FEM-Based Dynamic Subdivision Splines

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Abstract

Recent years have witnessed dramatic growth in the use of subdivision schemes for graphical modeling and animation, especially for the representation of smooth, oftentimes complex shapes of arbitrary topology. Nevertheless, conventional interactive approaches to subdivision objects can be extremely laborious and inefficient. Users must carefully specify the initial mesh and/or painstakingly manipulate the control vertices at different levels of the subdivision hierarchy to satisfy a diverse set of functional requirements and aesthetic criteria in the modeled object. This modeling drawback results from the lack of direct manipulation tools for the limit geometric shape. To improve the efficiency of interactive design, we have developed a unified FEM-based dynamic methodology for arbitrary subdivision schemes by marrying principles of computational physics and finite element analysis with powerful subdivision geometry. Our dynamic framework permits users to directly manipulate the limit surface obtained from any subdivision procedure via simulated "force" tools. Our experiments demonstrate that the new unified FEM-based framework promises a greater potential for subdivision techniques in geometric modeling, finite element analysis, engineering design, computer graphics, and other visual computing applications.

Keywords: Physics-Based Modeling, Geometric Modeling, Computer Graphics, CAGD, Subdivision Splines, Deformable Models, Dynamics, Finite Elements, Interactive Techniques.

1 Motivation

Efficiently modeling and intuitively manipulating complex shapes are fundamental to computer graphics, engineering design, manufacturing, animation and simulation, analysis and evaluation, rapid and virtual prototyping, visualization, and interaction with virtual environments. Therefore, the success of future visual computing technology and system development hinges upon the advancement of powerful modeling methods, efficient design tools coupled with natural human-computer interaction techniques.

For geometric and visual modeling communities, the industry-standard Non-Uniform Rational B-Spline (NURBS) cannot represent surfaces of arbitrary topological genus in one piece due to their limited (rectangular) parameterization. In principle, modeling surfaces of arbitrary topology requires NURBS trimming and/or patching. Although possible in commercial CAD systems, creating complex objects based on NURBS patching and trimming suffers from several difficulties: (1) Trimming two NURBS patches to match their shared boundary frequently involves the computation of surface-surface intersection (SSI), SSI algorithms generally are both computationally expensive and prone to numerical approximation errors; (2) Complex and less intuitive continuity constraints must be imposed along the common boundary of trimmed patches in order to ensure the smoothness requirements; and (3) Enforcing (or even approximating) the smoothness criteria of non-static models throughout a sculpting session is very difficult. In principle, considerable amount of human intervention is required to guarantee the seamlessness of the underlying NURBS patchwork.

By contrast, recursive subdivision schemes produce a visually pleasing, smooth surface in the limit through the repeated application of a fixed set of refinement rules on a user-specified control mesh. They have the potential to overcome the aforementioned difficulties associated with NURBS for the following reasons:

- Subdivision geometry naturally generalizes B-spline and NURBS representations. In principle, a single subdivision surface can model an object of arbitrary topology. It requires neither trimming nor patching operations, and smoothness requirements along the patch boundary can be automatically guaranteed.

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- Subdivision surfaces allow designers to arrange control vertices in a more natural way, which facilitates the creation of geometric features without the need to maintain a rectangular structure as required by NURBS. This can significantly reduce modeling and design times.

- Subdivision potentially allows the model to be refined locally. Local refinement is not possible with NURBS, since an entire row/column of control points must be added to preserve the rectangular parameterization.

Despite the diversity of subdivision schemes in the literature as well as the dramatic growth in the use of subdivision techniques for graphical modeling and animation during the past ten years, it remains almost impossible to manipulate the limit surface (obtained through procedure-based subdivision) in a direct and natural way. The current state-of-the-art permits modelers to interactively obtain the desired effects (e.g., functional requirements and aesthetic criteria) on the limit surface only through the kinematic manipulation of control vertices at various levels of the subdivision hierarchy. This shape design process is extremely clumsy and laborious in essence, despite modern interaction devices. Moreover, existing subdivision surfaces are not yet amenable to data exchange with industry standard formats such as B-splines and NURBS, hampering their widespread penetration in visual modeling and engineering design. The full potential of subdivision geometry has yet to be realized, because intuitive and flexible 3D interaction techniques between designers and subdivision geometry have not yet been adequately explored.

To improve the efficiency of interactive geometric modeling and engineering design, in this paper we offer a novel FEM-based solution which transforms the purely geometric subdivision procedure to geometric splines and, furthermore, which integrates subdivision splines with powerful FEM-based dynamic techniques. Consequently, our methodology and algorithms permit designers to physically modify subdivision splines at arbitrary region/location directly via simulated *forces*. This provides designers an intuitive, natural feeling analogous to modeling with real clay or play-dough.

2 Subdivision Geometry Overview

Chaikin [7] first introduced the subdivision concept to the modeling community for curve generation from an arbitrary control polygon. Following Chaikin's pioneering work, a wide variety of subdivision schemes for modeling smooth surfaces of arbitrary topology have been derived during the past two decades. In general, the existing subdivision schemes can be categorized into two distinct classes: (1) approximating subdivision techniques, and (2) interpolating subdivision techniques.

Among the approximating schemes, Doo and Sabin [9], and Catmull and Clark [4] generalized the idea of obtaining uniform biquadratic and bicubic B-spline patches from a rectangular mesh, respectively. Catmull and Clark [4] developed an algorithm that recursively generates a smooth surface from a polyhedral mesh of arbitrary topology. The Catmull-Clark subdivision surface can be reduced to a set of standard B-spline patches except at a finite number of degenerate points. Loop [15] presented a similar subdivision scheme based on the generalization of quartic triangular B-splines for triangular meshes. Hoppe *et al.* [13] further extended Loop's work to produce piecewise smooth surfaces with selected discontinuities. Halstead *et al.* [12] proposed an algorithm to construct a Catmull-Clark subdivision surface that interpolates the vertex mesh of arbitrary topology. Recently, non-uniform Doo-Sabin and Catmull-Clark surfaces that generalize non-uniform tensor-product B-spline surfaces to arbitrary topologies were introduced by Sederberg *et al.* [23]. Various issues involved in the use of these approximating subdivision schemes for character animation were discussed by DeRose *et al.* [8].

The most well-known interpolating subdivision scheme is the "butterfly" algorithm [11]. Butterfly method, like other subdivision schemes, makes use of a small number of neighboring vertices for subdivision. It requires simple data structure and is rather straightforward to implement. Nevertheless, it needs a topologically regular setting of the initial (control) mesh in order to obtain a smooth C^1 limit surface. Zorin *et al.* [31] have developed an improved interpolatory subdivision scheme that retains the simplicity of the butterfly scheme and results in much smoother surfaces even from irregular initial meshes. These interpolatory subdivision schemes have extensive applications in wavelets on manifolds, multiresolution decomposition of polyhedral surfaces, and multiresolution editing.

The derivation of various mathematical properties of the limit surface generated by the subdivision algorithms is rather complex. Doo and Sabin [10] first analyzed the smoothness behavior of the limit surface using the Fourier transform and an eigen-analysis of the subdivision matrix. Ball and Storry [2] and Reif [22] further extended Doo and Sabin's work on continuity properties of subdivision surfaces by deriving various necessary and sufficient smoothness conditions for different subdivision schemes. Specific subdivision schemes were also analyzed by several other researchers [18, 1, 14, 29]. Most recently, Stam [25] developed an exact point evaluation algorithm for Catmull-Clark subdivision scheme.

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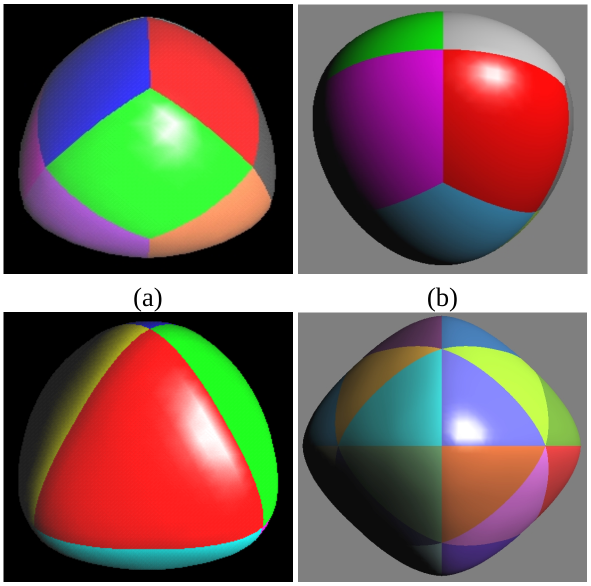

(a) (b) (c) (d)

Figure 1. The finite element decomposition of four subdivision examples, each color represents one individual element in the limit surface: (a-b) two limit surfaces of Catmull-Clark subdivision; (c-d) two limit surfaces of Butterfly subdivision.

3 FEM-Based Techniques

Subdivision splines have offered users extraordinary power and flexibility, especially when utilized in modeling complex shapes of arbitrary topology. Nonetheless, they constitute a purely geometric substrate whose design methodology does not exploit the full potential of the underlying geometric formulation because of the following problems:

- Modelers are faced with the tedium of shape refinement through time-consuming operations on a large number of topologically irregular control vertices and less-intuitive modification on various subdivision rules. In general, conventional techniques remains clumsy and laborious for effectively representing and deforming highly complex objects.
- The editing on control points is unnatural, since they generally do not reside on the sculpted geometric object. Hence, this indirect approach often requires designers to make many nonintuitive decisions, and it is even more difficult to accurately quantify the effects of refinement in arbitrary localized regions.
- Typical design requirements may be posed in both quantitative and qualitative terms. For example, a certain number of local features such as bulges or inflections may be strongly desired while requiring geometric objects to satisfy global smoothness criteria. Therefore, it can be very frustrating to enforce a set of diverse, heterogeneous criteria simultaneously via the indirect approach.

In contrast, physics-based modeling can ameliorate the geometric design process. Dynamic models are governed by differential equations that continuously evolve all of their intrinsic degrees of freedom (DOFs) in response to simulated forces. The new dynamic approach **augments (rather than supersedes)** standard geometry and geometric design, offering attractive extra advantages:

- Dynamics facilitates interaction, especially direct manipulation and interactive sculpting of complex geometric models in real-time. The dynamic approach subsumes all of the geometric capabilities in an elegant formulation that grounds shape variation in real-world physics.
- The equilibrium shape of a geometric object is characterized by a minimum of its potential energy, subject to imposed constraints. It is possible to formulate potential energy functionals that satisfy local and global design criteria. In particular, elastic energy functionals will allow the imposition of global qualitative "fairness" criteria through quantitative means.
- Geometric design is a time-varying process because designers are often interested in not only the final static equilibrium shape but the intermediate shape variation as well. Dynamic models produce smooth, natural motions that are familiar and can easily be controlled.
- Practical design processes span from conceptual design to the fabrication of mechanical parts. Physics-based modeling techniques integrate geometry with physics in a natural and coherent way. The unified formulation is potentially relevant throughout the entire modeling, simulation, analysis, and manufacturing process. More importantly, it is possible to introduce manufacturing constraints in the earlier design stage.
- Modeling systems with dynamic interfaces should be of great interests to scientists, engineers, as well as to non-expert users. For example, physics-based shape design can free designers from having to make non-intuitive decisions. Non-expert users are able to concentrate on visual shape variation without necessarily comprehending the underlying mathematical formulation. Physics-based interaction, in short, should appeal to everyone.

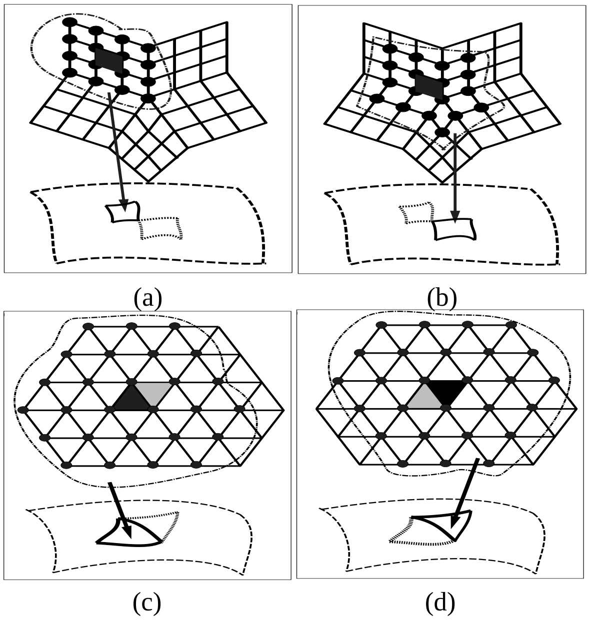

Figure 2. Control point configurations for the finite element patches and their corresponding parametric domains of arbitrary subdivision schemes: (a-b) Catmull-Clark subdivision and its rectangular elements; (c-d) Butterfly subdivision and its triangular elements.

4 Dynamic Modeling Formulation

Free-form deformable models were first introduced to the modeling community by Terzopoulos *et al.* [27], and were improved by a number of researchers during the past decade. Terzopoulos and Fleischer demonstrated simple interactive sculpting using viscoelastic and plastic models [26]. Celniker and Gossard developed an interesting prototype system [5] for interactive free-form design based on the finite-element optimization of energy functionals proposed in [26]. The system combines geometric constraints with sculpting operations based on forces and loads to yield fair shapes. However, this approach does not provide interactive mechanisms in dealing with forces and loads. Bloor and Wilson developed related models using similar energies and numerical optimization [3]. Subsequently, Celniker and Welch investigated deformable B-splines with linear constraints [6]. Welch and Witkin extended the approach to trimmed hierarchical B-splines for interactive modeling of free-form surface with constrained variational optimization [30]. We proposed and developed D-NURBS [28, 21]. D-NURBS offer the advantage of interactive and direct manipulation of NURBS curves and surfaces, resulting in physically meaningful thus intuitively predictable motion and shape variation. The D-NURBS formulation permits hard and soft geometric constraints to be imposed through Lagrange multipliers or penalty methods, respectively.

Despite the popularity of our D-NURBS models, they cannot easily be adapted to model surfaces of arbitrary genus due to their rectangular structure. By contrast, the subdivision scheme is a powerful and superior candidate for the next generation of physics-based models that can bridge the gap between D-NURBS and conventional finite element models. We have formulated FEM-based dynamic models for the Catmull-Clark, Butterfly, and Loop subdivision schemes [20, 16, 17]. In addition, we have developed a general and systematic mechanism for converting the limit surface of any subdivision scheme to the physics-based formulation using a single type of finite element [17].

We shall now present the formulation of our FEM-based subdivision splines and develop a dynamic framework that permits users to directly manipulate the limit surface obtained from any subdivision procedure via simulated "force" tools. The most significant contribution of our unified approach is the formulation of the limit surface of an arbitrary subdivision scheme as being composed of a single type of novel finite element. The limit surface of an arbitrary subdivision scheme can be viewed as a function of the initial control mesh. The shape parameters (initial control vertices) of subdivision splines play the key role of generalized (physical) coordinates in a finite element formulation. We introduce time, mass, and deformation energy into the procedure-based subdivision splines and employ Lagrangian dynamics to arrive at a system of differential equations that governs the shape and motion of subdivision splines. The typical subdivision surface $\mathbf{s}$ can be decomposed into a set of finite elements (each color denotes an individual finite element in Fig. 1):

$$\mathbf{s} = \sum_{i=1}^{k} \mathbf{s}_i, \tag{1}$$

where k is the number of faces defined by the initial control mesh. We concatenate all the coordinates of initial control vertices into the vector:

$$\mathbf{p}(t) = \begin{bmatrix} \cdots & \mathbf{p}_i^\top & \cdots \end{bmatrix}^\top,$$

where $\mathbf{p}(t)$ are functions of time, also known as the DOF vector of the limit surface $\mathbf{s}$. Now, we can explicitly express the velocity and position of the limit surface as

$$\mathbf{s}(\mathbf{x}, \mathbf{p}) = \mathbf{J}(\mathbf{x})\mathbf{p},$$

and

$$\dot{\mathbf{s}}(\mathbf{x}, \mathbf{p}) = \mathbf{J}(\mathbf{x})\dot{\mathbf{p}},$$

where an over-struck dot denotes a time derivative, $\mathbf{x} \in S^0$ and S^0 define the domain of the initial mesh. Note that S^0

is the parametric domain of the limit surface. Fig. 2 illustrates several examples of finite elements and their control vertices. The matrix $\mathbf{J}(\mathbf{x})$ of dimension $(3, 3n)$ is the Jacobian matrix of the limit surface with respect to $\mathbf{p}$. It is also the concatenation of basis functions for the corresponding vertices in the initial mesh

$$\mathbf{J} = \left[\begin{array}{ccc} \cdots & \mathbf{B}_i(\mathbf{x}) & \cdots \end{array} \right].$$

Fig. 3 shows several examples of basis functions associated with specific subdivision schemes. The equations governing the motion of subdivision splines are derived from the work-energy version of Lagrangian dynamics:

$$\mathbf{M}\ddot{\mathbf{p}} + \mathbf{D}\dot{\mathbf{p}} + \mathbf{K}\mathbf{p} = \mathbf{f}_{\mathbf{p}}, \tag{2}$$

where $\mathbf{M}$, $\mathbf{D}$, and $\mathbf{K}$ are the mass, damping and stiffness matrices of the physical model, respectively. All matrices can be formulated explicitly. The mass and damping matrices can be expressed as

$$\mathbf{M}(\mathbf{p}) = \int_{\mathbf{x}\in S^0} \mu(\mathbf{x})\mathbf{J}(\mathbf{x})^\top \mathbf{J}(\mathbf{x}) d\mathbf{x}, \tag{3}$$

and

$$\mathbf{D}(\mathbf{p}) = \int_{\mathbf{x}\in S^0} \gamma(\mathbf{x})\mathbf{J}(\mathbf{x})^\top \mathbf{J}(\mathbf{x}) d\mathbf{x}, \tag{4}$$

where $\mu(\mathbf{x})$ is the prescribed mass density function over the subdivision splines and $\gamma(\mathbf{x})$ is the prescribed damping density function. To define an elastic potential energy $E(\mathbf{s})$ for the surface, we can adopt a large variety of functional formulations (such as the simple *thin-plate-under-tension* energy model [28] or complex non-quadratic curvature-based energy [27]). In general, the energy functional $E(\mathbf{s})$ involves derivative quantities (up to order n) such as $\mathbf{s}_u$, $\mathbf{s}_v$, $\mathbf{s}_{uu}$, etc., where (u, v) is a local parametric coordinate of $\mathbf{s}(\mathbf{x})$. The internal force and the stiffness matrix can be formulated as

$$\mathbf{f}_{int} = \frac{\partial E}{\partial \mathbf{p}} = \mathbf{K}\mathbf{p}. \tag{5}$$

This allows our FEM-based subdivision splines to exhibit a wide range of material and physical behavior such as linear elastic and/or non-linear plastic deformation. The generalized force

$$\mathbf{f}_{\mathbf{p}} = \int_{\mathbf{x}\in S^0} \mathbf{J}^\top \mathbf{f}(\mathbf{s}(\mathbf{x}), t) d\mathbf{x} \tag{6}$$

is obtained through the principle of virtual work done by the applied force distribution $\mathbf{f}(\mathbf{s}(\mathbf{x}), t)$. Because of the generality of our deformation functional, our FEM-based model is applicable for modeling isotropic as well as anisotropic phenomena.

To sculpt subdivision splines interactively in a modeling system, it is vital to provide users with real-time feedback. Dynamic simulation sessions involving real-world complex shapes are computationally expensive due to the typically large number of DOFs. Therefore, rather than using costly time-integration methods that take the largest possible time steps, it is more important to provide a smoothly animated display by maintaining the continuity of the dynamics from one step to the next. Hence, less costly yet robust and stable time integration methods that take modest time steps are desirable.

Note that, the equations of motion in (2), which determine the evolution of $\mathbf{p}$, cannot be solved analytically in general. Instead, we pursue an efficient numerical implementation. Standard finite element procedures explicitly assemble the global matrices. Instead, we use an iterative matrix solver to avoid the cost of assembling the global matrices $\mathbf{M}$, $\mathbf{D}$, and $\mathbf{K}$. A patch (e.g., $\mathbf{s}_i$) of subdivision models (refer to Fig.1 and Fig.2) is considered to be a novel type of finite element. The element data structure contains the geometric specification of the patch element along with its physical properties. A complete subdivision object consists of an ordered array of subdivision elements with additional information. The element structure includes pointers to appropriate components of the global vector $\mathbf{p}$ (initial control points). Neighboring elements will share some generalized coordinates. We also allocate in each element an elemental mass, damping, and stiffness matrix, and include in the element data structure the quantities needed to compute these matrices. These quantities include the mass $\mu(\mathbf{x})$, damping $\gamma(\mathbf{x})$, and elasticity (used to define E) density functions, which may be represented as either analytic functions or parametric arrays of sample values. Our finite element data structure fully supports parallel assembly and evaluation of individual elements.

The integral expressions for the mass, damping, and stiffness matrices associated with each element can be evaluated numerically using standard discretization techniques such as Gaussian quadrature [19]. Assuming the parametric domain of the element is Ω, the expression for entry m_{ij} of the mass matrix takes the integral form

$$m_{ij} = \int_\Omega \mu(\mathbf{x}) f_{ij}(\mathbf{x}) d\mathbf{x}.$$

Given integers N_g, we can find Gauss weights a_g, and abscissas $\mathbf{x}_g$ within Ω such that m_{ij} can be approximated by

$$m_{ij} \approx \sum_{g=1}^{N_g} a_g \mu(\mathbf{x}_g) f_{ij}(\mathbf{x}_g).$$

The computation of damping and stiffness matrices follows suit. Note that, the efficient assembly of the above material matrices (and associated quadrature computation) requires the rapid and precise evaluation of subdivision splines at arbitrary location in their parametric domain. For Catmull-Clark surfaces and Loop subdivision schemes, all basis

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functions as well as their derivatives up to order n can be evaluated (at least in an approximate sense if no closed-form analytic expression exists) [25, 24, 20].

Alternatively, the continuum of element $\mathbf{s}_i$ may be discretized into a user-defined point set $\mathbf{d}$ and the associated mass-spring mesh:

$$\mathbf{d} = \left[\begin{array}{ccc} \cdots & \mathbf{d}_i^\top & \cdots \end{array}\right]^\top = \mathbf{A}\mathbf{p},$$

where $\mathbf{A}$ is a discretized Jacobian $\mathbf{J}(\mathbf{x})$ of $\mathbf{s}_i$ (evaluated at $\mathbf{d}$). Assuming a discrete mass density function $\mu(\mathbf{d}_i)$ which is non-zero only at $\mathbf{d}_i$, we can construct a diagonal matrix $\mathbf{M}_\mathbf{d}$ for mass-set $\mathbf{d}$, whose non-zero diagonal entries take the form: $\mu(\mathbf{d}_i)$. The elemental mass matrix $\mathbf{M}$ of $\mathbf{s}_i$ can be approximated through:

$$\mathbf{M} = \mathbf{A}^\top \mathbf{M}_\mathbf{d} \mathbf{A}. \tag{7}$$

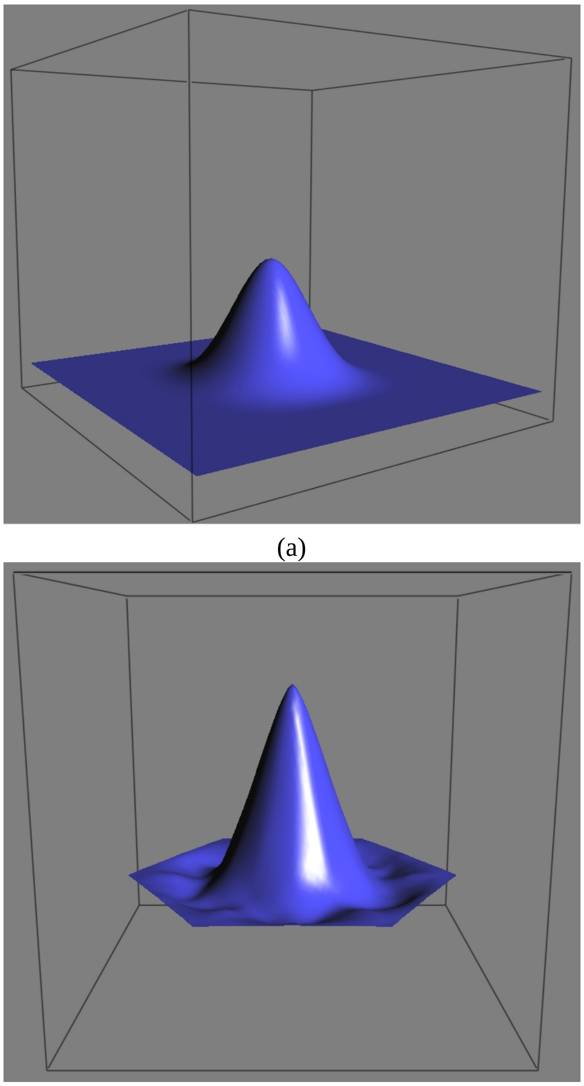
The elemental damping and stiffness matrices can be derived analogously. The mass-spring approximation of subdivision splines is proven to be feasible in order to achieve the goal of real-time interaction without sacrificing the accuracy of the model.

To solve (2) within an interactive modeling environment, the state of a subdivision spline finite element at time $t+\Delta t$ can be integrated using prior states at time t and $t-\Delta t$. To maintain the stability of the integration scheme, we use an implicit time integration method, which employs

$$\begin{aligned}(2\mathbf{M} + \Delta t\mathbf{D} + 2\Delta t^2\mathbf{K})\,\mathbf{p}^{(t+\Delta t)} &= \\ 2\Delta t^2(\mathbf{f_p}) + 4\mathbf{M}\mathbf{p}^{(t)} &- (2\mathbf{M} - \Delta t\mathbf{D})\mathbf{p}^{(t-\Delta t)},\end{aligned} \tag{8}$$

where the superscripts denote evaluation of the quantities at the indicated times. The matrices and forces are evaluated at time t. The conjugate gradient method can then be employed to obtain an iterative solution.

Our dynamic subdivision splines not only permit designers to manipulate the individual DOF with conventional geometric methods, but they also allow users to modify its shape with interactive sculpting forces. Fig. 1 illustrates the results of four interactive sculpting sessions using simple spring forces (also refer to Fig. 4 for more complex shapes). More importantly, the FEM-based design methodology provides designers an intuitively natural way to automatically determine topologically irregular control vertices in accordance with functional constraints and dynamic sculpting.



(a)

(b)

Figure 3. Basis functions of arbitrary subdivision schemes: (a) Catmull-Clark subdivision; (b) Butterfly subdivision.

5 Conclusion

The novel, dynamic framework of FEM-based subdivision splines will not only augment (rather than replace) well established NURBS-based modeling technologies but also generalize newly-developed theory and methodology of physics-based modeling (e.g., D-NURBS) in industrial

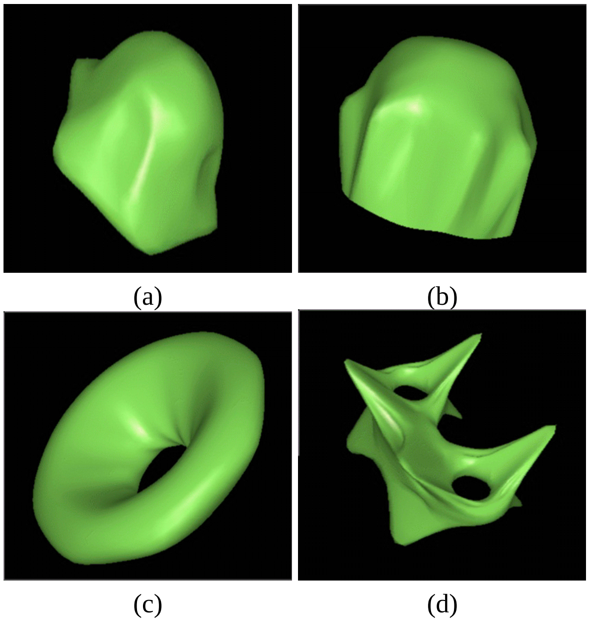


Figure 4. Four interactive sculpting sessions of subdivision splines objects using simple spring forces.

practice. In particular, the FEM-based subdivision splines can (1) contribute both to the geometric design and finite element analysis communities, (2) integrate their superior geometric features with the many demonstrated conveniences of physics-based interaction, (3) promise a greater potential to bridge the large gaps among interactive modeling, geometric design, finite element analysis, and manufacturing, (4) serve as a solid basis for future theories and techniques which can eventually unify all aspects of modeling, design, and manufacturing, and (5) further foster the applicability of subdivision geometry in a wider range of visual computing applications such as visualization, virtual reality, computer vision, robotics, and medical imaging. Our methodology and its associated empirical system with physics-based sculpting capabilities will appeal to a spectrum of users ranging from highly-trained engineering designers, computer professionals, artists, to naive users.

Acknowledgments

This research was supported in part by the NSF CAREER award CCR-9896123, the NSF grant DMI-9896170, and a research grant from Ford Motor Company.

References

[1] W. Dahmen A.S. Cavaretta and C.A. Micchelli. Stationary subdivision. *Memoirs of the American Mathematical Society*, 93(453), September 1991. American Mathematical Society.

[2] A.A. Ball and D.J.T. Storry. An investigation of curvature variations over recursively generated b-spline surfaces. *ACM Transactions on Graphics*, 9(4):424–437, 1990.

[3] M.I.G. Bloor and M.J. Wilson. Using partial differential equations to generate free-form surfaces. *Computer-Aided Design*, 22(4):202–212, 1990.

[4] E. Catmull and J. Clark. Recursively generated B-spline surfaces on arbitrary topological meshes. *Computer-Aided Design*, 10(6):350–355, 1978.

[5] G. Celniker and D. Gossard. Deformable curve and surface finite elements for free-form shape design. *Computer Graphics*, 25(4):257–266, 1991. (Proc. ACM Siggraph'91).

[6] G. Celniker and W. Welch. Linear constraints for deformable B-spline surfaces. In *Proceedings, Symposium on Interactive 3D Graphics*, pages 165–170, 1992.

[7] G.M. Chaikin. An algorithm for high speed curve generation. *Computer Vision, Graphics and Image Processing*, 3(4):346–349, 1974.

[8] T. DeRose, M. Kass, and T. Truong. Subdivision surfaces in character animation. In *Computer Graphics Proceedings, Annual Conference Series, ACM SIGGRAPH*, pages 85–94, July 1998.

[9] D. Doo. A subdivision algorithm for smoothing down irregularly shaped polyhedrons. In *Proceedings on Interactive Techniques in Computer Aided Design*, pages 157–165, 1978.

[10] D. Doo and M. Sabin. Behaviour of recursive division surfaces near extraordinary points. *Computer-Aided Design*, 10(6):356–360, 1978.

[11] N. Dyn, D. Levin, and J. Gregory. A butterfly subdivision scheme for surface interpolation with tension control. *ACM Transactions on Graphics*, 9(2):160–169, 1990.

[12] M. Halstead, M. Kass, and T. DeRose. Efficient, fair interpolation using Catmull-Clark surfaces. In *Computer Graphics* Proceedings, Annual Conference Series, Proc. ACM Siggraph'93 (Anaheim, CA, Aug., 1993), pages 35–44, 1993.

[13] H. Hoppe, T. DeRose, T. Duchamp, M. Halstead, H. Jin, J. McDonald, J. Schweitzer, and W. Stuetzle.

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Piecewise smooth surface reconstruction. In *Computer Graphics Proceedings, ACM SIGGRAPH Annual Conference Series*, pages 295–302, July 1994.

[14] L. Kobbelt and P. Schröder. Constructing variationally optimal curves through subdivision. Technical report, California Institute of Technology Computer Science Department, 1997. Tech. Rep. CS-TR-97-05.

[15] C. Loop. Smooth subdivision surfaces based on triangles. Master's thesis, University of Utah, Department of Mathematics, 1987.

[16] C. Mandal, H. Qin, and B.C. Vemuri. Dynamic modeling of butterfly subdivision surfaces. *IEEE Transactions on Visualization and Computer Graphics*, 2000. To appear (also available UFL-CISE Technical Report TR98-009).

[17] C. Mandal, H. Qin, and B.C. Vemuri. A novel FEM-based dynamic framework for subdivision surfaces. *Computer Aided Design*, 2000. To appear in a special issue on solid modeling.

[18] C.A. Micchelli and H. Prautzsch. Uniform refinement of curves. *Linear Algebra and Its Applications*, 114/115:841 – 870, 1989.

[19] W. Press, B. Flanney, S. Teukolsky, and W. Verttering. *Numerical Recipes: The Art of Scientific Computing*. Cambridge University Press, Cambridge, 1986.

[20] H. Qin, C. Mandal, and B.C. Vemuri. Dynamic Catmull-Clark subdivision surfaces. *IEEE Transactions on Visualization and Computer Graphics*, 4(3):215–229, July 1998.

[21] H. Qin and D. Terzopoulos. D-NURBS: A physics-based geometric design framework. *IEEE Transactions on Visualization and Computer Graphics*, 2(1):85–96, March 1996.

[22] U. Reif. A unified approach to subdivision algorithms near extraordinary points. *Computer Aided Geometric Design*, 12(2):153–174, 1995.

[23] T.W. Sederberg, J. Zheng, D. Sewell, and M. Sabin. Non-uniform recursive subdivision surfaces. In *Computer Graphics Proceedings, Annual Conference Series, ACM SIGGRAPH*, pages 387–394, July 1998.

[24] J. Stam. Evaluation of Loop subdivision surfaces. In *Computer Graphics Proceedings CDROM, Annual Conference Series, ACM SIGGRAPH*, July 1998.

[25] J. Stam. Exact evaluation of Catmull-Clark subdivision surfaces at arbitrary parameter values. In *Computer Graphics Proceedings, Annual Conference Series, ACM SIGGRAPH*, pages 395–404, July 1998.

[26] D. Terzopoulos and K. Fleischer. Deformable models. *The Visual Computer*, 4(6):306–331, 1988.

[27] D. Terzopoulos, J. Platt, A. Barr, and K. Fleischer. Elastically deformable models. *Computer Graphics*, 21(4):205–214, 1987.

[28] D. Terzopoulos and H. Qin. Dynamic NURBS with geometric constraints for interactive sculpting. *ACM Transactions on Graphics*, 13(2):103–136, 1994.

[29] H. Weimer and J. Warren. Subdivision schemes for thin plate splines. *Computer Graphics Forum*, 17(3):303–313, 1998. (Proceedings of EuroGraphics'98).

[30] W. Welch and A. Witkin. Variational surface modeling. *Computer Graphics*, 26(2):157–166, 1992. (Proc. ACM Siggraph'92).

[31] D. Zorin, P. Schröder, and W. Sweldens. Interpolating subdivision for meshes with arbitrary topology. In *Computer Graphics Proceedings, Annual Conference Series, ACM SIGGRAPH*, pages 189–192, August 1996.