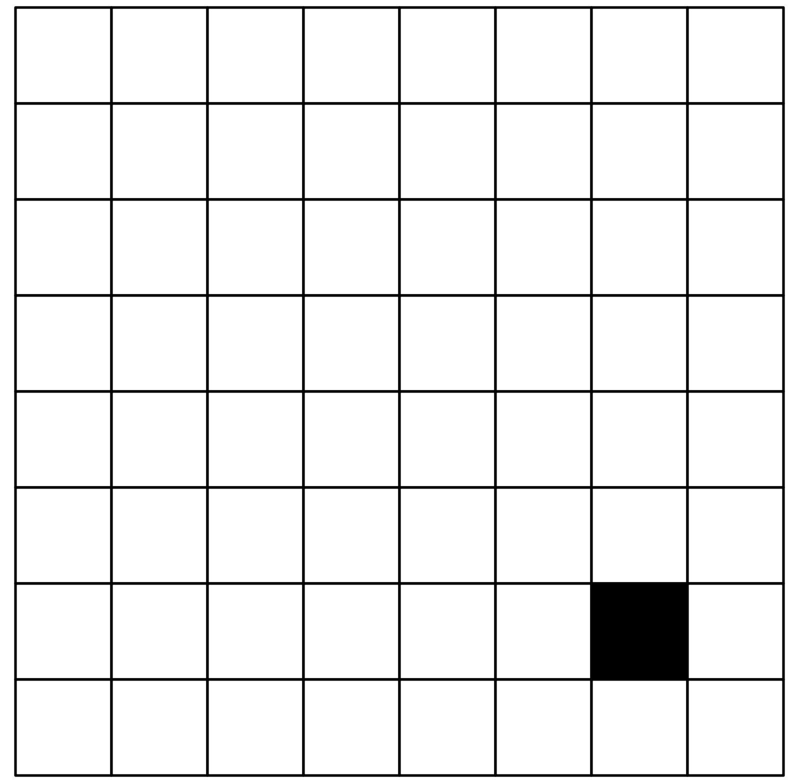
– the trominoes must not overlap



$2^3 \times 2^3$ board

Tromino Cover

Steps



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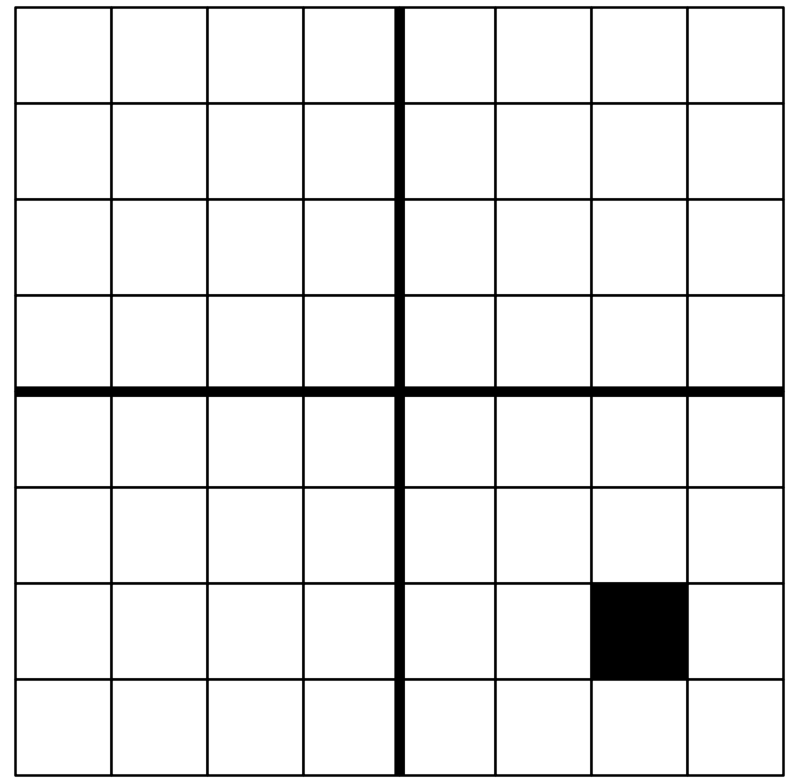

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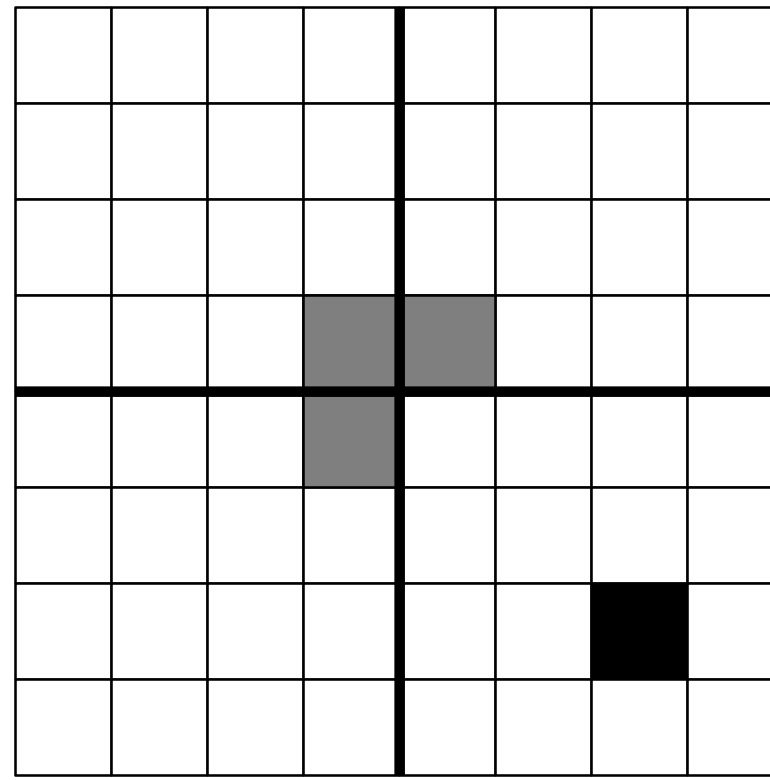

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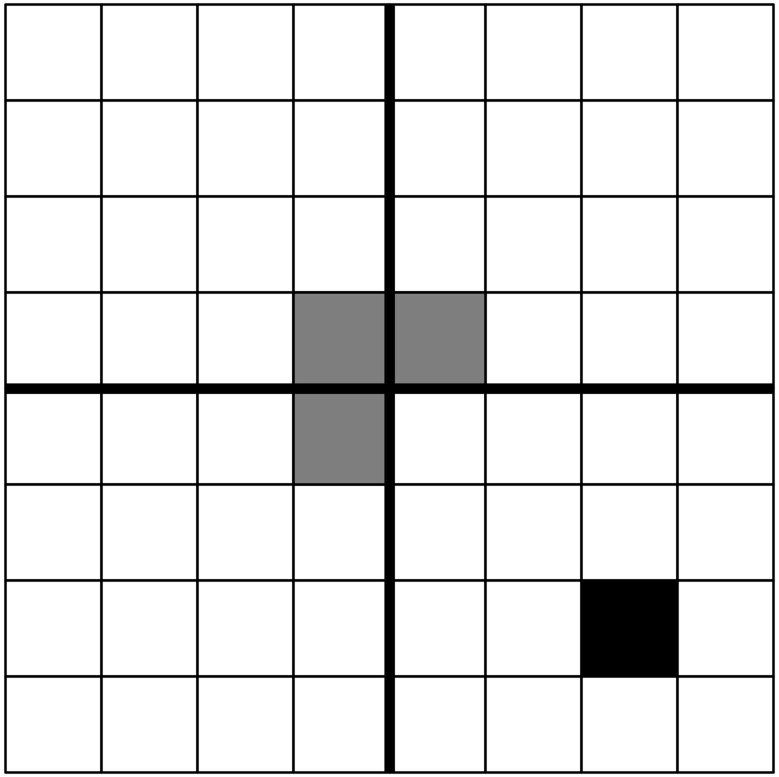

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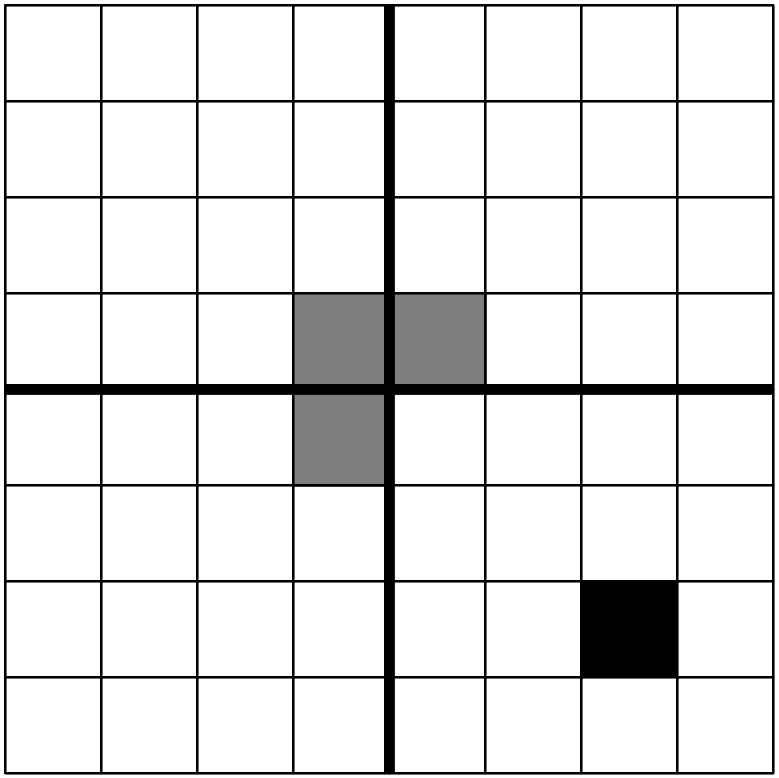

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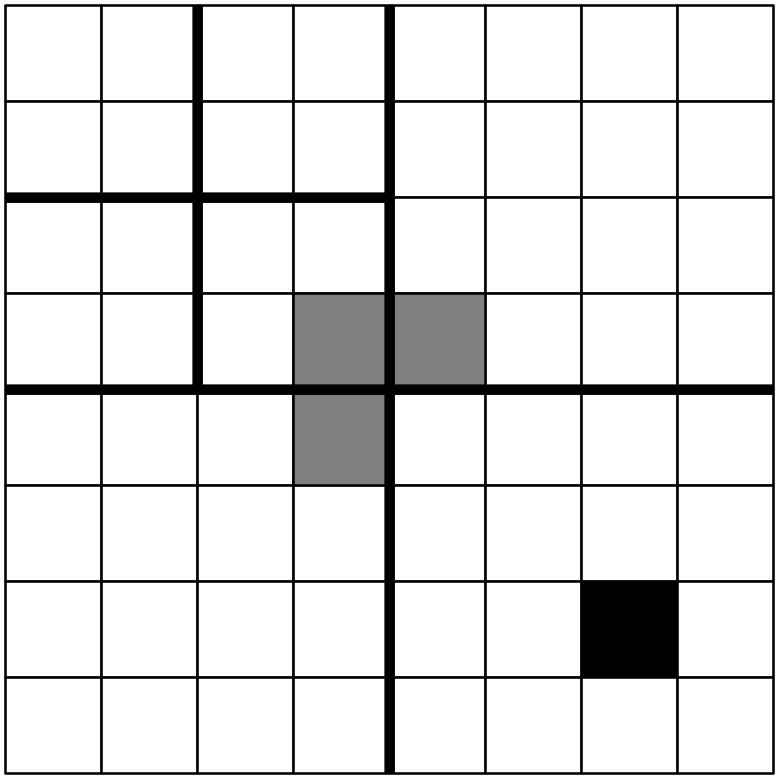

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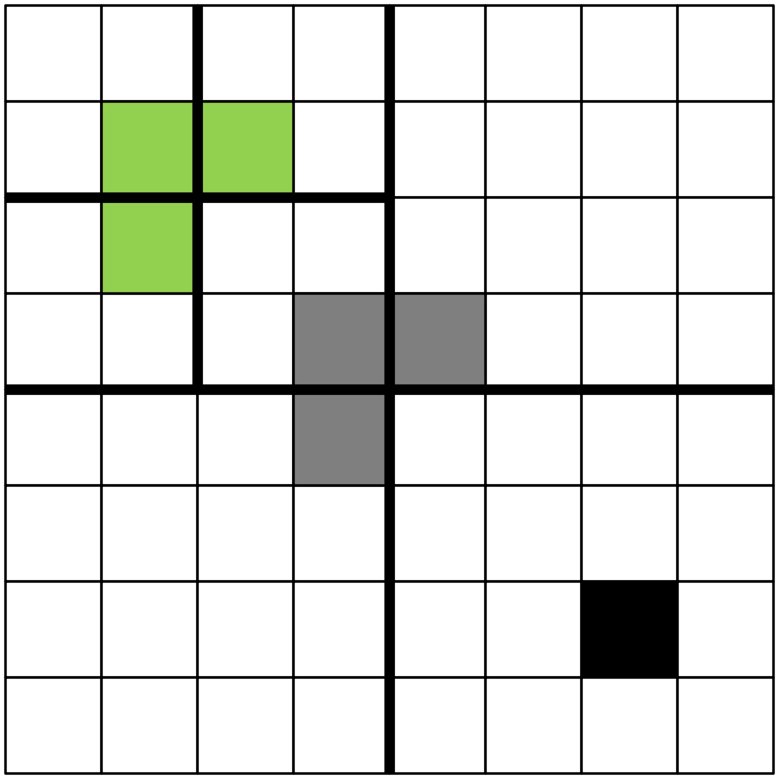

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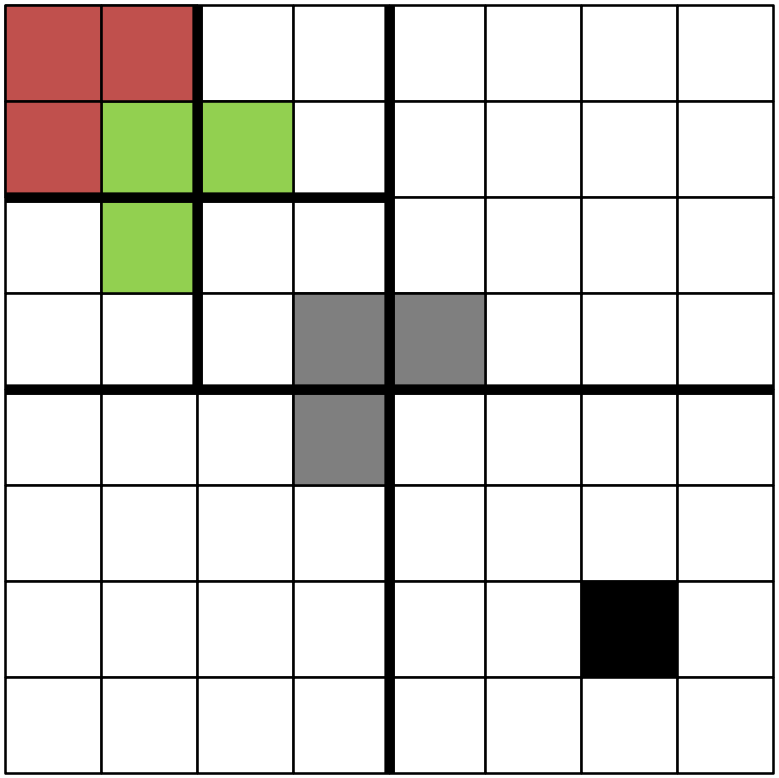

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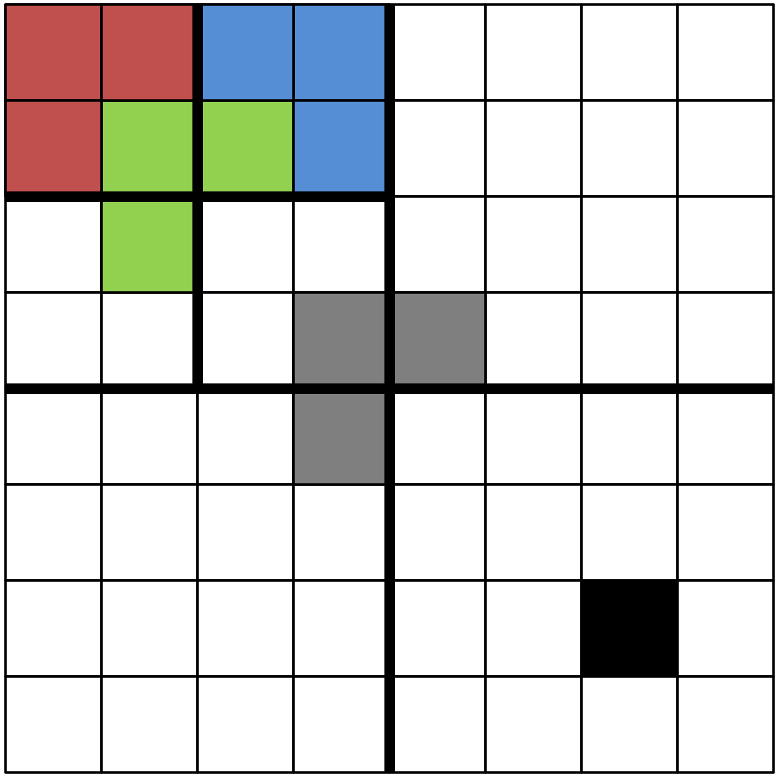

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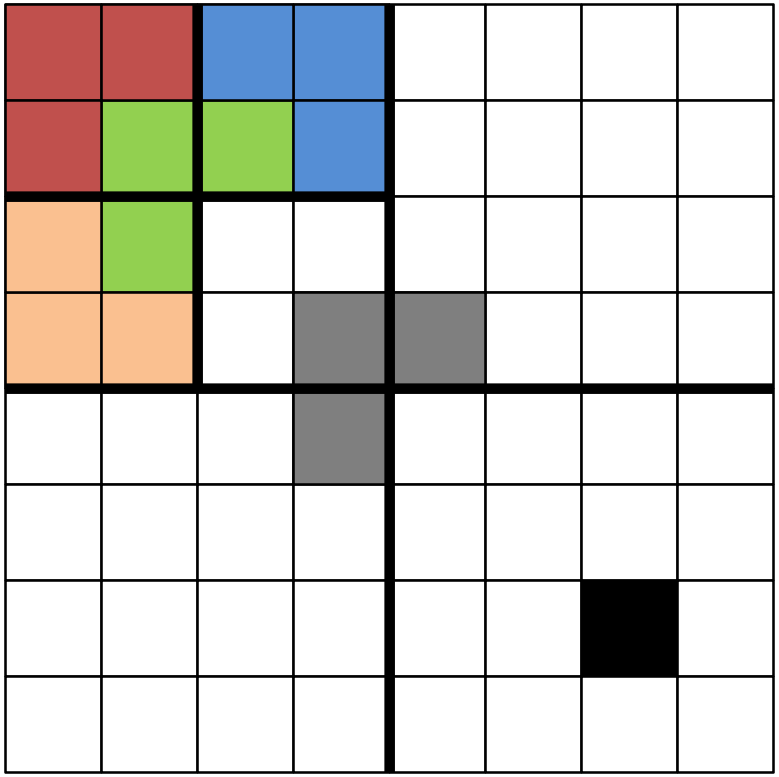

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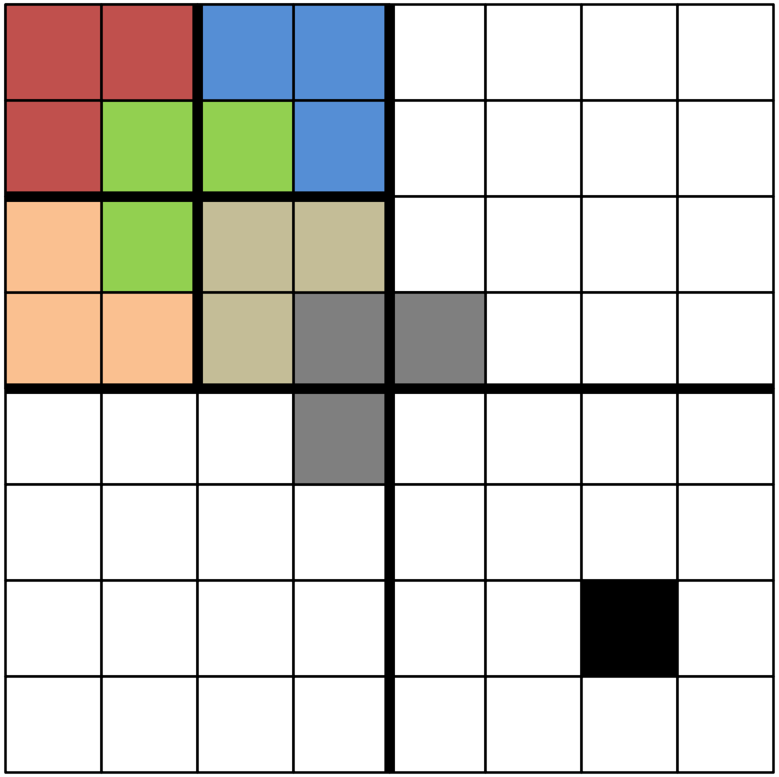

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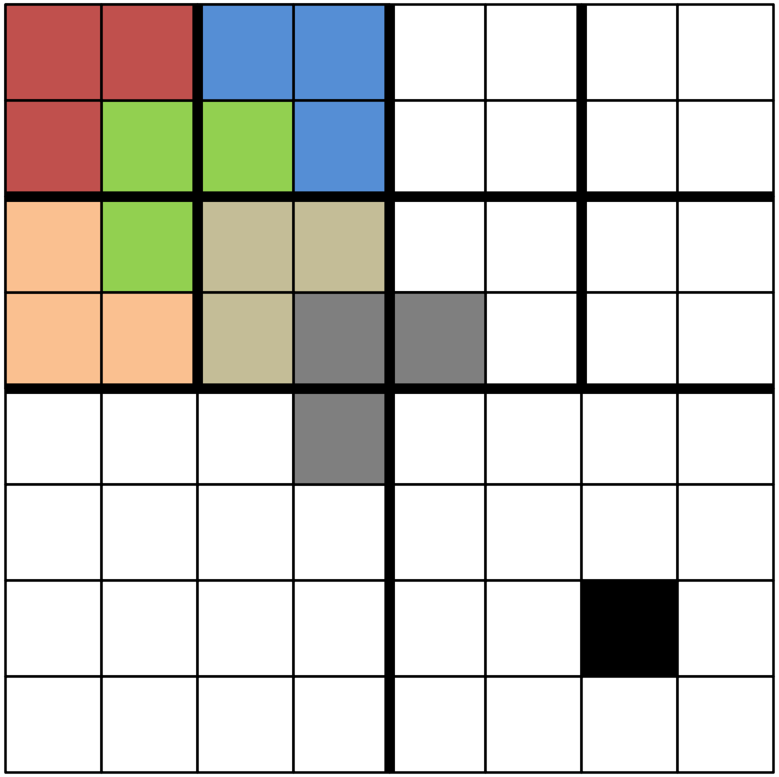

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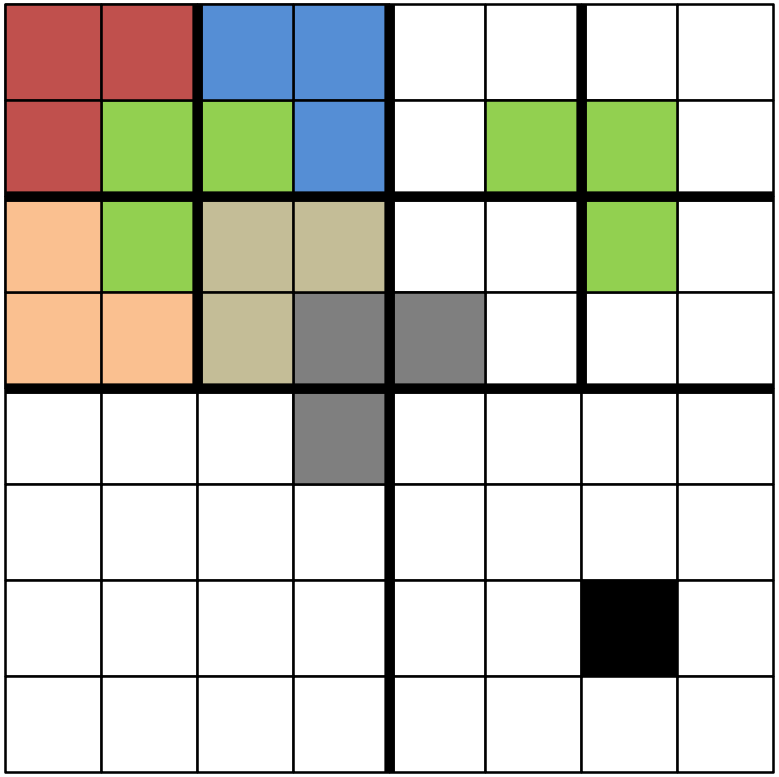

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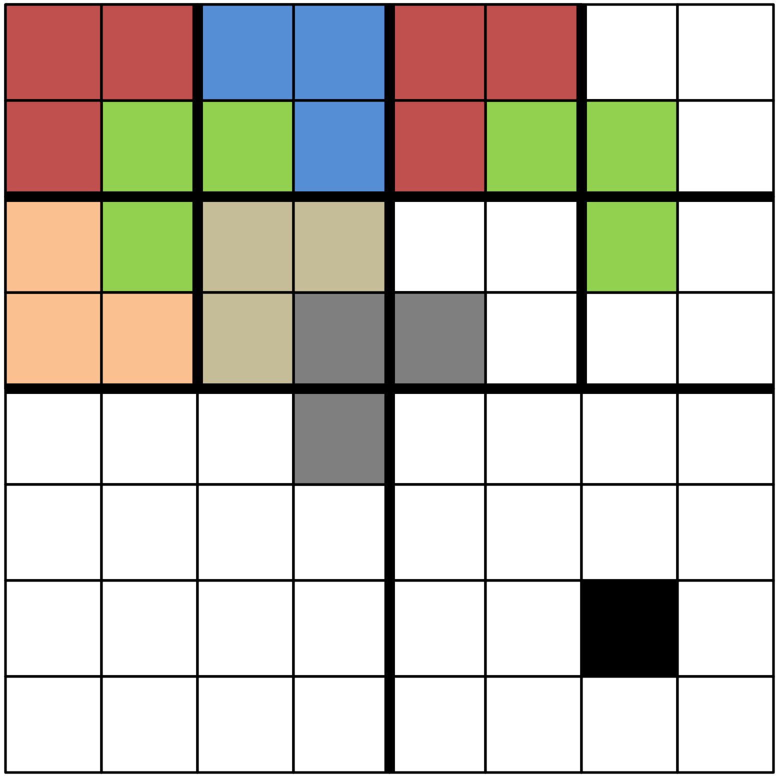

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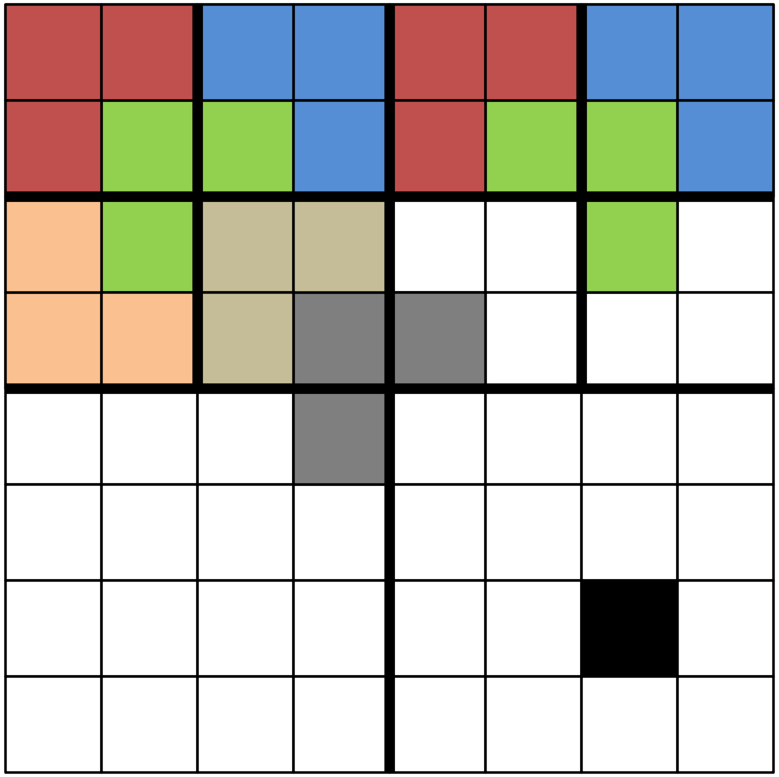

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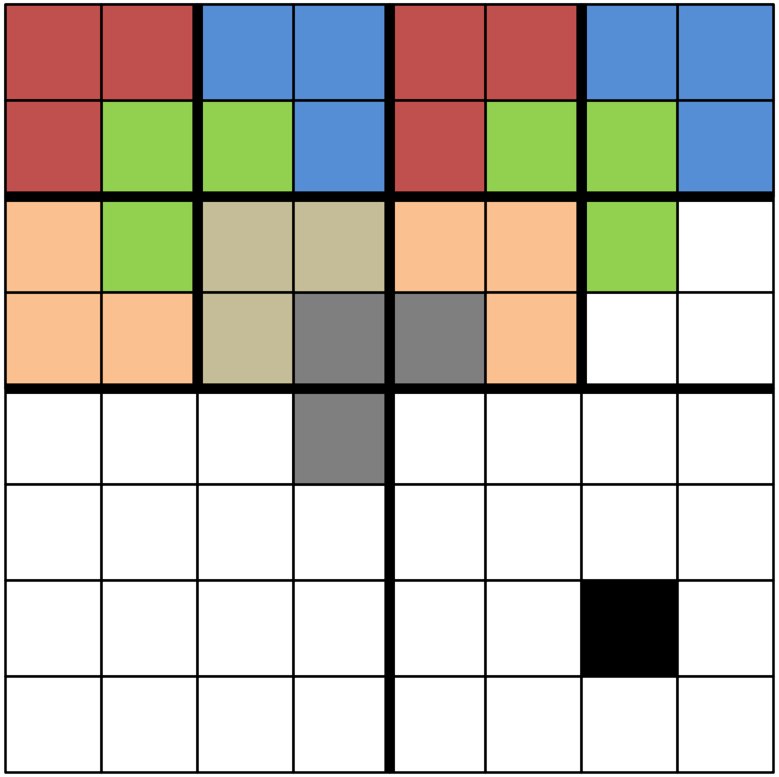

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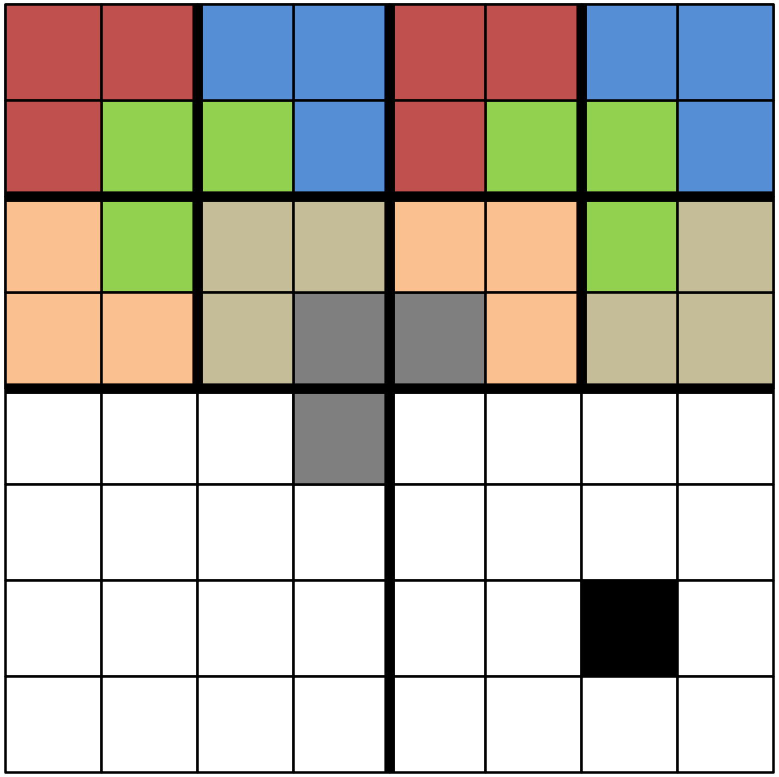

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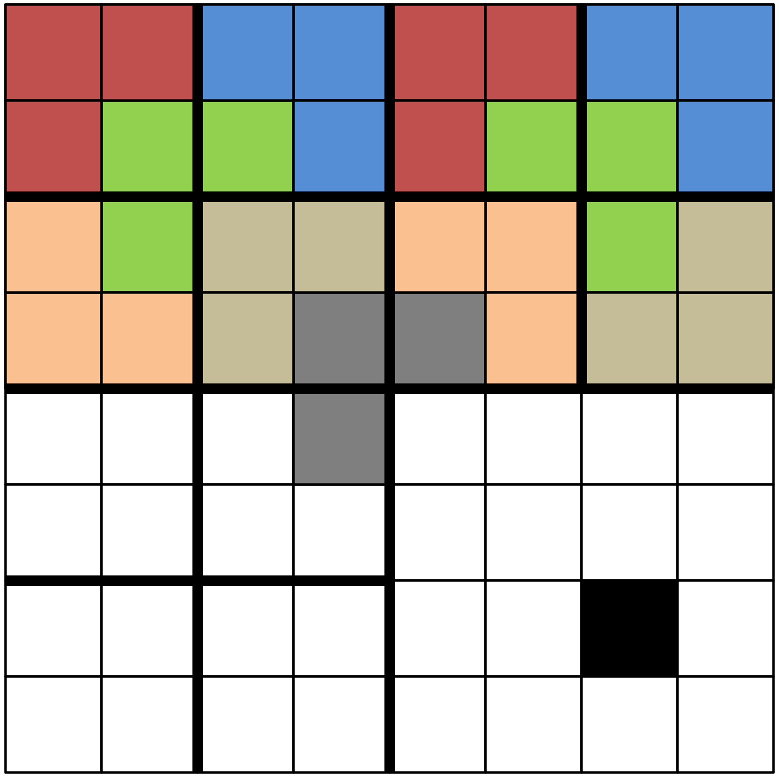

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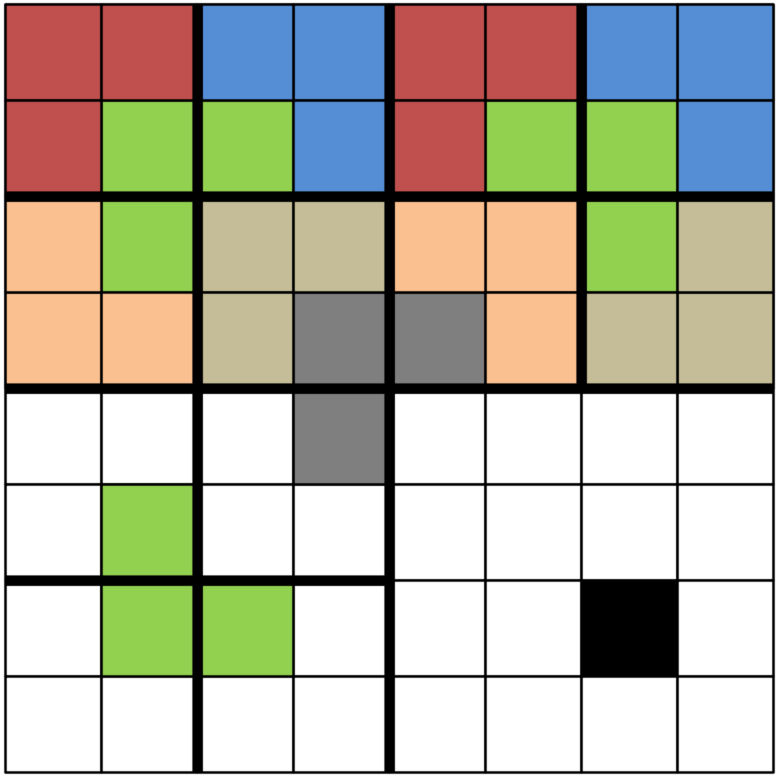

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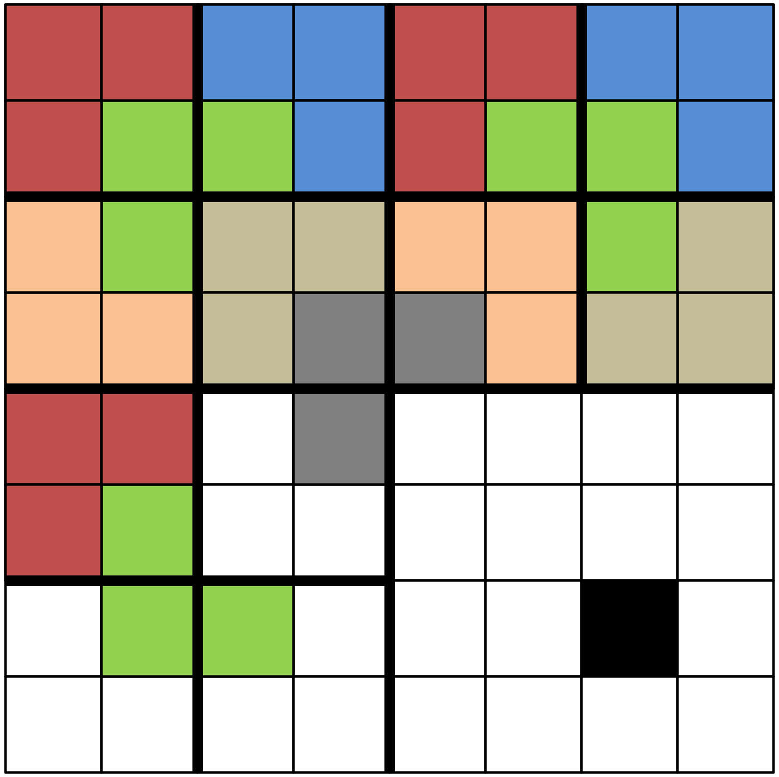

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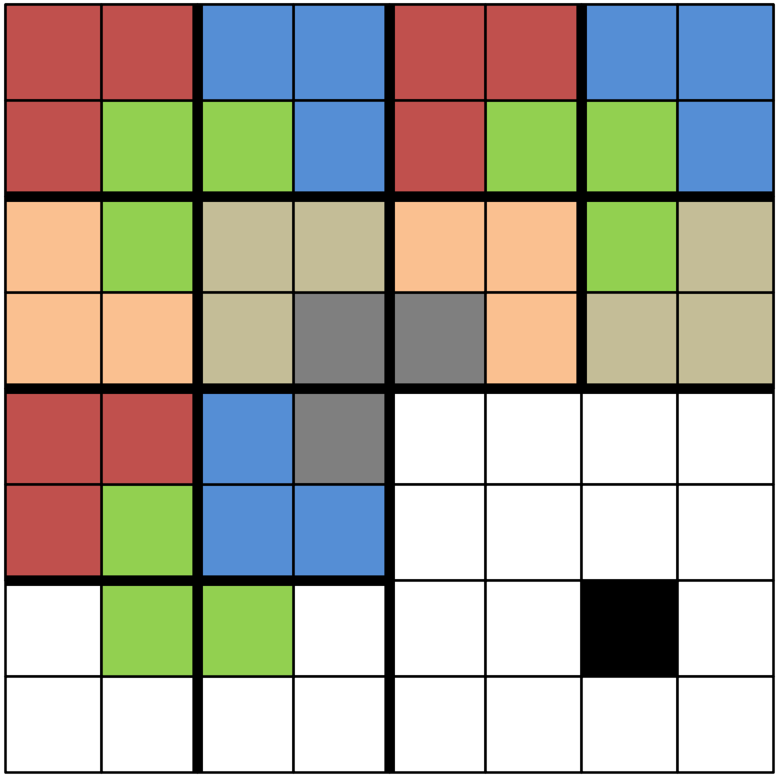

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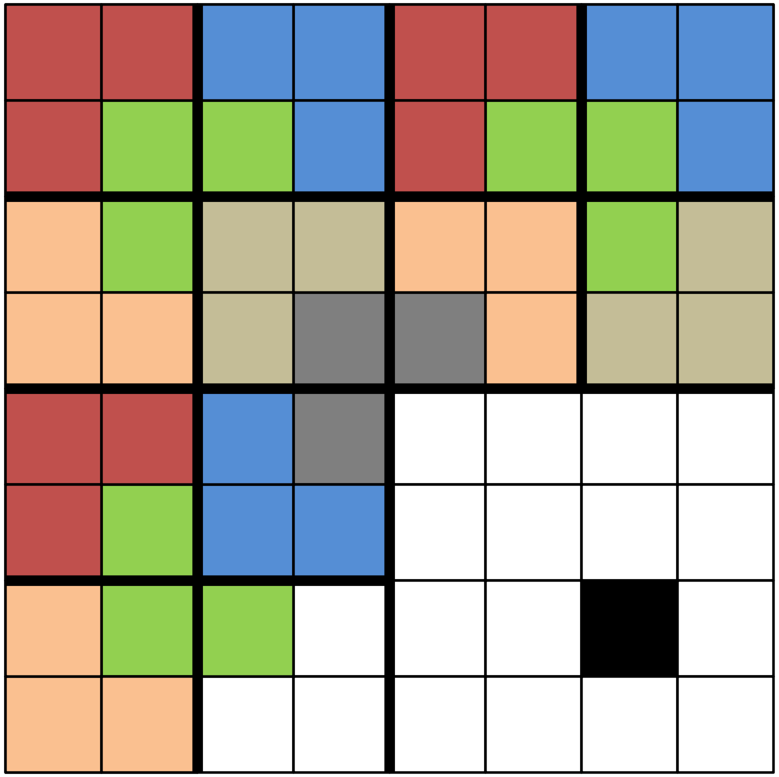

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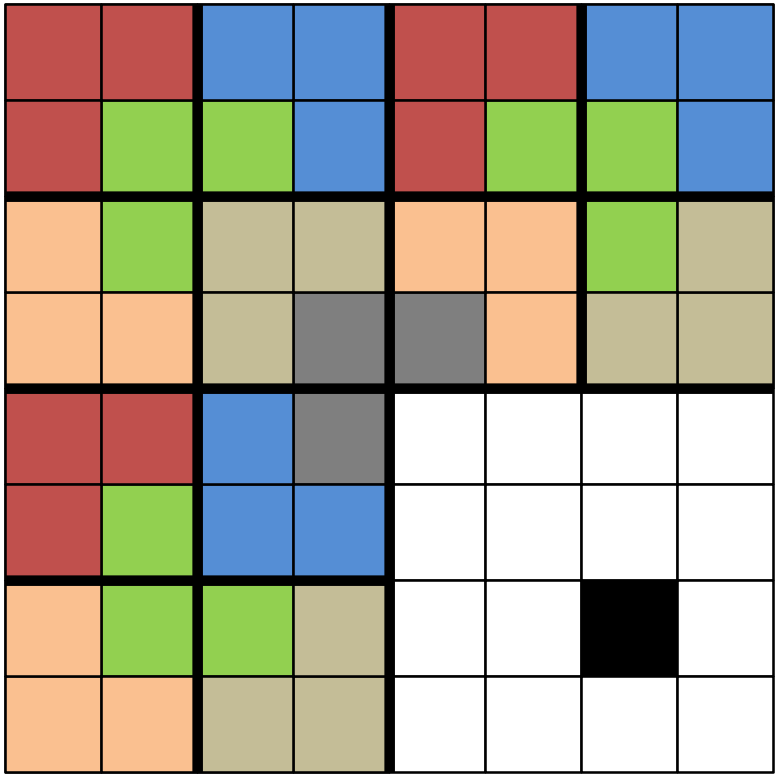

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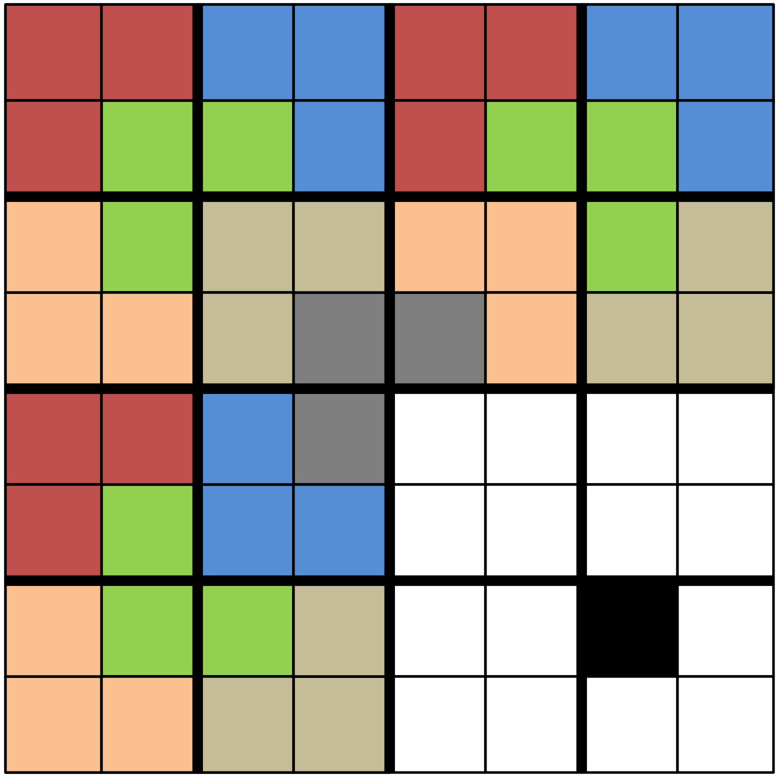

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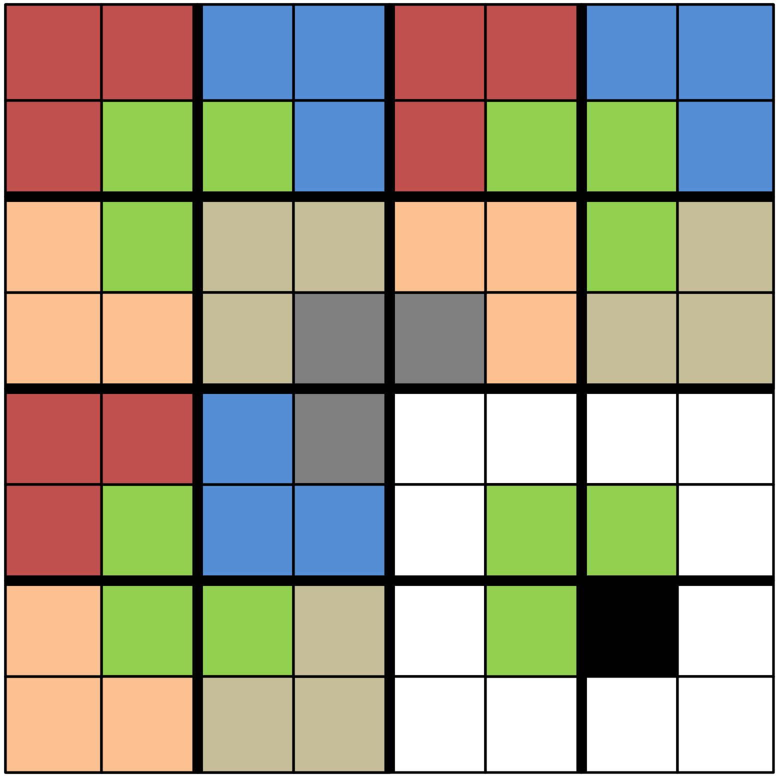

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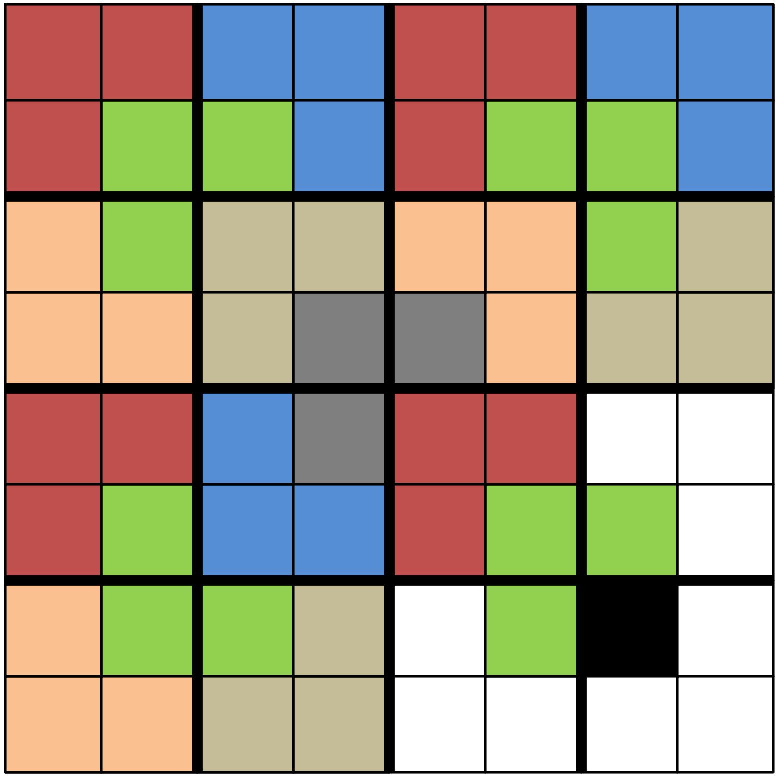

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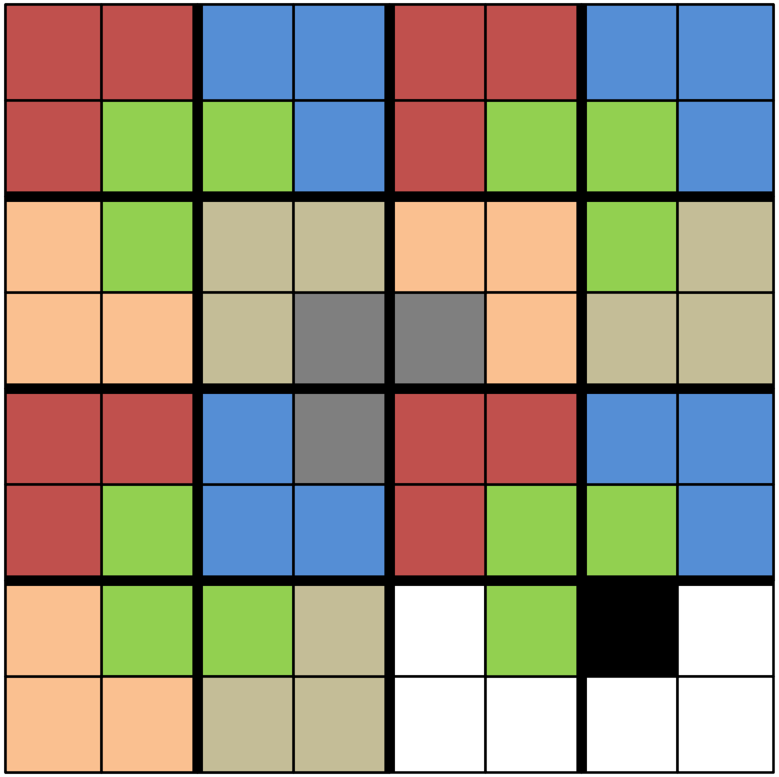

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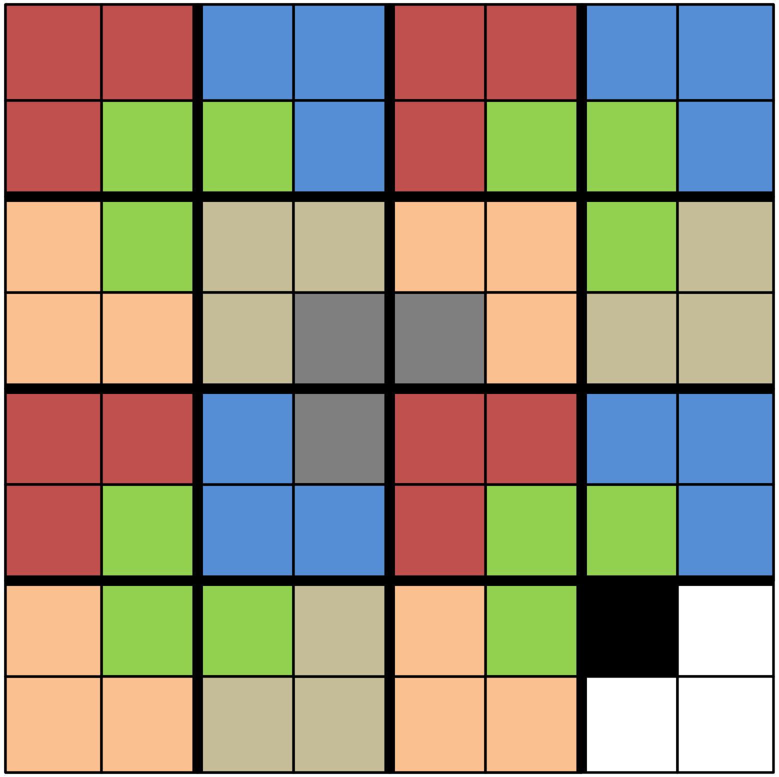

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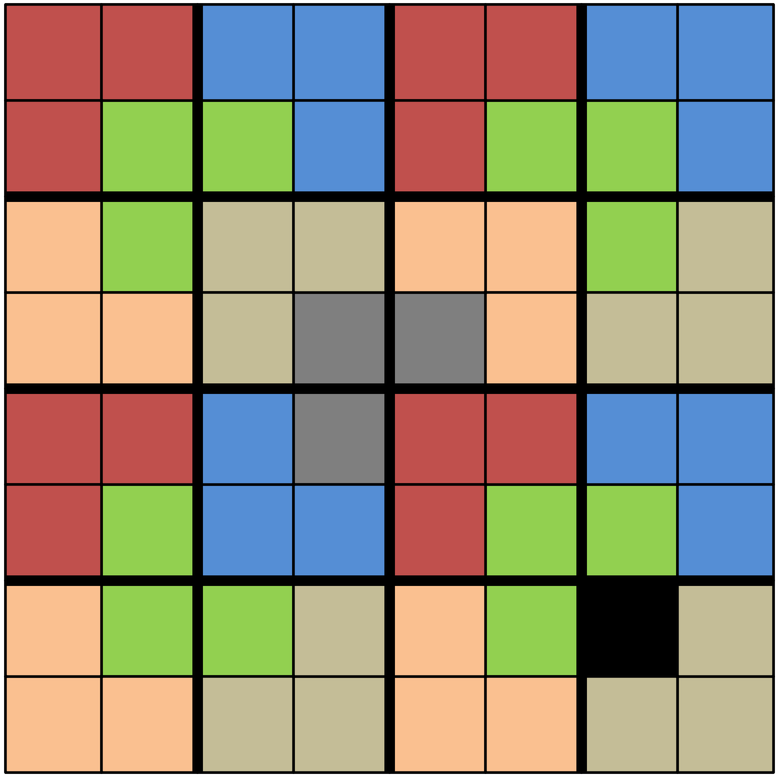

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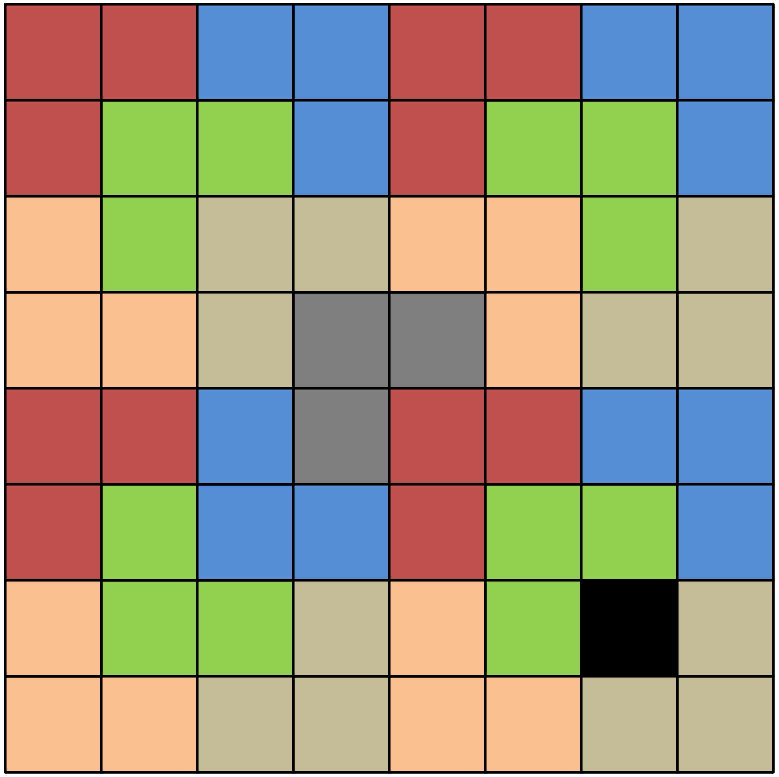

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– This algorithm design technique is called *recursive divide & conquer.*

A Latin Phrase

"Divide et impera"

(meaning: "divide and rule" or "divide and conquer")

— Philip II, king of Macedon (382-336 BC),
describing his policy toward the Greek city-states
(some say the Roman emperor Julius Caesar,
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Unsurprisingly, this is also a very powerful problem solving strategy in computer science.

Divide-and-Conquer

1. **Divide:** divide the original problem into smaller subproblems that are easier are to solve
2. **Conquer:** solve the smaller subproblems (perhaps recursively)
3. **Merge:** combine the solutions to the smaller subproblems to obtain a solution for the original problem

Integer Multiplication

Multiplying Two n-bit Numbers

$$x = \underbrace{\overbrace{\boxed{\quad x_L \quad}}^{\frac{n}{2}\,bits}\overbrace{\boxed{\quad x_R \quad}}^{\frac{n}{2}\,bits}} = 2^{n/2}x_L + x_R$$

$$y = \underbrace{\boxed{\quad y_L \quad}\boxed{\quad y_R \quad}}_{n\ bits} = 2^{n/2}y_L + y_R$$

$$xy = \left(2^{n/2}x_L + x_R\right)\left(2^{n/2}y_L + y_R\right) = 2^n x_L y_L + 2^{n/2}(x_L y_R + x_R y_L) + x_R y_R$$

So # $\frac{n}{2}$-bit products: 4

bit shifts (by n or $\frac{n}{2}$ bits): 2

additions (at most $2n$ bits long) : 3

We can compute the $\frac{n}{2}$-bit products recursively.

Let $T(n)$ be the overall running time for n-bit inputs. Then

$$T(n) = \begin{cases} \Theta(1) & if\ n = 1, \\ 4T\left(\frac{n}{2}\right) + \Theta(n) & otherwise. \end{cases} = \Theta\left(n^2\right) \quad \text{(how? derive)}$$

Multiplying Two *n*-bit Numbers Faster (Karatsuba's Algorithm)

$$x = \underbrace{\overbrace{\boxed{x_L}}^{\frac{n}{2}\,bits}\overbrace{\boxed{x_R}}^{\frac{n}{2}\,bits}} = 2^{n/2}x_L + x_R$$

$$y = \underbrace{\boxed{y_L}\boxed{y_R}}_{n\,bits} = 2^{n/2}y_L + y_R$$

$$\begin{aligned} xy &= \left(2^{n/2}x_L + x_R\right)\left(2^{n/2}y_L + y_R\right) \\ &= 2^n x_L y_L + 2^{n/2}(x_L y_R + x_R y_L) + x_R y_R \\ &= 2^n x_L y_L + 2^{n/2}\left((x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R\right) + x_R y_R \end{aligned}$$

So # $\frac{n}{2}$- or $\left(\frac{n}{2} + 1\right)$ -bit products: 3

Then the overall running time for n-bit inputs:

$$T(n) = \begin{cases} \Theta(1) & if\ n = 1, \\ 3T\left(\frac{n}{2}\right) + \Theta(n) & otherwise. \end{cases}$$

$$= \Theta\left(n^{\log_2 3}\right) = O\left(n^{1.59}\right)$$(how? derive)

Algorithms for Multiplying Two *n*-bit Numbers

Inventor	Year	Complexity
Classical	—	$\Theta\left(n^2\right)$
Anatolii Karatsuba	1960	$\Theta\left(n^{\log_2 3}\right)$
Andrei Toom & Stephen Cook (generalization of Karatsuba's algorithm)	1963 – 66	$\Theta\left(n 2^{\sqrt{2 \log_2 n}} \log n\right)$
Arnold Schönhage & Volker Strassen (Fast Fourier Transform)	1971	$\Theta(n \log n \log \log n)$
Martin Fürer (Fast Fourier Transform)	2005	$n \log n\, 2^{\mathrm{O}(\log^* n)}$

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Lower bound: $\Omega(n)$ (why?)