Homework #4

(Due: Dec 6)

Task 1. [60 Points] Jugs and Glasses

You have an infinite supply of water in one liter jugs $(J_1, J_2, \ldots)$, each of which is full of water. You will have to fill a finite number of glasses with water from the jugs. Say, there are $n > 0$ such glasses, and for $1 \leq i \leq n$, the capacity of the i-th glass is $g_i \in (0, 1]$ liter. Each glass must be filled to the brim from exactly one jug. What is the minimum number of jugs needed to fill all glasses?

Fill-Glasses($g_1,\ g_2,\ \ldots, g_n$)
Input: Capacities (in liter) of $n > 0$ glasses: $g_1, g_2, \ldots, g_n$, where for $1 \leq i \leq n$, $g_i \in (0, 1]$.
Output: Number m of one liter jugs needed to fill all glasses to the brim, where m is not necessarily optimal. Each glass must be completely filled using water from exactly one jug.

1. $m \leftarrow 0,\ \ c \leftarrow 0$
2. ***for*** $i \leftarrow 1$ ***to*** n ***do***
3. if $c < g_i$ ***then***
4. $m \leftarrow m + 1,\ \ c \leftarrow 1$
5. $c \leftarrow c - g_i$
6. ***return*** m

Figure 1: Filling glasses with water.

(a) [**45 Points**] Consider the algorithm given in Figure 1 for solving the "Jugs and Glasses" problem. Prove that the number of jugs returned by the algorithm is within a factor of 2 of optimal.

(b) [**15 Points**] Give an example to show that the bound you proved in part (a) is tight.

Task 2. [140 Points] Reducing Unhappiness

Your task is to divide $n \geq 2$ people into two disjoint groups in order to reduce their unhappiness as explained below.

We identify each person by a unique integer between 1 and n. For $1 \leq i \neq j \leq n$, we say that $\langle i, j \rangle$ is an unhappy pair provided person i does not like to be in the same group as person j, and vice versa. The *unhappiness score* of a group is given by the number of unhappy pairs in that group, and the unhappiness score of a collection of groups is the sum of the unhappiness scores of all groups in that collection.

You will be given the set P of all m unhappy pairs among the given n people, where $0 \leq m \leq \frac{n(n-1)}{2}$.

DETERMINISTIC-REDUCTION-OF-UNHAPPINESS(n, m, P)

Input: Number of people n, number of unhappy pairs m, and set P of all m unhappy pairs.
Output: Returns two groups G_1 and G_2 such that $|G_1| + |G_2| = n$, $G_1 \cap G_2 = \emptyset$, and the reduction in the unhappiness score due to this grouping is within a factor of 2 of optimal.

1. Let G_1 be an arbitrary subset of $\{1, 2, \ldots, n\}$, and let $G_2 \leftarrow \{1, 2, \ldots, n\} \setminus G_1$
2. *done* $\leftarrow$ FALSE
3. ***while*** *done* $=$ FALSE ***do***
4.   ***if*** $\exists x \in G_1$ such that moving x from G_1 to G_2 reduces total unhappiness score of G_1 and G_2 ***then***
5.   move x from G_1 to G_2
6.   ***elif*** $\exists y \in G_2$ such that moving y from G_2 to G_1 reduces total unhappiness score of G_1 and G_2 ***then***
7.   move y from G_2 to G_1
8.   ***else***
9.   *done* $\leftarrow$ TRUE
10. ***endwhile***
11. ***return*** $\langle G_1, G_2 \rangle$

Figure 2: Deterministic algorithm for reducing unhappiness.

Let G_1 and G_2 be the two groups you have created with unhappiness score m_1 and m_2, respectively. Let $\Delta = m - (m_1 + m_2)$. Your goal is to make Δ as large as possible.

(*a*) [**10 Points**] Argue that Δ can never be negative, and that $\Delta > 0$ provided $m > 0$.

(*b*) [**20 Points**] Argue that the ***while*** loop in lines 3–10 of DETERMINISTIC-REDUCTION-OF-UNHAPPINESS given in Figure 2 iterates at most m times, and as a result the algorithm runs in time polynomial in n.

(*c*) [**40 Points**] Prove that the Δ value corresponding to sets G_1 and G_2 returned by DETERMINISTIC-REDUCTION-OF-UNHAPPINESS is within a factor of 2 of optimal.

(*d*) [**10 Points**] Argue that the solution returned by RANDOMIZED-REDUCTION-OF-UNHAPPINESS given in Figure 3 is within a factor of 2 of optimal.

(*e*) [**60 Points**] Show that the expected running time of RANDOMIZED-REDUCTION-OF-UNHAPPINESS is polynomial in n. Find a high probability (upper) bound (w.r.t. n) on the running time of the algorithm.

Randomized-Reduction-of-Unhappiness(n, m, P)

Input: Number of people n, number of unhappy pairs m, and set P of all m unhappy pairs.
Output: Returns two groups G_1 and G_2 such that $|G_1| + |G_2| = n$, $G_1 \cap G_2 = \emptyset$, and the reduction in the unhappiness score due to this grouping is within a factor of 2 of optimal.

$done \leftarrow$ FALSE
while $done =$ FALSE ***do***
    $G_1 \leftarrow \emptyset$
    for $x \leftarrow 1$ ***to*** n ***do***
        set $G_1 \leftarrow G_1 \cup \{x\}$ with probability $\frac{1}{2}$
    $G_2 \leftarrow \{1, 2, \ldots, n\} \setminus G_1$
    $m_1 \leftarrow$ unhappiness score of G_1
    $m_2 \leftarrow$ unhappiness score of G_2
    if $m_1 + m_2 \leq \frac{m}{2}$ ***then***
        $done \leftarrow$ TRUE
endwhile
return $\langle G_1, G_2 \rangle$

Figure 3: Randomized algorithm for reducing unhappiness.