# A $C^1$ Globally Interpolatory Spline of Arbitrary Topology

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**Abstract.** Converting point samples and/or triangular meshes to a more compact spline representation for arbitrarily topology is both desirable and necessary for computer vision and computer graphics. This paper presents a $C^1$ manifold interpolatory spline that can exactly pass through all the vertices and interpolate their normals for data input of complicated topological type. Starting from the Powell-Sabin spline as a building block, we integrate the concepts of global parametrization, affine atlas, and splines defined over local, open domains to arrive at an elegant, easy-to-use spline solution for complicated datasets. The proposed global spline scheme enables the rapid surface reconstruction and facilitates the shape editing and analysis functionality.

## 1 Introduction

Constructing smooth interpolatory spline surfaces from any data input in 3D is frequently needed in visual computing. Given a scattered point cloud, $\{\mathbf{P}_i = (x_i, y_i, z_i)\}_{i=1}^m$, and associated normal vectors $\{\mathbf{n}_i = (nx_i, ny_i, nz_i)\}_{i=1}^m$, the goal of this paper is to find a smooth surface $\mathbf{F}$ that interpolates both the vertex positions and their normals simultaneously of complicated topological type.

Unlike most of the conventional methods which typically trim parametric spline surfaces defined over open planar domains, stitch them along their trimmed boundaries with care, and enforce the smoothness requirements of certain degree across their common boundaries, our spline scheme is global and interpolatory. It can faithfully reconstruct smooth shapes of any manifold from geometric input without resorting to any patching and/or trimming operations. The technical core of our new approach is the Powell-Sabin spline defined over any open, triangulated domain. The primary goal is the exact interpolation (for both vertices and their normals), therefore, the Powell-Sabin spline scheme is an ideal candidate for this requirement. Nonetheless, the technical challenge is how to generalize the Powell-Sabin spline defined over planar, triangulated domains to a global spline spanning over domain of complicated topology without any cutting and patching work. We accomplish this mission through the following steps: (1) The initial, raw data input is globally parameterized in order to map the 3D geometry onto a 2D domain; (2) For any 3D point, we are only interested in a certain localized 2D region in its vicinity; (3) We decompose the entire 3D geometry into a suite of overlapping regions and construct their corresponding affine atlases on 2D; (4) These affine

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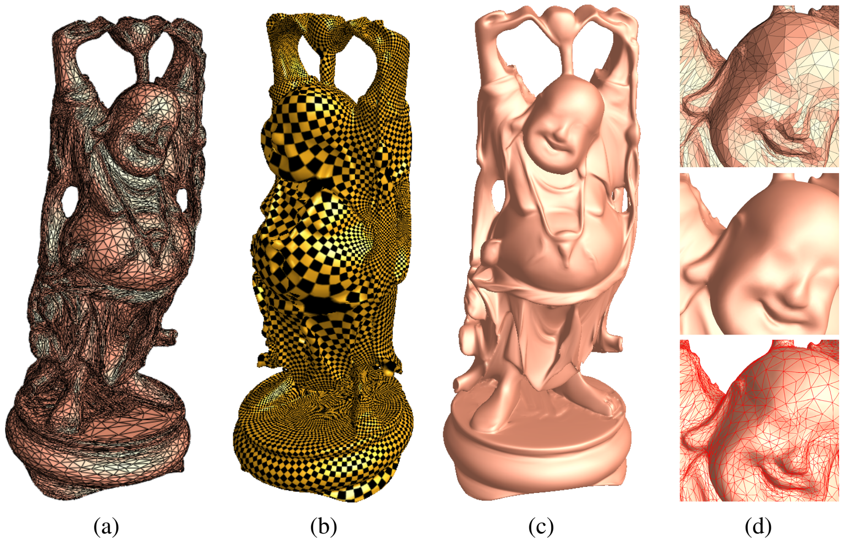



**Fig. 1.** Globally interpolatatory spline: (a) A genus-6 Buddha model with 25K vertices; (b) Global conformal parameterization; (c) A global $C^1$ spline surface which interpolates all the vertices and their normals of (a); (d) Close-up view: *top*, original mesh; *middle*, spline surface; *bottom*, spline surface with the red curves corresponding to the edges in the mesh

charts in 2D constitute all the local parametric domain for defining all the open Powell-Sabin spline surfaces that interpolate only a subset of data points; (5) These locally defined spline surfaces span across their neighbors and share some common regions; and finally (6) We build a globally interpolatory spline by collecting all the control points and using all the affine atlases as their global domain.

## 2 Previous Work

### 2.1 Planar Powell-Sabin Spline

Powell-Sabin splines are functions in the space $S_2^1(\Delta_{ps})$ of $C^1$ continuous piecewise quadratic functions on a Powell-Sabin refinement [1]. Such a refinement $\Delta_{ps}$ can be obtained from an arbitrary triangulation $\Delta$ by splitting each triangle into six subtriangles with a common interior point. In contrast to triangular Bézier splines, where imposing smoothness conditions between the patches requires a great number of nontrivial relations between the control points to be satisfied, the $C^1$ continuity of a Powell-Sabin spline is guaranteed for any choice of the control points.

The first B-spline representation of Powell-Sabin spline was derived by Shi et al. [2]. However, their construction approach had serious drawbacks from the numerical point of view. Dierckx [3] resolved the numerical problem by constructing a normalized *B*-spline basis for Powell-Sabin splines. This representation has a very nice geometric interpretation involving the tangent control triangles for manipulating the Powell-Sabin surfaces. Since then, the normalized Powell-Sabin spline has been receiving much attention in the computer aided geometric design community. Surface approximation and interpolation using Powell-Sabin spline have been reported in [4,5,6]. Windmolders and Dierckx solved the subdivision problem for uniform Powell-Sabin splines, that is on triangulations with all equilateral triangles [7]. Recently, Vanraes et al. present the subdivision rule for general Powell-Sabin spline [8].

### 2.2 Interpolatory Spline

Interpolation is a very useful and intuitive feature in computer aided geometric design. Two different research directions have been pursued. One is based on the subdivision surfaces that recursively subdivide the control mesh, such as the butterfly scheme [9] or modified butterfly scheme [10]. The other direction consists of building a patch of smoothly joined parametric patches. This paper focuses on the spline based interpolation scheme. There exists a vast literature on interpolation by splines over triangulations (see the survey [11] and the references therein). In the interest of the space, we only cite few of them which are closely related to our work.

Hahmann and Bonneau [12] presented a piecewise quintic $G^1$ spline surface interpolating the vertices of a triangular surface mesh of arbitrary topological type. They further improved the method without imposing any constraint on the first derivatives and thus avoid any unwanted undulations when interpolating irregular triangulations [13]. Nürnberger and Zeilfelder presented [14] a local Lagrange interpolation scheme for $C^1$-splines of degree $q \geq 3$ on arbitrary triangulations. This interpolating spline yields optimal approximation order and can be computed with linear complexity.

### 2.3 Manifold Construction

There are some related work on defining functions over manifold. Grimm and Hugues [15] pioneered a generic method to extend B-splines to surfaces of arbitrary topology, based on the concept of overlapping charts. Cotrina et al. proposed a $C^k$ construction on manifold [16,17]. Ying and Zorin [18] presented a manifold-based smooth surface construction method which has $C^\infty$-continuous with explicit nonsingular parameterizations. Recently, Gu, He and Qin [19] developed a general theoretical framework of manifold splines in which spline surfaces defined over planar domains can be systematically generalized to any manifold domain of arbitrary topology (with or without boundaries). Manifold spline is completely different from the above methods in that: 1) The transition functions of manifold spline must be affine. Therefore, the requirements of manifold spline is much stronger. That is why topological obstruction plays an important role in the construction. 2) Manifold spline produces the polynomial or rational polynomials. On any chart, the basis functions are always polynomials or rational polynomials, and represented as $B$-splines or rational $B$-splines.

In [19], Gu et al. defined the manifold spline based on triangular $B$-spline [20]. This construction requires a complicated data fitting procedure when converting points to splines. Inspired by [19], we strive to devise a globally interpolatory splines that are founded upon the original work of [3]. Our method is different from the above methods in that: 1) All the existing developments of Powell-Sabin splines are defined on the planar domain; 2) The existing global interpolatory splines need patching and stitching work; 3) All the manifold constructions except the manifold splines do not produce globally polynomials or rational polynomials. Our work generalizes the planar Powell-Sabin spline to arbitrary manifold without any patching and stitching work. Also, due to the nice properties of the normalized Powell-Sabin spline, our method can interpolate both positions and normals.

## 3 The Globally Interpolatory Spline

This section first reviews the normalized planar Powell-Sabin $B$-spline [3] and then presents all the necessary components for our global spline scheme.

### 3.1 Powell-Sabin Spline on the Planar Domain

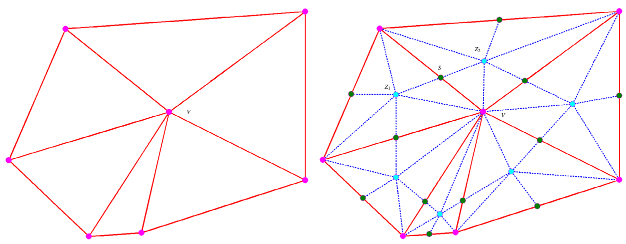
Let $\Omega$ be a polygonal domain in $\mathbb{R}^2$ and let $\Delta$ be a conforming triangulation of $\Omega$, comprising triangles $\rho_j$, $j = 1,\ldots,N_t$, having vertices $V_i := (x_i, y_i), i = 1,\ldots,N_v$. A Powell-Sabin refinement, $\Delta_{ps}$ of $\Delta$ is the refined triangulation, obtained by subdividing each triangle of $\Delta$ into six sub-triangles as follows. Select an interior point $Z_j$ in each triangle $\rho_j$ and connect it with the three vertices of $\rho_j$ and with the points $Z_{j_1}, Z_{j_2}, Z_{j_3}$ where $\rho_{j_1}, \rho_{j_2}, \rho_{j_3}$ are the triangles adjacent to $\rho_j$ (See Figure 2). We denote by $S_2^1(\Delta_{ps})$ the space of piecewise $C^1$ continuous quadratic polynomials on $\Delta_{ps}$. Powell and Sabin [1] proved that the dimension of the space $S_2^1(\Delta_{ps})$ equals to $3N_v$ and any element of $S_2^1(\Delta_{ps})$ is uniquely determined by its value and its gradient at the vertices of $\Delta$, i.e., there exists a unique solution $s(x,y) \in S_2^1(\Delta_{ps})$ for the interpolation problem

$$s(V_i) = f_i,\ \frac{\partial}{\partial x} s(V_i) = f_{x,i},\ \frac{\partial}{\partial y} s(V_i) = f_{y,i},\ i = 1,\ldots,N_v. \tag{1}$$

So given the function and its derivative values at each vertex $V_i$, the Bézier ordinates on the domain sub-triangles are uniquely defined and the continuity conditions between sub-triangles are automatically enforced.

Dierckx [3] showed that each piecewise polynomial $s(x,y) \in S_2^1(\Delta_{ps})$ has a unique representation

$$s(x,y) = \sum_{i=1}^{N_v} \sum_{j=1}^{3} c_{ij} B_i^j(x,y),\ (x,y) \in \Omega \tag{2}$$



(a) (b)

**Fig. 2.** The Powell-Sabin refinement $\Delta^*$ (b) of a triangulation $\Delta$ (a)

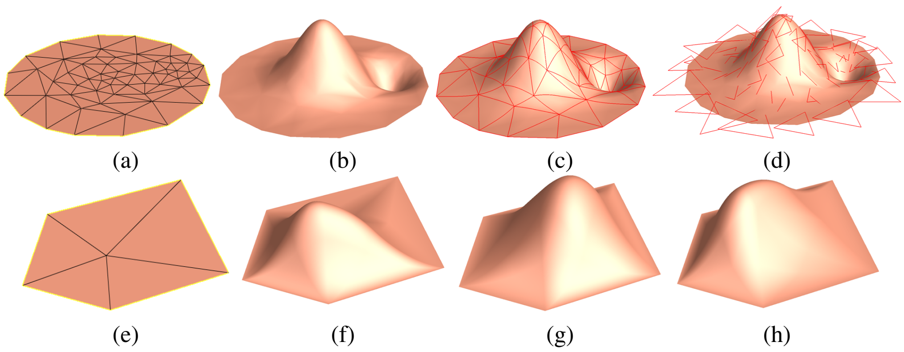



**Fig. 3.** Powell-Sabin spline over a planar domain: (a) Domain etriangulation; (b) Spline surface; (c) Spline surface, the red curves correspond to the edges in the domain triangulation; (d) Spline surface overlaid by the control triangles (shown in red) which are tangent to the surface; (e) The molecule of one vertex **v**; (f)-(h) Three basis functions associated with vertex **v**

where the basis functions form a partition of unity, i.e.,

$$B_i^j(x,y) \geq 0 \tag{3}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{3} B_i^j(x,y) = 1 \text{ for all } x,y \in \Omega \tag{4}$$

Furthermore, these basis functions have local support: $B_i^j(x,y)$ vanishes outside the so-called molecule $M_i$ of vertex $V_i$, which is the union of all triangles $\mathcal{T}_k$ containing $V_i$.

The basis functions $B_i^j(x,y)$ can be obtained by finding three linearly independent triplets $(\alpha_{ij},\beta_{ij},\gamma_{ij})$, $j=1,2,3$ for each vertex $V_i$. $B_i^j(x,y)$ is the unique solution of the interpolation problem with $(f_k,f_{xk},f_{yk}) = (\delta_{ki}\alpha_{ij},\delta_{ki}\beta_{ij},\delta_{ki}\gamma_{ij})$, where $\delta_{ki}$ is the Kronecker delta. The triplets $(\alpha_{ij},\beta_{ij},\gamma_{ij})$, $j=1,2,3$ are determined by the following Dierckx's algorithm [3,21]:

1. For each vertex $v_i$, find its Powell-Sabin triangle points, which are the immediately surrounding Bézier domain points of the vertex $v_i$ and vertex $v_i$ itself.
2. For each vertex $v_i$, find a triangle $t_i(Q_{i1},Q_{i2},Q_{i3})$ which contains all the Powell-Sabin triangle points of $v_i$ from all the triangles in the molecule $M_i$. Denote $Q_{ij} = (X_{ij},Y_{ij})$ the position of vertex $Q_{ij}$.
3. Three linearly independent triplets of real numbers $\alpha_{ij},\beta_{ij},\gamma_{ij}$, $j=1,2,3$ can be derived from the Powell-Sabin triangle $t_i$ of a vertex $v_i$ as follows:
   $(\alpha_{i1},\alpha_{i2},\alpha_{i3})$ = Barycentric coordinate of $v_i$ with respect to $t_i$,
   $(\beta_{i1},\beta_{i2},\beta_{i3}) = ((Y_{i2}-Y_{i3})/h,(Y_{i3}-Y_{i1})/h,(Y_{i1}-Y_{i2})/h)$,
   $(\gamma_{i1},\gamma_{i2},\gamma_{i3}) = ((X_{i3}-X_{i2})/h,(X_{i1}-X_{i3})/h,(X_{i2}-X_{i1})/h)$,
   where $h = \det\begin{pmatrix} 1 & 1 & 1 \\ X_{i1} & X_{i2} & X_{i3} \\ Y_{i1} & Y_{i2} & Y_{i3} \end{pmatrix}$.

We then define the control triangles as $T_i(\mathbf{C}_{i1}, \mathbf{C}_{i2}, \mathbf{C}_{i3})$. Dierckx proved that the normalized Powell-Sabin spline has a very nice geometric interpretation that the control triangle is tangent to the spline surface [3].

Figure 3 illustrates an example of Powell-Sabin spline surface over a planar triangulated domain. Note that, their basis functions $B_i^j(\mathbf{u})$ vanish outside the molecule $M_i$ (see Figure 3(e-h)). Furthermore, the control points $(\mathbf{C}_{i1}, \mathbf{C}_{i2}, \mathbf{C}_{i3})$ form a control triangle which is always tangent to the spline surface at $\mathbf{s}(\mathbf{v}_i)$ (see Figure 3(d)).

### 3.2 Generalizing Powell-Sabin Spline to Arbitrary Topology

In [19], Gu et al. addressed several key technical issues of manifold splines in which spline surfaces defined over planar domains can be systematically extended to manifold domains of arbitrary topology. In a nutshell, a manifold spline can be intuitively interpreted as a set of spline patches that are automatically glued in a coherent and consistent way without any gap, such that all the patches collectively cover the entire manifold. The surface evaluation can be easily conducted using the control points and corresponding basis functions of any overlapping patches, without leading to any inconsistency. The followings are the necessary theoretical results which enable our global spline scheme based on Powell-Sabin's approach.

**Theorem 1.** *The sufficient and necessary condition for a manifold $M$ to admit manifold spline is that $M$ must be an affine manifold.*

This theorem implies that the existence of manifold splines solely depends on the existence of affine atlas. If the domain manifold $M$ is an affine manifold, we will be able to directly generalize the local spline patches to a global spline defined on $M$. Details about the affine manifold and affine atlas can be found in the Appendix.

**Theorem 2.** *The only closed surface admitting affine atlas is of genus one. All oriented open 2-manifolds admit an affine atlas.*

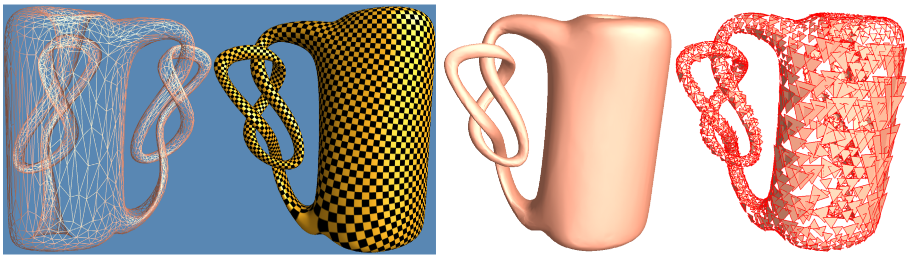
Theorem 2 points out that not all surfaces admit the affine atlas. The topological obstruction of a global affine atlas is the Euler class. In fact, by removing one point from the closed domain manifold, we can convert it to an affine manifold.

**Theorem 3 (Affine atlas deduced from conformal structure).** *Given a closed genus $g$ surface $M$, and a holomorphic 1-form $\omega$. Denote by $Z = \{zeros\ of\ \omega\}$ the zero points of $\omega$. Then the size of $Z$ is no more than $2g-2$, and there exists an affine atlas on $M/Z$ deduced by $\omega$.*

Essentially, Theorem 3 indicates that an affine atlas of a manifold $M$ can be deduced from its conformal structure in a straightforward fashion.

### 3.3 Algorithmic Details

Given a triangle mesh $M$ of arbitrary topological type, we want to find a manifold Powell-Sabin spline which interpolates the vertices of $M$ and their normals. Our spline surface construction algorithm consists of two consecutive steps: (1) compute the global conformal parameterization; and (2) construct the global spline.



(a) (b) (c) (d)

**Fig. 4.** Interpolation of a genus-2 model. (a) The two-hole bottle model with 2K vertices; (b) Global conformal parameterization; (c) Spline surface; (d) Control triangles.

**Compute Global Conformal Parameterization and Affine Atlas.** As mentioned above, the domain manifold $M$ admits a manifold spline if and only if it has affine atlas, which can be deduced from the conformal structure of $M$ directly. Thus, in order to construct our global spline, we shall first compute the conformal structures of the domain manifold $M$. A conformal atlas is an atlas such that all transition functions are analytic. Two conformal atlases are compatible if their union is still a conformal atlas. All compatible conformal atlases form conformal structure. It is known that all surfaces have conformal structure and are called Riemann surfaces. The algorithm to compute global conformal parameterization and affine atlas is as follows:

1. Compute the holomorphic 1-form $\omega$ of $M$ using Gu-Yau's algorithm [22].
2. Remove the zero points $Z$ of $\omega$ and the adjacent faces.
3. Construct an open covering for $M/Z$. For each vertex $\mathbf{V}_i$, take the union of all faces within its molecule as an open set, denoted by $U_i$.
4. Test if the union of any two $U_i$'s is a topological disk by checking the Euler number. If not, subdivide $U_i$.

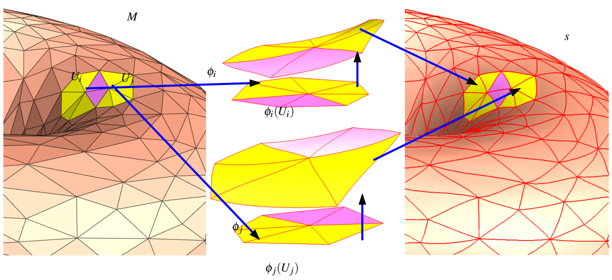



**Fig. 5.** Constructing local spline patches: The parametric domain $M$ is a triangular mesh of arbitrary topology as shown on the left. The polynomial spline surface $\mathbf{s}$ is shown on the right. Two overlapping spline patches are magnified and highlighted in the middle. On each parameter chart $(U_i, \phi_i)$, $(U_j, \phi_j)$, the surface is a locally defined planar Powell-Sabin spline patch. For the overlapping part, its two planar domains differ only by an affine transformation $\phi_{ij}$.

5. Pick one vertex $p_i \in U_i$, for any vertex $p \in U_i$, define $\phi_i(p) = \int_{p_i}^{p} \omega$.
6. Compute coordinate transition functions $\phi_{ij} = \int_{p_i}^{p_j} \omega$.

**Global Spline Construction.** Note that the evaluation of Powell-Sabin spline over any planar region relies on the computation of Barycentric coordinates of the parameter with respect to the domain triangles. If we change the parameter by an affine transformation, the evaluation is invariant and the final shape of the spline surface will not be changed. Figure 5 highlights the transition from local patches to the global spline. The algorithm to construct the global spline is as follows:

1. Prepare the underlying parameteric domain (For any vertex $\mathbf{V}_i \in M$, denote by $(U_i, \phi_i)$ its parametric chart which contains the molecule of $\mathbf{V}_i$).
2. Compute the three linear independent triplets, $(\alpha_{ij}, \beta_{ij}, \gamma_{ij})$, $j = 1,2,3$. Build the basis functions using the above Dierckx's algorithm.
3. Assign the control points $(\mathbf{C}_{i1}, \mathbf{C}_{i2}, \mathbf{C}_{i3})$ which satisfy

$$\mathbf{V}_i = \sum_{j=1}^{3} \alpha_{ij} \mathbf{C}_{ij} \tag{5}$$

and

$$\frac{(\mathbf{C}_{i1} - \mathbf{C}_{i2}) \times (\mathbf{C}_{i1} - \mathbf{C}_{i3})}{\| (\mathbf{C}_{i1} - \mathbf{C}_{i2}) \times (\mathbf{C}_{i1} - \mathbf{C}_{i3}) \|} = \mathbf{n}_i = (nx_i, ny_i, nz_i)^T \tag{6}$$

One can prove that the control triangle $(\mathbf{C}_{i1}, \mathbf{C}_{i2}, \mathbf{C}_{i3})$ is tangent to the spline surface $\mathbf{s}$ at $\mathbf{V}_i$, i.e.,

$$\mathbf{s}(\phi_i(\mathbf{V}_i)) = \mathbf{V}_i \tag{7}$$

$$\frac{\mathbf{s}_u(\phi_i(\mathbf{V}_i)) \times \mathbf{s}_v(\phi_i(\mathbf{V}_i))}{\| \mathbf{s}_u(\phi_i(\mathbf{V}_i)) \times \mathbf{s}_v(\phi_i(\mathbf{V}_i)) \|} = \mathbf{n}_i \tag{8}$$

The detailed proof is in the Appendix.

**Variational Shape Design.** In the Powell-Sabin spline scheme, each vertex of the domain triangulation is associated with three control points. In the above spline construction step, we require the control points satisfying Equation (5) and (6). Therefore, there are still three degrees of freedom remaining. We can use these free variables for variational shape design. For example, we can fair the spline surface by minimizing the following energy functional subject to the interpolation constraints:

$$\min \ \alpha \iint_M (\mathbf{s}_u^2 + \mathbf{s}_v^2) dudv + \beta \iint_M (\mathbf{s}_{uu}^2 + 2\mathbf{s}_{uv}^2 + \mathbf{s}_{vv}^2) dudv \tag{9}$$

$$\begin{aligned} \text{subject to } \ \mathbf{V}_i &= \sum_{j=1}^{3} \alpha_{ij} \mathbf{C}_{ij} \\ \langle \mathbf{C}_{i1} - \mathbf{C}_{i2}, \mathbf{n}_i \rangle &= 0 \\ \langle \mathbf{C}_{i2} - \mathbf{C}_{i3}, \mathbf{n}_i \rangle &= 0, \ \text{ for each vertex } \mathbf{V}_i \in M, \end{aligned}$$

where $\langle , \rangle$ is the inner product, $u$ and $v$ are parameters on the local charts. The objective function is the standard thin-plate energy with membrane terms, which can be written

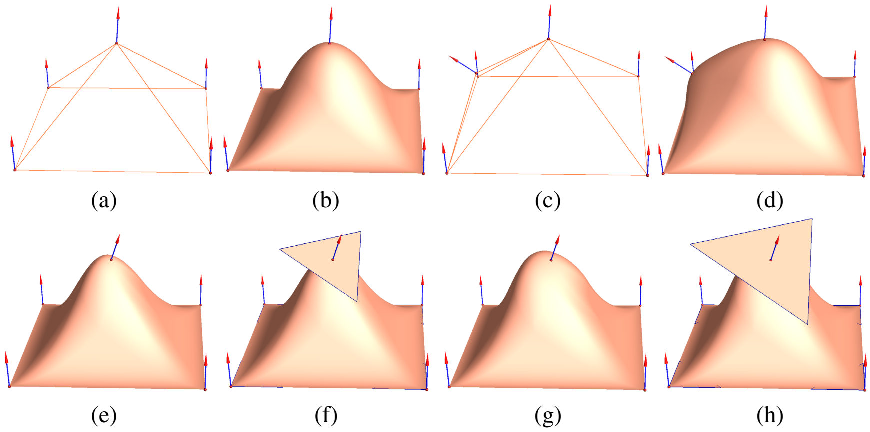



**Fig. 6.** Manipulation of manifold Powell-Sabin spline: The input is a triangular mesh $M$ with normal information as shown in (a). We construct a manifold Powell-Sabin spline $\mathbf{S}$ to interpolate both the positions and normals of $M$ (shown in (b)). We insert a new vertex $v$ in the original mesh and assign a normal to $v$ (shown in (c)). The corresponding spline is shown in (d). We can also change the normal but fix the positions, the spline and control triangles are shown in (e) and (f), respectively. We can even fix the vertices positions and their normals but change the size of the control triangles without violating the interpolation property. In (h), we enlarge the control triangle of the top-most vertex and get a new surface shown in (g). Note that the new spline still interpolates the positions and normals.

as a quadratic form of control points. Therefore, the above optimization problem can be solved efficiently using the Lagrange multiplier method.

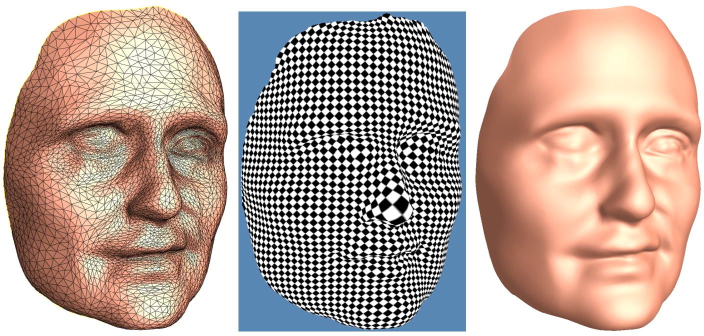
**Handling the Singular Points.** In [19], Gu et al. showed the manifold splines must have singular points if the domain manifold is closed and not a torus. The number of singular points is no more than $2g-2$ for a genus $g$ domain manifold $M$. The singular points $Z$ can be automatically detected from the conformal structure of $M$ by checking the winding number. Then the molecule of $Z$ is removed from $M$. No spline patches are defined on the molecule of $Z$. Therefore, there exist holes in the spline surface. For each hole, we compute a minimal surface spanning the hole such that it satisfies the given boundary condition.

### 3.4 Properties

The proposed globally interpolatory spline (based on Powell-Sabin spline over the planar domain) exhibits the following features:

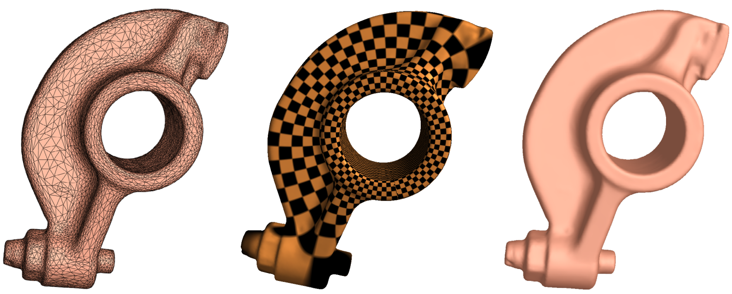
1. *Piecewise polynomial.* The global spline surface is a quadratic piecewise polynomial defined on the manifold $M$ which has arbitrary triangulation. It is globally $C^1$-continuous and very efficient to evaluate.
2. *Local support.* It has local support since the basis functions $B_i^j(\mathbf{u})$ vanish outside the molecule of $\mathbf{v}_i$.

3. *Tangent plane control/Interpolation/Local shape modification.* The control triangle $(\mathbf{C}_{i0}, \mathbf{C}_{i1}, \mathbf{C}_{i2})$ is tangent to the spline surface $\mathbf{s}$ at $\mathbf{V}_i$. Thus, by manipulating the control triangle, the spline surface can interpolate both positions and normals. Furthermore, besides interpolation of the positions and normals, the control triangle still has three degrees of freedom which can be used for local shape modification and variational shape design.
4. *Convex hull.* The polynomial surface is inside the convex hull of the control points.
5. *Local adaptive refinement.* Since there is no restriction on the triangulation of $M$, the spline surface can be locally refined by knot insertion, e.g., inserting a new vertex inside the existing triangle, or splitting any edge.
6. *Minimal number of singular points.* The number of singular points depends only on the topology of the manifold $M$, i.e., no more than $2g-2$ singular points for a genus $g$ domain manifold.



(a) (b) (c)

**Fig. 7.** Example of a genus-0 open surface: (a) The face model with 4K vertices; (b) Global conformal parameterization; (c) The globally interpolatory spline



(a) (b) (c)

**Fig. 8.** Example of a genus-1 surface: (a) The rockerarm model with 10K vertices; (b) Global conformal parameterization; (c) The globally interpolatory spline

## 4 Results

We have implemented a prototype system on a 3GHz Pentium IV PC with 1GB RAM. Figure 6 illustrates the various properties of manifold Powell-Sabin spline which is useful in computer aided geometric design. We perform experiments on several models of various topological types, i.e., a genus-0 face (Figure 7), a genus-1 rockerarm (Figure 8), a genus-2 bottle (Figure 4), and a genus-6 Happy Buddha (Figure 1). The overall computational procedure requires about $6 \sim 30$ minutes for our test models.

## 5 Conclusion

In this paper, our goal is to seek a global spline solution that will allow us to interpolate all vertices and their normals using one-piece spline representation without any cutting and stitching operations. Founded upon the Powell-Sabin spline, we have developed a new globally interpolatory spline which is truly one-piece formulation without generating any seams when crossing triangular edges on its domain mesh. The interpolation property is valuable for the reverse engineering task that can effectively convert point-cloud raw data to the compact spline formulation. Our globally interpolatory spline is also relevant to surface modeling, variational design, and interactive editing.

## Acknowledgements

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## Appendix: Proof of the Interpolation Property

We prove that our global spline (based on Powell-Sabin spline) interpolates the domain manifold $M$ and its normals, i.e., for a vertex $\mathbf{V}_i \in M$, $\mathbf{s}(\phi_i(\mathbf{V}_i)) = \mathbf{V}_i$ and $\mathbf{n}(\phi_i(\mathbf{V}_i)) = \mathbf{n}_i$ where $\phi_i : U_i \to \mathbb{R}^2$ maps the molecule of $\mathbf{V}_i$ to the planar domain.

The basis functions of vertices $\mathbf{V}_k$ have local support, i.e., they vanish outside the molecule of $\phi_k(\mathbf{V}_k)$. Therefore,

$$\mathbf{s}(\phi_i(\mathbf{V}_i)) = \sum_{i=1}^{N_v} \sum_{j=1}^{3} \mathbf{C}_{ij} B_i^j(\phi_i(\mathbf{V}_i)) = \sum_{j=1}^{3} \mathbf{C}_{ij} B_i^j(\phi_i(\mathbf{V}_i)) = \sum_{j=1}^{3} \mathbf{C}_{ij} \alpha_{ij} = \mathbf{V}_i.$$

The last equation results from the fact that $\alpha_{ij}$, $j = 1,2,3$ are also the Barycentric coordinate of $\mathbf{V}_i$ with respect to $(\mathbf{C}_{i1}, \mathbf{C}_{i2}, \mathbf{C}_{i3})$. Similarly, the normal $\mathbf{n}(\phi(\mathbf{V}_i))$ can be calculated as

$$\mathbf{n}(\phi_i(\mathbf{V}_i)) \propto \mathbf{s}_u(\phi_i(\mathbf{V}_i)) \times \mathbf{s}_v(\phi_i(\mathbf{V}_i)) = (\sum_{j=1}^{3} \mathbf{C}_{ij}\beta_{ij}) \times (\sum_{j=1}^{3} \mathbf{C}_{ij}\gamma_{ij})$$
$$= \lambda(\mathbf{C}_{i1} \times \mathbf{C}_{i2} + \mathbf{C}_{i2} \times \mathbf{C}_{i3} + \mathbf{C}_{i3} \times \mathbf{C}_{i1})$$
$$= \lambda(\mathbf{C}_{i1} - \mathbf{C}_{i2}) \times (\mathbf{C}_{i1} - \mathbf{C}_{i3}) \propto \mathbf{n}_i,$$

where $\lambda = \beta_{i1}\gamma_{i2} - \beta_{i2}\gamma_{i1} = \beta_{i3}\gamma_{i1} - \beta_{i1}\gamma_{i3} = \beta_{i2}\gamma_{i3} - \beta_{i3}\gamma_{i2}$. Therefore, the control triangle $(\mathbf{C}_{i1}, \mathbf{C}_{i2}, \mathbf{C}_{i3})$ is tangent to the surface $\mathbf{s}$ at vertex $\mathbf{V}_i$.