Data Interpolation

- Why interpolation?
- We acquire discrete observations/measurements for continuous systems, and we would like to convert discrete measurements to continuous representations
- We definitely need the ability to interpolate values "in-between" discrete points

Data Interpolation

- One simple example
- Our goal is to find the value of a function between known values
- Let us consider the two pairs of values (x,y):

  (0.0, 1.0), and (1.0, 2.0)

What is y at $x = 0.5$? That is, what's $(0.5, y)$?

Linear Interpolation

- Given two points, (x_1,y_1), (x_2,y_2):

  Fit a straight line between the points

$$y(x) = a\,x + b$$

$$a=(y_2-y_1)/(x_2-x_1), \quad b= (y_1 x_2-y_2 x_1)/(x_2-x_1),$$

  Use this equation to find y values for any

$$x_1 < x < x_2$$

Another Example

- What about four points ?
- (0, 2), (1, 0.3975), (2, -0.1126), (3, -0.0986)

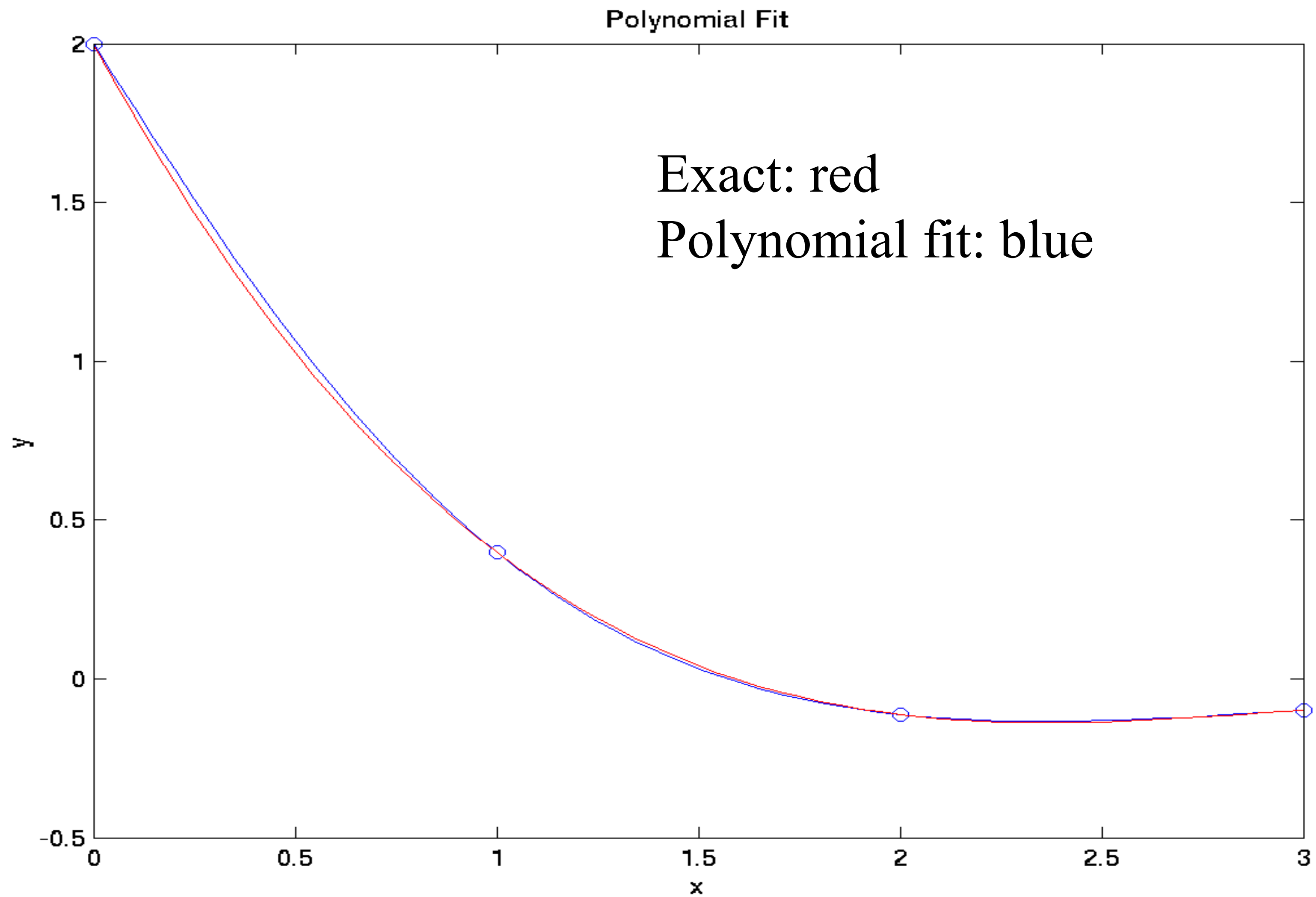
Another Example

Data points are: (0,2), (1,0.3975), (2, -0.1126), (3, -0.0986).

Fitting a cubic polynomial through the four points gives:

$$y_p(x) = 2.0 - 2.3380x + 0.8302x^2 - 0.0947x^3$$

Polynomial Fit to Example



Polynomial Interpolants

- Now given n (n=4) data points $(x_i, y_i), i = 1,4$
- Find the interpolating function that goes through these points, will need a cubic polynomial

$$y_p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

- If there are n+1 data points, the function will become (with n+1 unknown variables)

$$y_p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_N x^N$$

Polynomial Interpolant

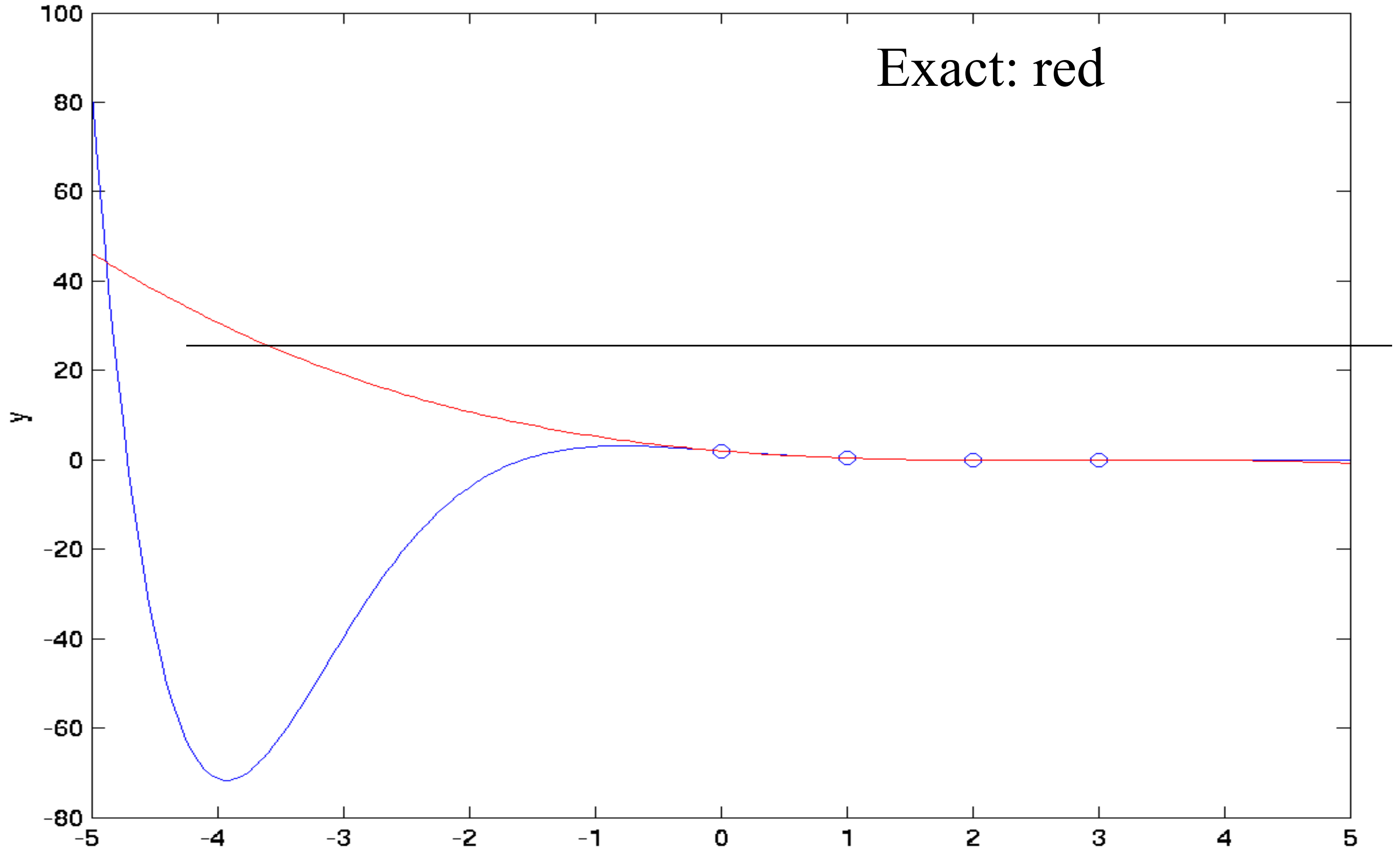
- The polynomial must pass through the four points, resulting in the following constraints

$$\begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix} \begin{pmatrix} a_o \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

$$\mathbf{p}\,\mathbf{a} = \mathbf{y}$$

$$\mathbf{a} = \mathbf{p}^{-1}\,\mathbf{y}$$

Caution: Extrapolation

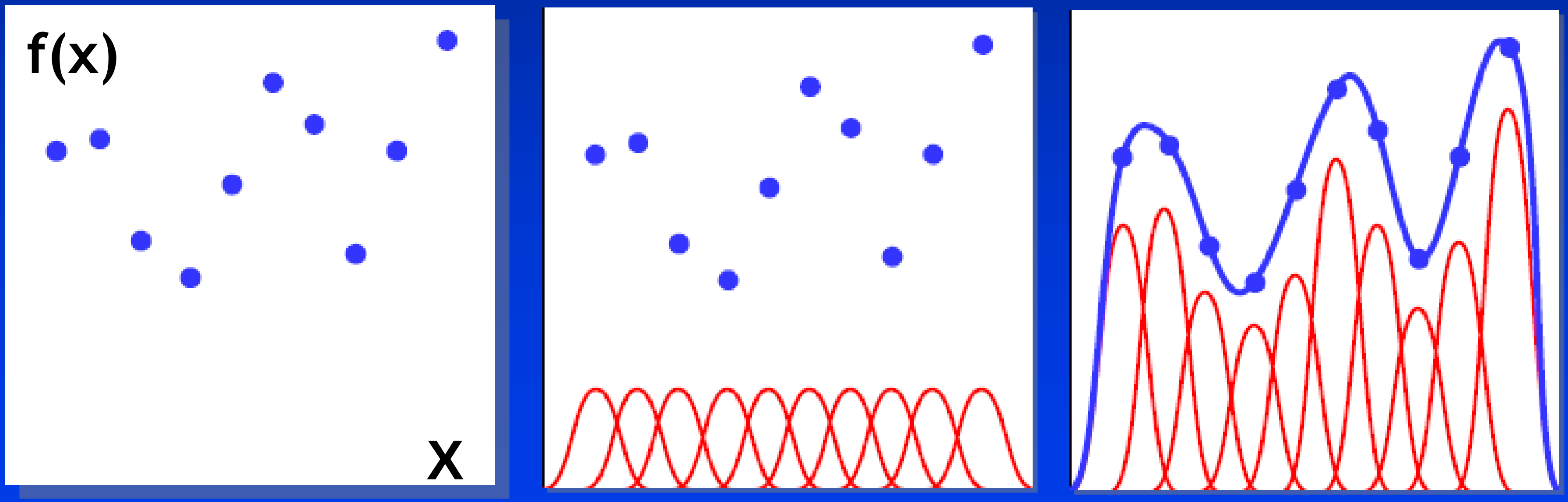


An N^{th} order polynomial has N roots!

Scattered Data Fitting and Applications in Data Visualization

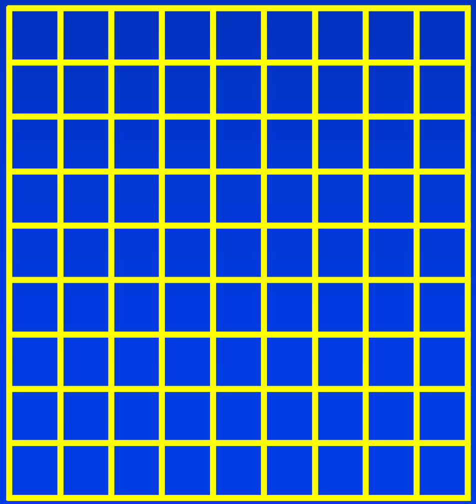
The Scattered Data Fitting Problem

- λ_i define the influence of the center
- After constructing s(x), the interpolation or extrapolation can be easily performed



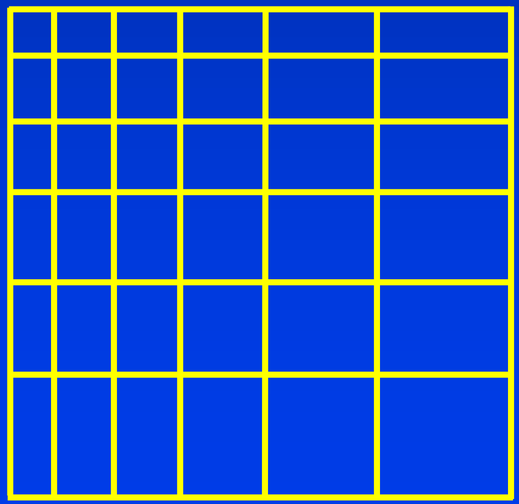
Uniform Field (Domain)

- Measurements stored in a rectangular grid
- Equal spacing between rows and columns
- Images – each grid square is a pixel



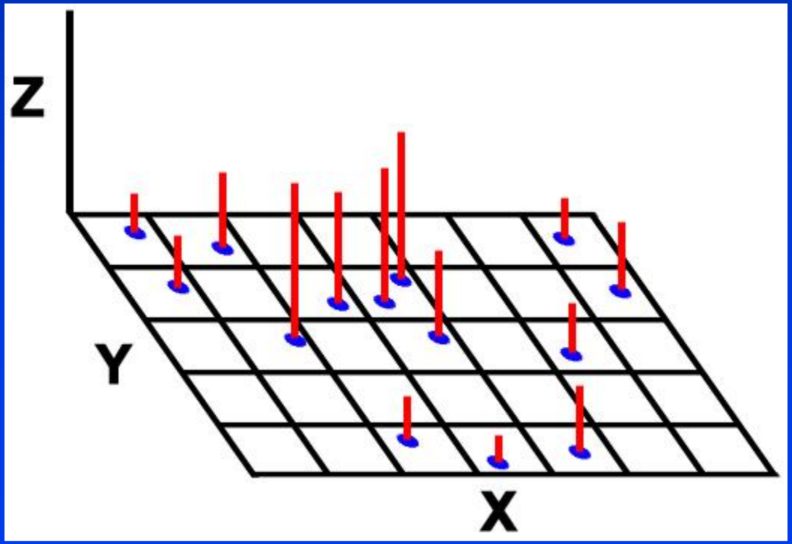
Rectilinear Fields

- Data samples not equally spaced along the coordinate axes
- Rectangular grid with varying distances between rows and columns

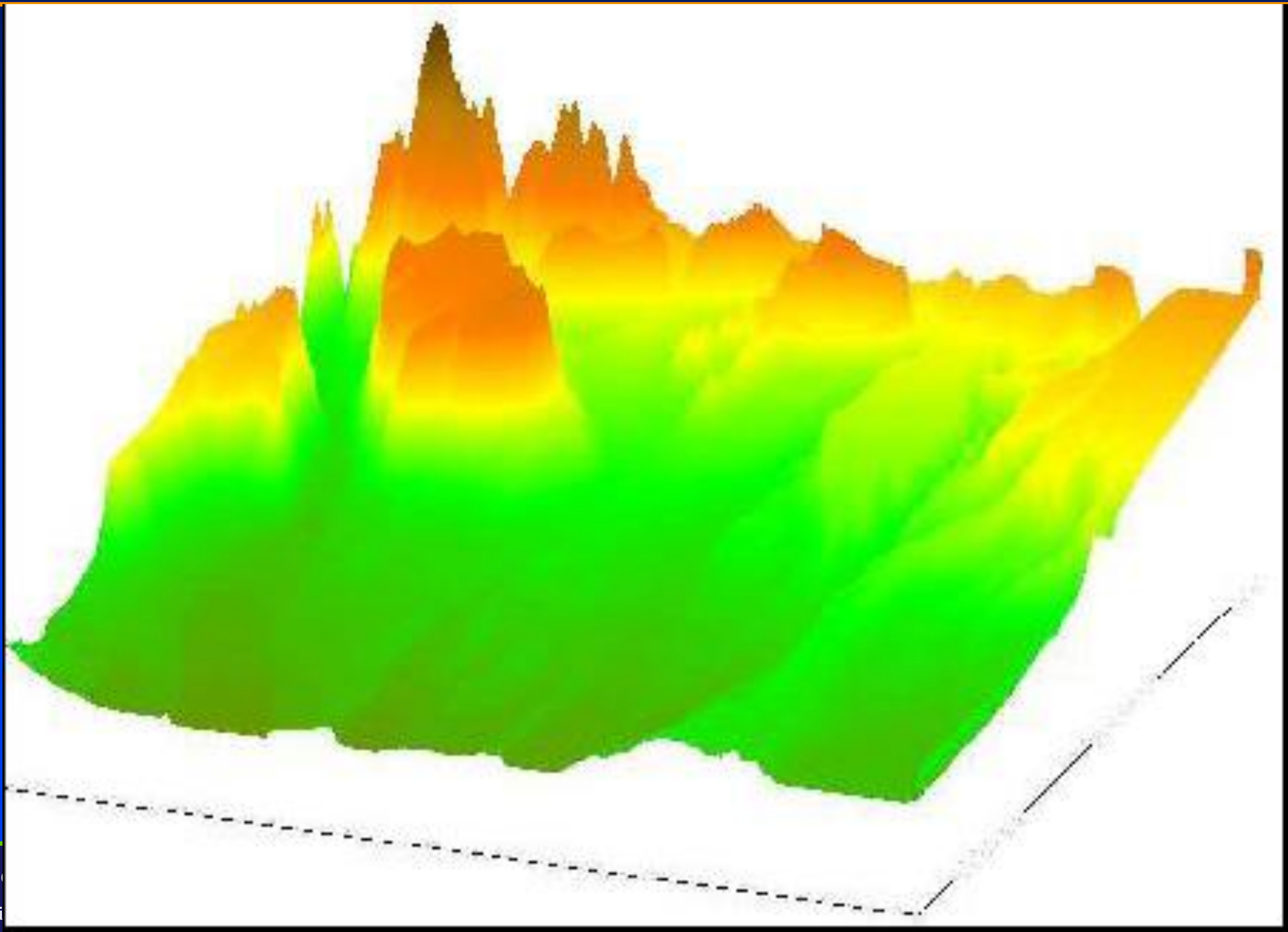


The Scattered Data Fitting Problem

- Irregular Fields
  - Contain scattered measurements not corresponding to a rectilinear structure
  - No overall organizational structure
  - Similar to coordinate systems used in standard mathematics



Scattered Data Fitting



Scattered Data Interpolation/Fitting

- Given N samples ($\mathbf{x}_i$, f_i), such that $s(\mathbf{x}_i)=f_i$, We would like to reconstruct a function s(x)
  - $\mathbf{x}_i$ are the *points from measurement*
  - Reconstructed function is denoted s(**x**)

- Actually, there are infinite number of solutions
- We have specific constraints:
  - s(**x**) should be continuous over the entire domain
  - We want a 'smooth' surface

- Radial basis functions are popularly used solutions

Data Fitting: Scattered Data Interpolation

- Characteristics
  - Interpolation vs. Extrapolation
  - Linear Interpolation vs. Higher Order
  - Structured vs. Scattered
  - 1-Dimensional vs. Multi-Dimensional
- Techniques
  - Splines (cubic, B-splines, ...)
  - Series (polynomial, radial basis functions, ...)
  - ....
  - Exact solution, minimization, fitting, approximation

Scattered Data Interpolation

- Radial Basis Functions (RBFs) are a powerful solution to the Problem of *Scattered Data Fitting*
  - N point samples are given as data inputs, we want to interpolate, extrapolate, approximate

- This problem occurs in many areas:
  - Mesh repair and model completion
  - Surface reconstruction
    - Range scanning, geographic surveys, medical data
  - Field visualization (2D and 3D)
  - Image warping, morphing, registration
  - Artificial Intelligence
  - Etc.

Graphics Applications

- Given a set of samples, what are the in-between values ?

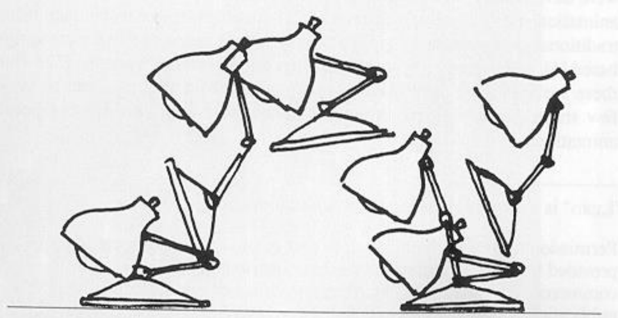


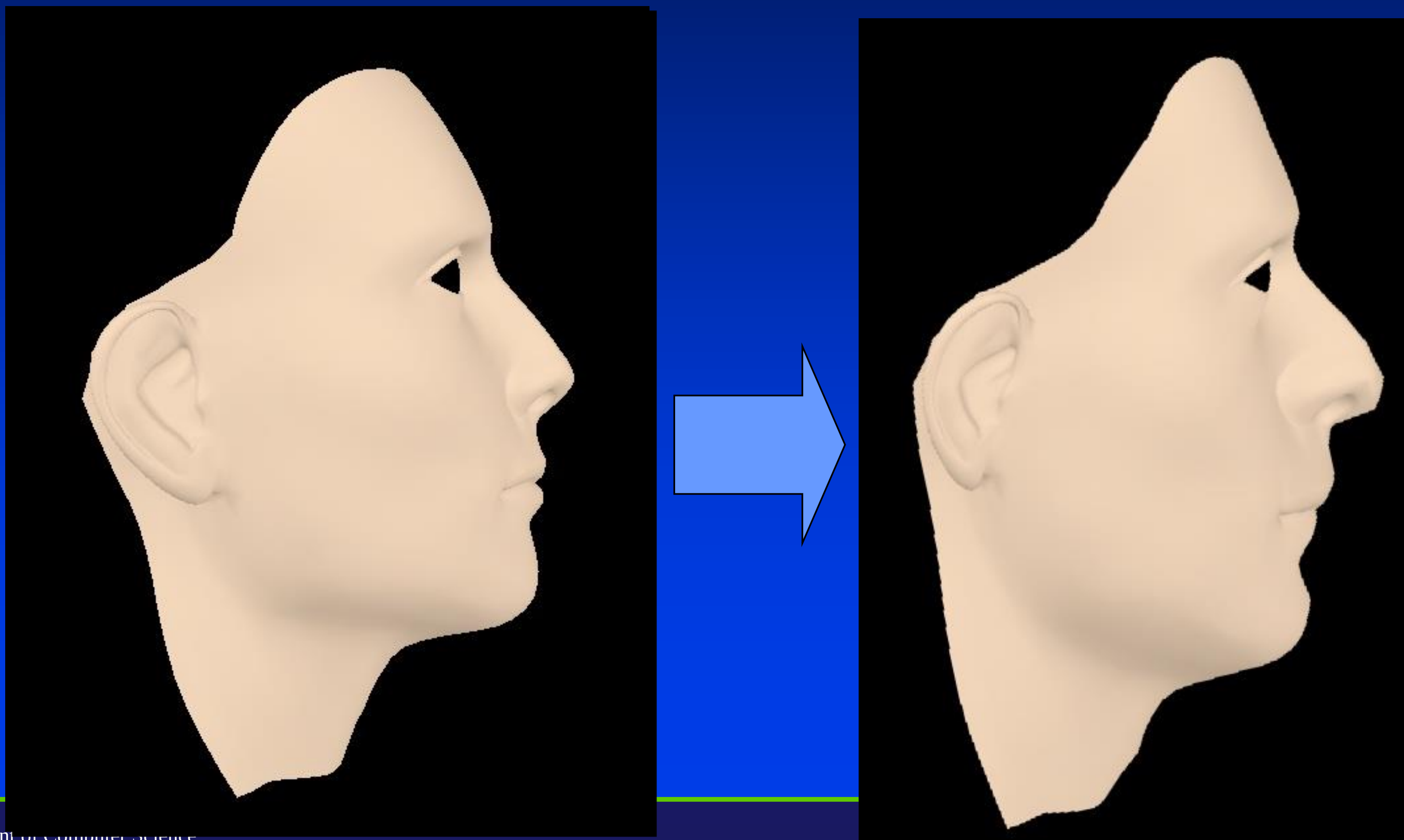
FIGURE 3. Squash & stretch in Luxo Jr.'s hop.

- Linear interpolation, interpolating by splines, …
  - It works for structured data.

How about unstructured or scattered data samples?

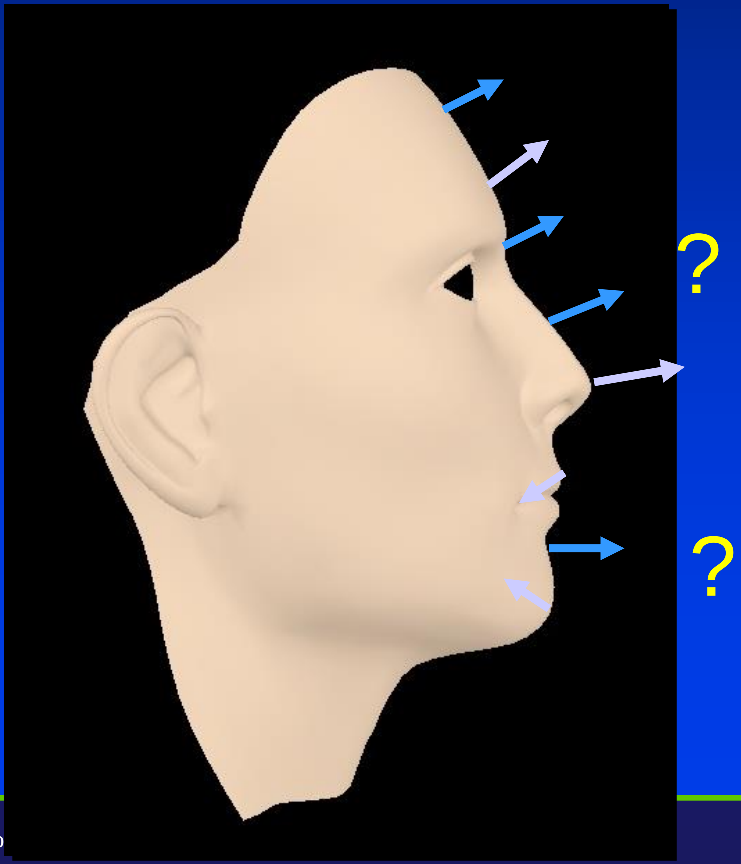
Scattered Data Interpolation

For instance, head model adjustment…



Scattered Data Interpolation

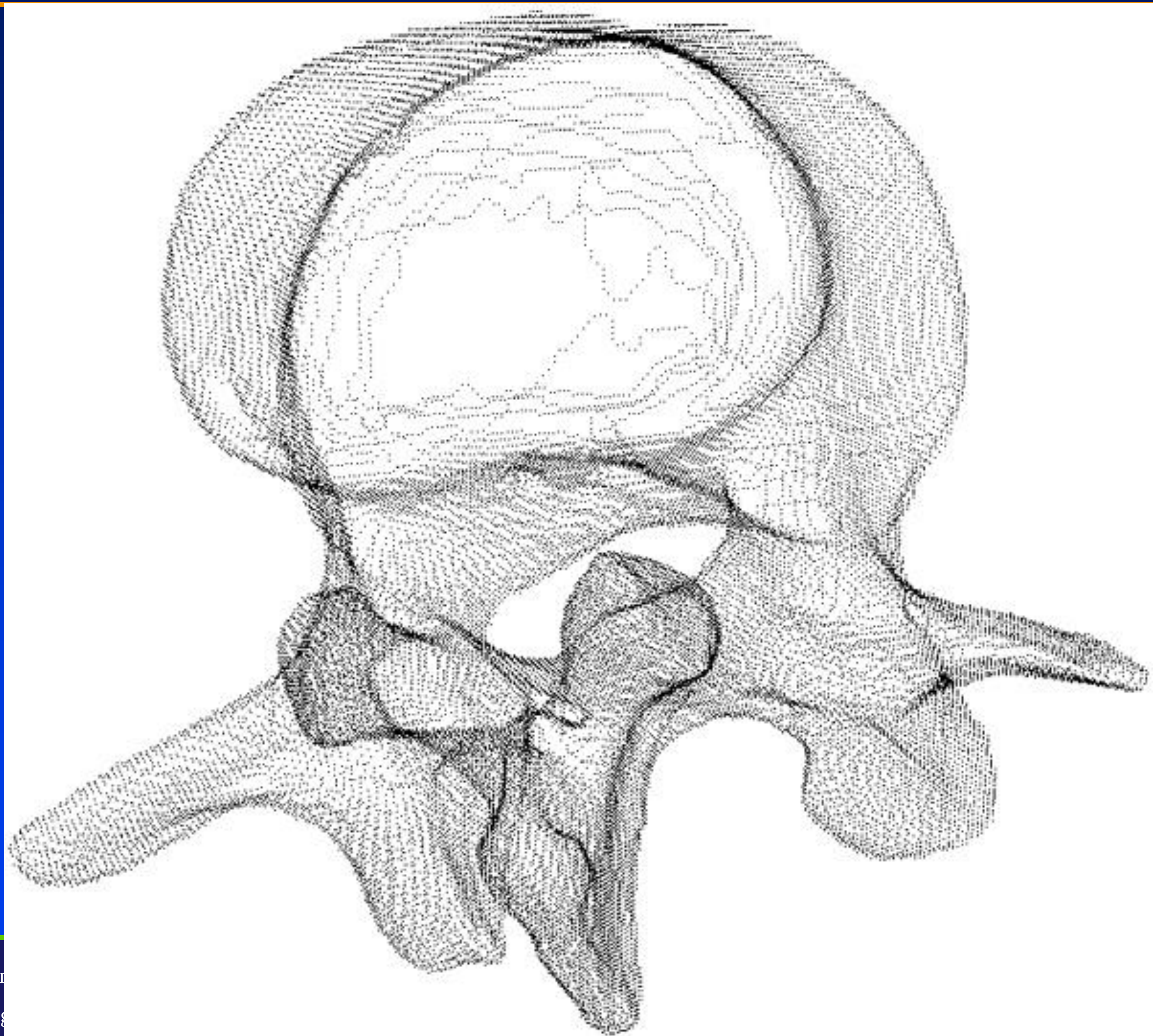
You can drag all vertices (more than 6000) or drag feature samples…



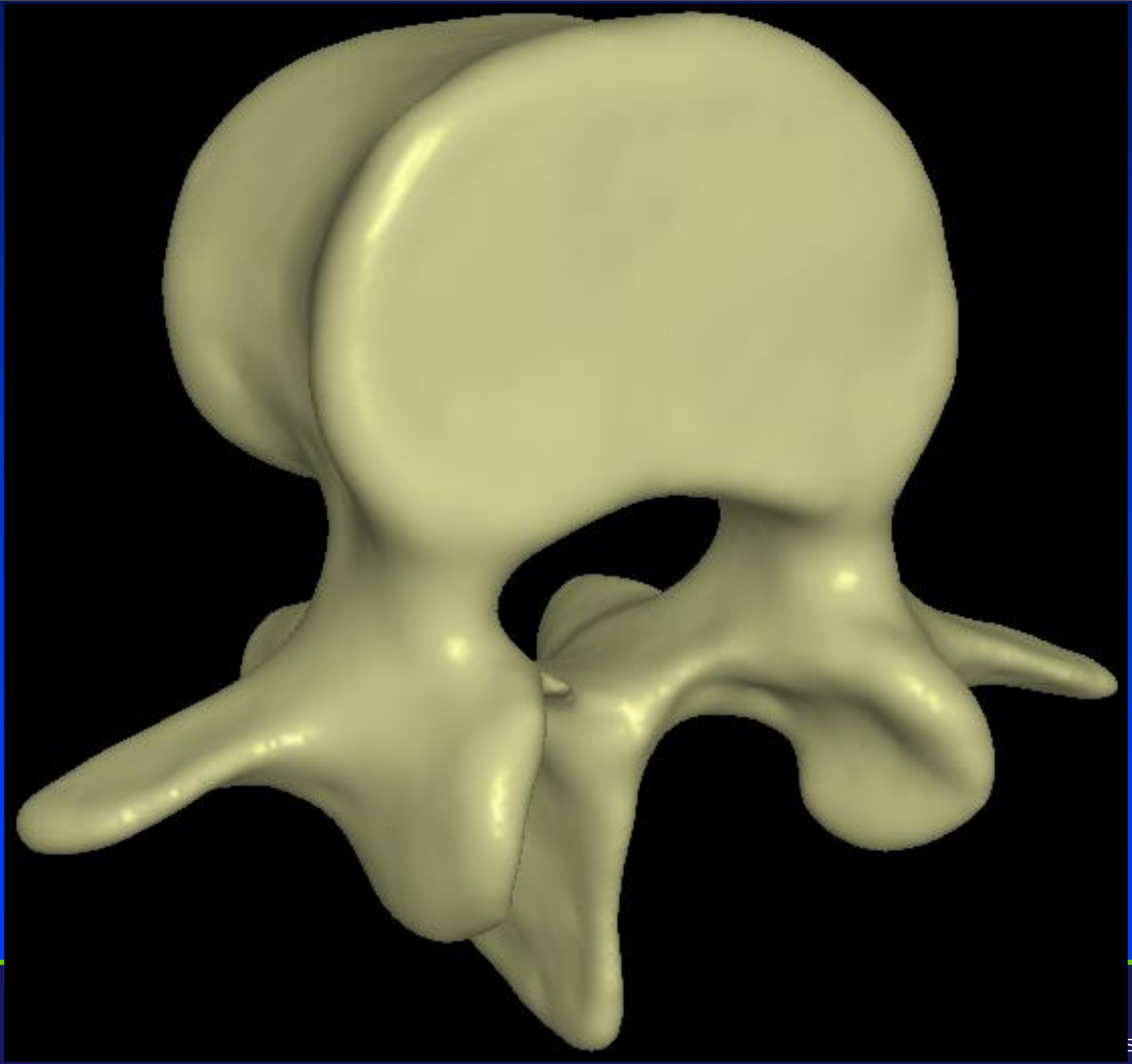
Drag by users

What are the displacement ?

Scattered Data Modeling



Smooth Surfaces



Scattered Data Approximation and Interpolation

- Scattered data: an arbitrary set of points in Rd space, and these scattered data carry scalar quantities (i.e., a scalar field in d dimensional parametric space)

Ordinary Least-Squares

Least Squares Interpolant

- For n points, we only have a fitting polynomial of order m (m < (n-1)), we want the least squares fitting polynomial is similar to the exact fit form:

$$\mathbf{y_p(x) = p\,a}$$

- Now p is becoming a n * m matrix. We have fewer unknowns than data points, the interpolant can not go through all the points exactly, we need to measure the total error

$$\sum_{i=1}^{N} \epsilon_i^2$$

$$\epsilon_i = y_p(x_i) - y_i$$

Least Squares Approximation

- Problem statement: we have n points in Rd space, and we want to obtain a globally defined function f(x) that can approximate the given scalar values at these points in the least-squares senses
- We are considering the space of polynomials of total degree m in d spatial dimensions

$$\min_{f \in P_m^d} \sum_i \left\| \left(f(x_i) - f_i \right\| \right.^2$$

$$f(\mathbf{x}) = \mathbf{b}(\mathbf{x})^T \bullet \mathbf{c}$$

$$\mathbf{b}(\mathbf{x}) = [b_1(\mathbf{x}) \quad b_2(\mathbf{x}) \quad \ldots \quad b_k(\mathbf{x})]^T$$

$$\mathbf{c} = [c_1 \quad c_2 \quad \ldots \quad c_k]^T$$

Least Squares Approximation

- Commonly-used basis functions include: quadratic, linear, constant terms
- For example:

$$\mathbf{b}(\mathbf{x}) = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 \end{bmatrix}^T$$

$$\mathbf{b}(\mathbf{x}) = \begin{bmatrix} 1 & x & y & z \end{bmatrix}^T$$

$$\mathbf{b}(\mathbf{x}) = \begin{bmatrix} 1 \end{bmatrix}$$

Solution

- Function minimization: the partial derivatives of the error functional must be set to zero
- We now obtain a linear system of equations

$$\frac{\partial E}{\partial \mathbf{c}} = \mathbf{0}$$

$$\sum_i 2b_j(\mathbf{x}_i)\left[\mathbf{b}(\mathbf{x}_i)^T\mathbf{c} - f_i\right] = 0$$

Solution

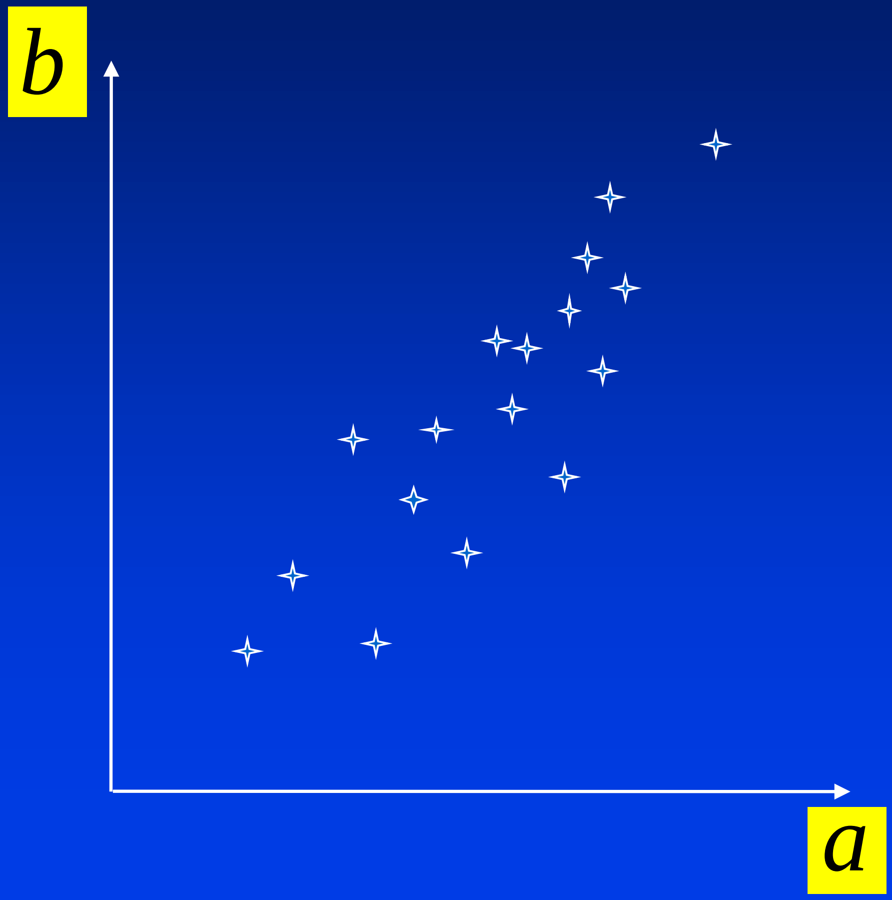
$$\sum_i \left[\mathbf{b}(\mathbf{x}_i)\mathbf{b}(\mathbf{x}_i)^T \mathbf{c} - \mathbf{b}(\mathbf{x}_i) f_i\right] = \mathbf{0}$$

$$\mathbf{c} = \left[\sum_i \mathbf{b}(\mathbf{x}_i)\mathbf{b}(\mathbf{x}_i)^T\right]^{-1} \sum_i \mathbf{b}(\mathbf{x}_i) f_i$$

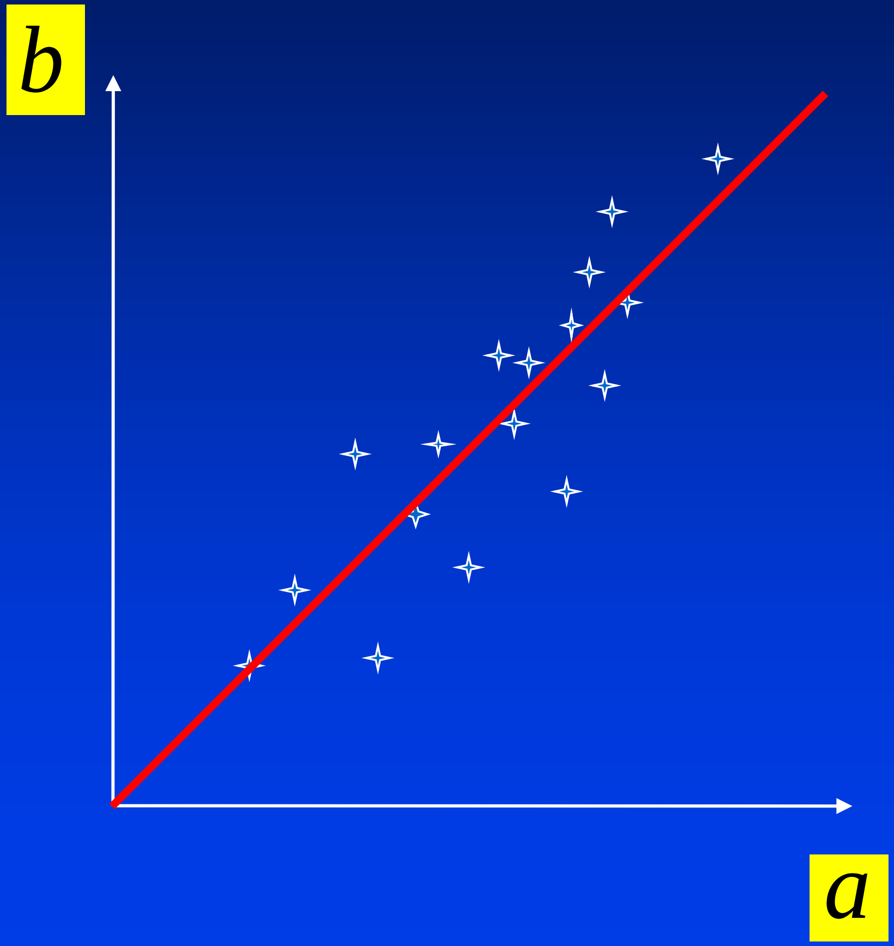
Outline

- Linear regression
- Geometry of least-squares
- Discussion of the Gauss-Markov theorem

Ordinary Least-Squares



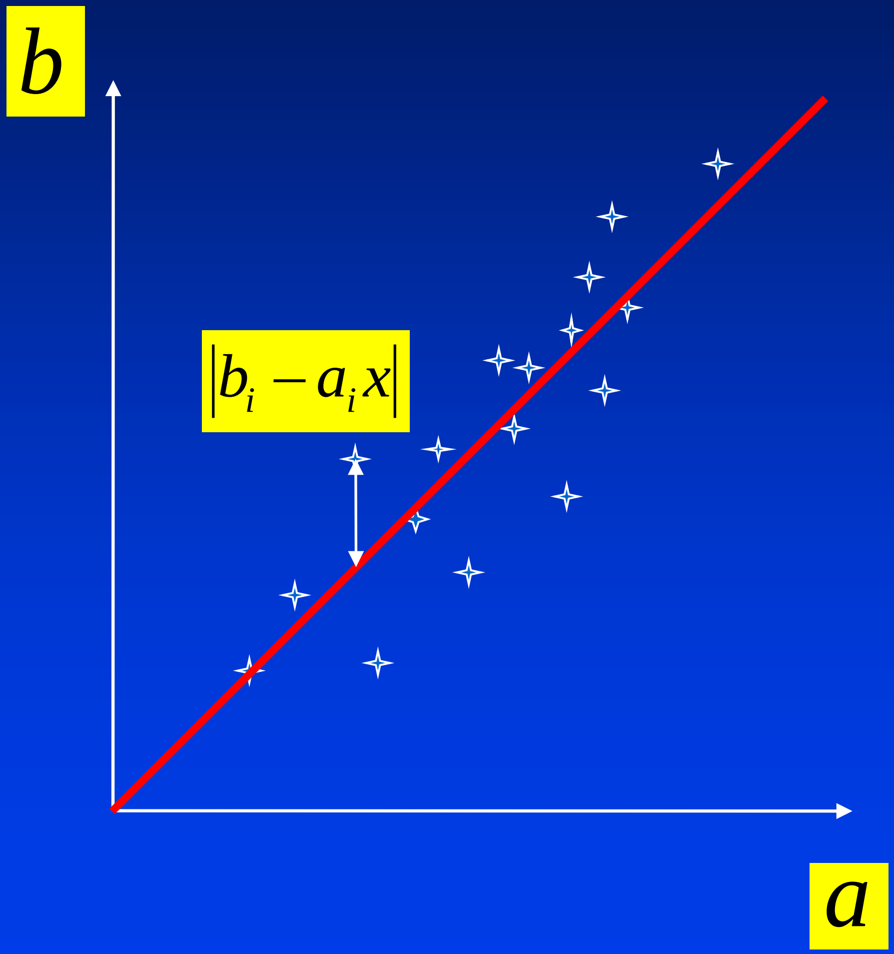
One-dimensional Regression



Find a line that represent the ”best” linear relationship:

$$b = ax$$

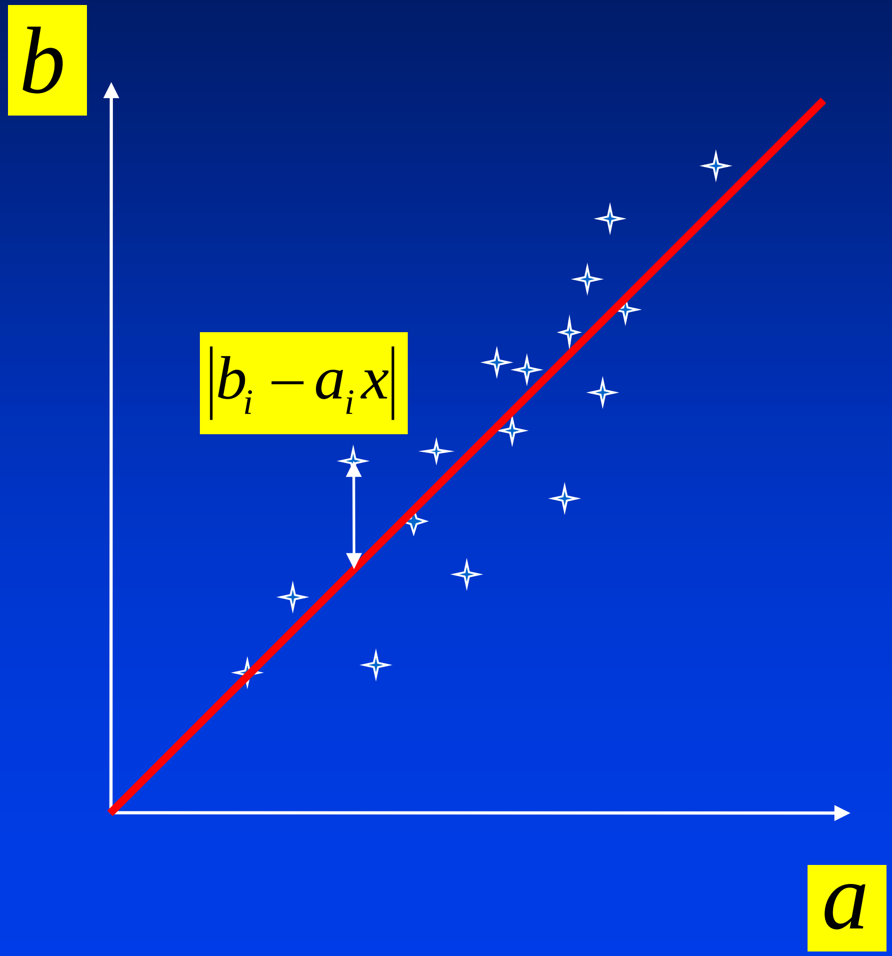
One-dimensional Regression



- Problem: the line does NOT go through all the data points exactly, so only approximation

$$e_i = b_i - a_i x$$

One-dimensional Regression



- Find the line that minimizes the sum of error squared:

$$\sum_i (b_i - a_i x)^2$$

Matrix Notation

Using the following notations

$$\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

we can rewrite the error function using linear algebra as:

$$e(x) = \sum_i (b_i - a_i x)^2$$

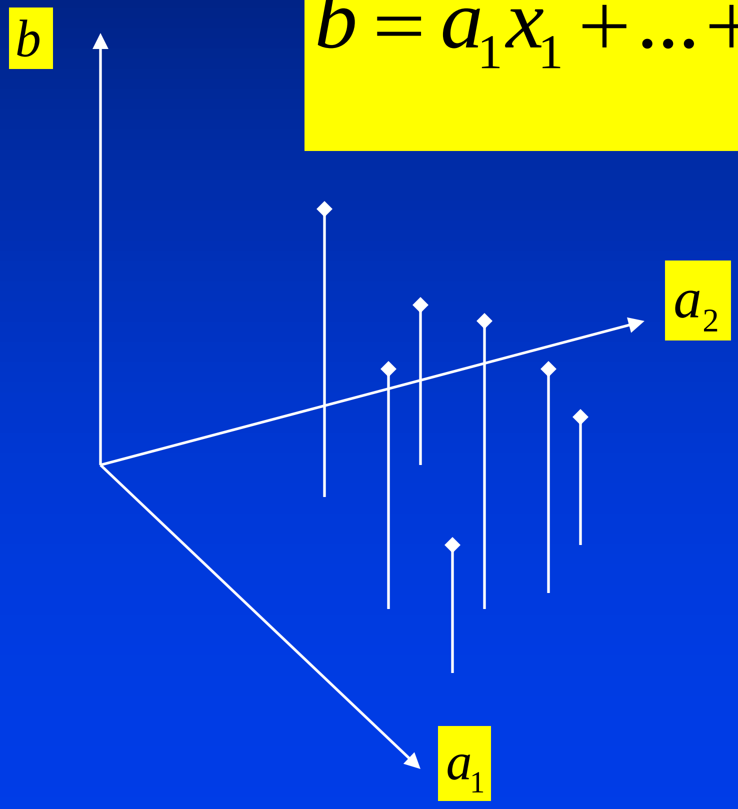
$$= (\mathbf{b} - x\mathbf{a})^T (\mathbf{b} - x\mathbf{a})$$

$$e(x) = \|\mathbf{b} - x\mathbf{a}\|^2$$

Multidimensional Linear Regression

Using a model with *m* parameters

$$b = a_1 x_1 + \ldots + a_m x_m = \sum_j a_j x_j$$



Multidimensional Linear Regression

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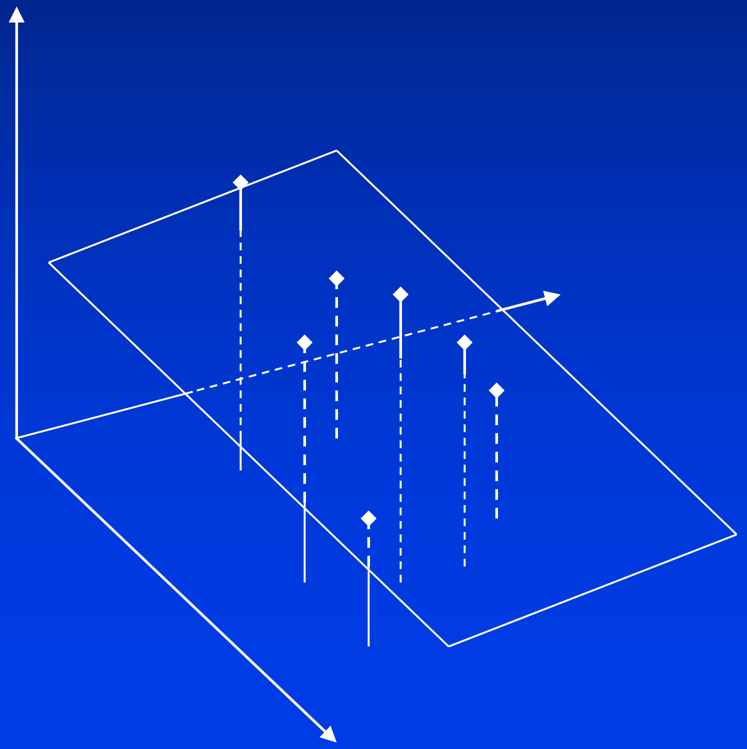

Multidimensional Linear Regression

Using a model with *m* parameters

$$b = a_1 x_1 + \ldots + a_m x_m = \sum_j a_j x_j$$

and *n* measurements

$$e(\boldsymbol{x}) = \sum_{i=1}^{n} \left(b_i - \sum_{j=1}^{m} a_{i,j} x_j\right)^2$$

$$= \left\| \boldsymbol{b} - \left[\sum_{j=1}^{m} a_{i,j} x_j \right] \right\|^2$$

$$e(\boldsymbol{x}) = \left\| \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x} \right\|^2$$

Matrix Notation

$$\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}$$

$$\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} - \begin{bmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$$

Matrix Notation

$$\boldsymbol{b} - \boldsymbol{Ax} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} - \begin{bmatrix} a_{1,1} & .. & a_{1,m} \\ \vdots & & \vdots \\ a_{n,1} & .. & a_{n,m} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$$

$$= \begin{bmatrix} b_1 - (a_{1,1}x_1 + \ldots + a_{1,m}x_m) \\ \vdots \\ b_n - (a_{n,1}x_1 + \ldots + a_{n,m}x_m) \end{bmatrix}$$

$\boldsymbol{b} - \boldsymbol{Ax}$

parameter 1

$$\boldsymbol{b} - \boldsymbol{Ax} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} - \begin{bmatrix} a_{1,1} & .. & a_{1,m} \\ \vdots & & \vdots \\ a_{n,1} & .. & a_{n,m} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

measurement *n*

$$= \begin{bmatrix} b_1 - (a_{1,1}x_1 + \ldots + a_{1,m}x_m) \\ \vdots \\ b_n - (a_{n,1}x_1 + \ldots + a_{n,m}x_m) \end{bmatrix}$$

Geometric Interpretation

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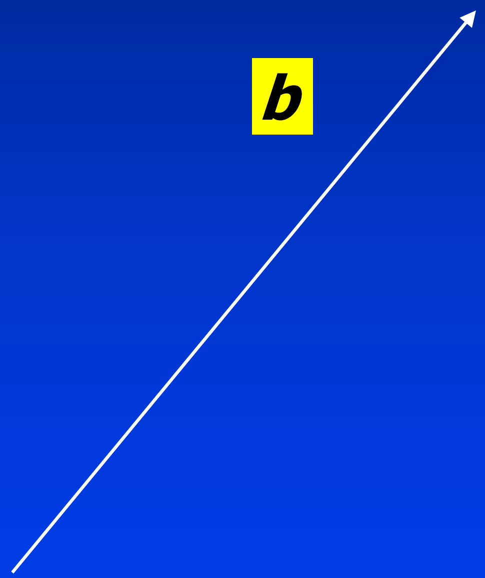
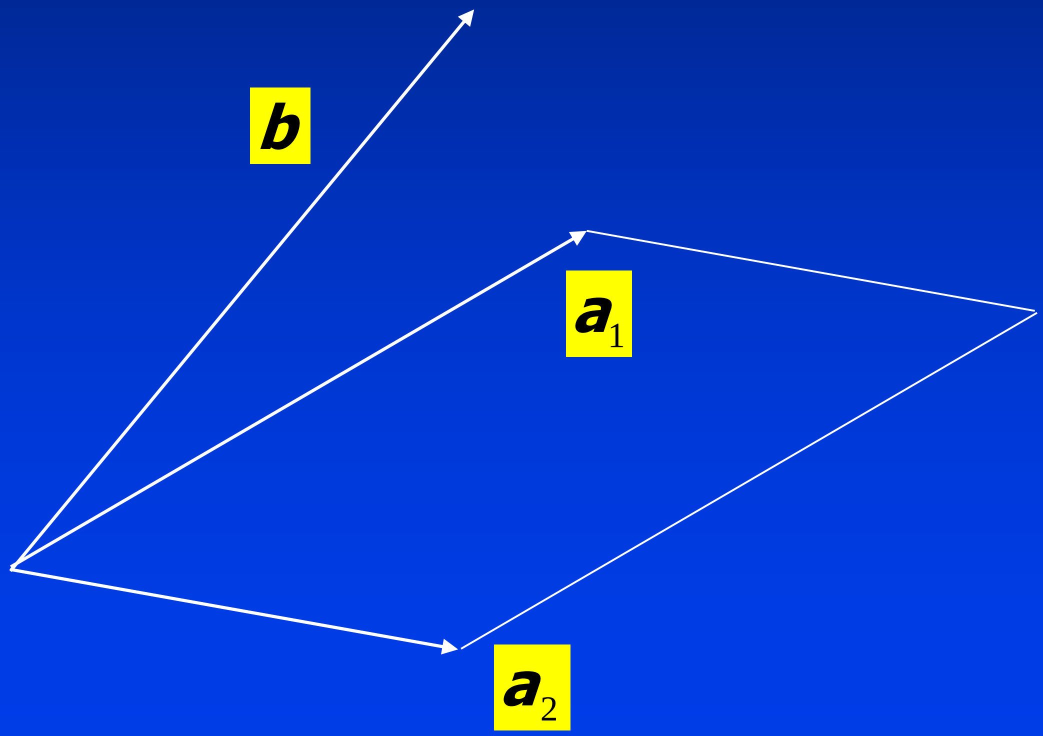
- ***b*** is a vector in R^n



Geometric interpretation

- ***b*** is a vector in R^n
- The columns of ***A*** define a vector space *range(A)*



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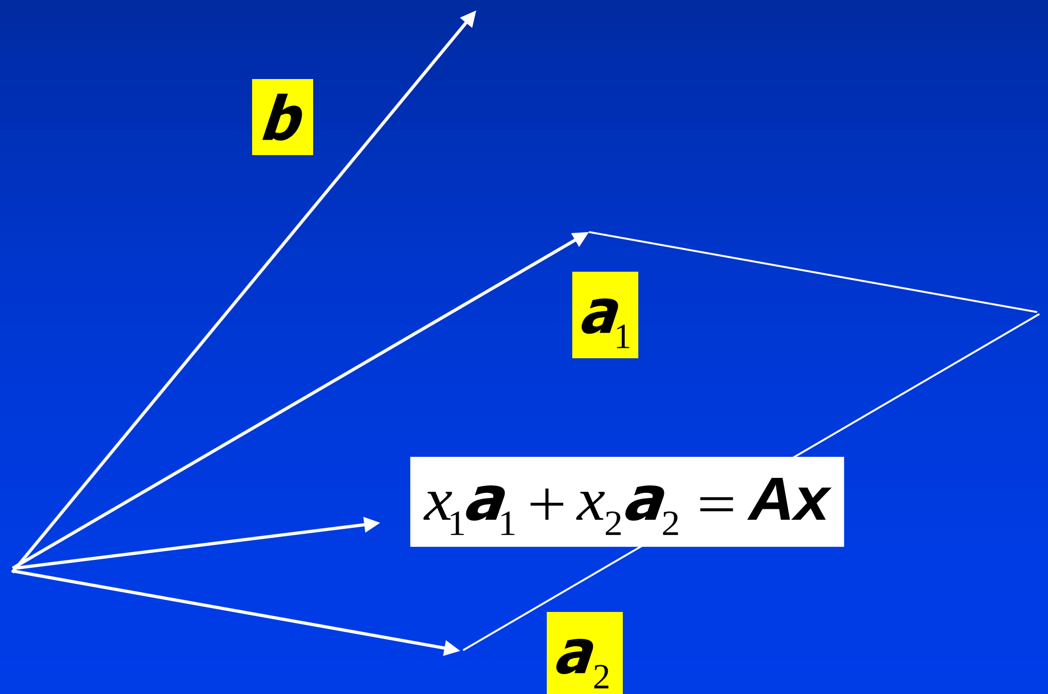

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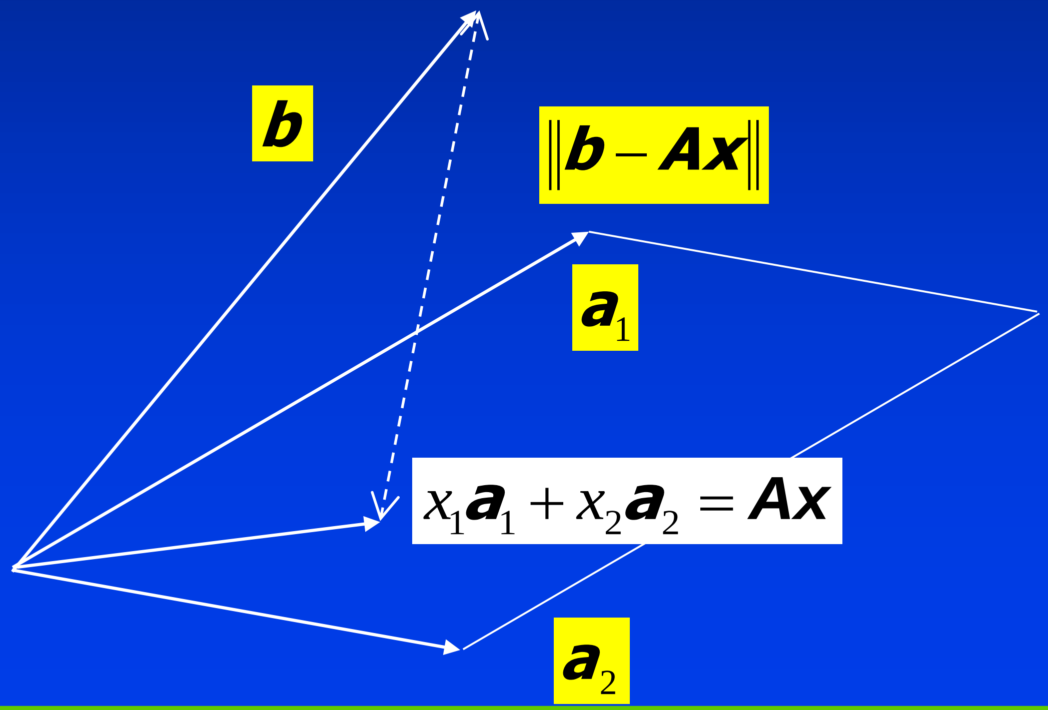
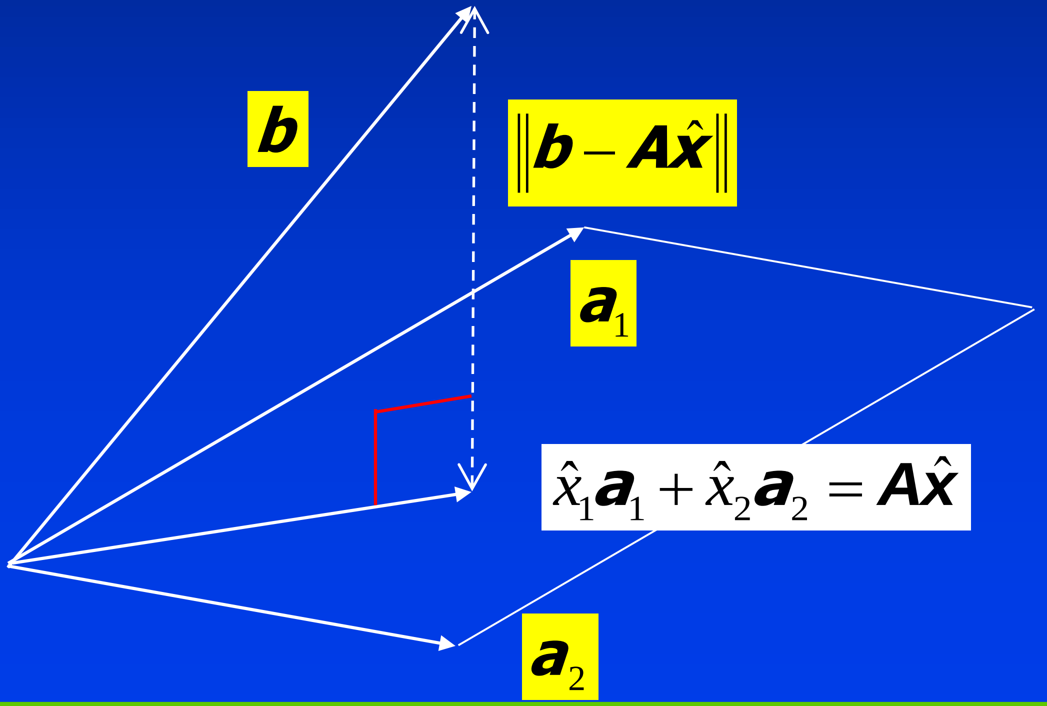


Geometric Interpretation

- $A\hat{x}$ is the orthogonal projection of ***b*** onto *range(**A**)*

$$\Leftrightarrow A^T\left(b - A\hat{x}\right) = 0 \Leftrightarrow A^T A\hat{x} = A^T b$$



The Normal Equation

$$\boldsymbol{A}^T\boldsymbol{A}\hat{\boldsymbol{x}} = \boldsymbol{A}^T\boldsymbol{b}$$

The Normal Equation: $A^T A\hat{x} = A^T b$

- Existence: $A^T A\hat{x} = A^T b$ has always a solution

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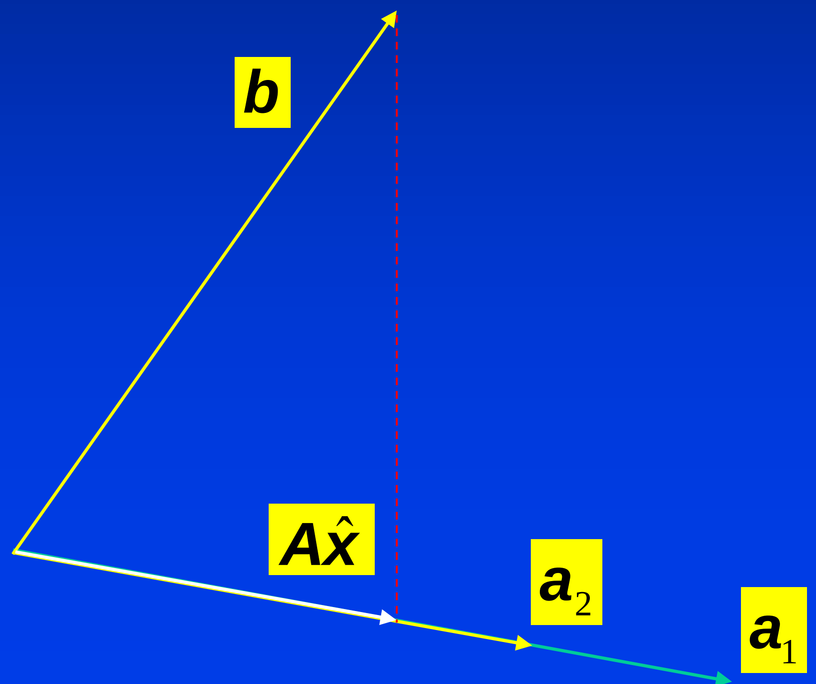
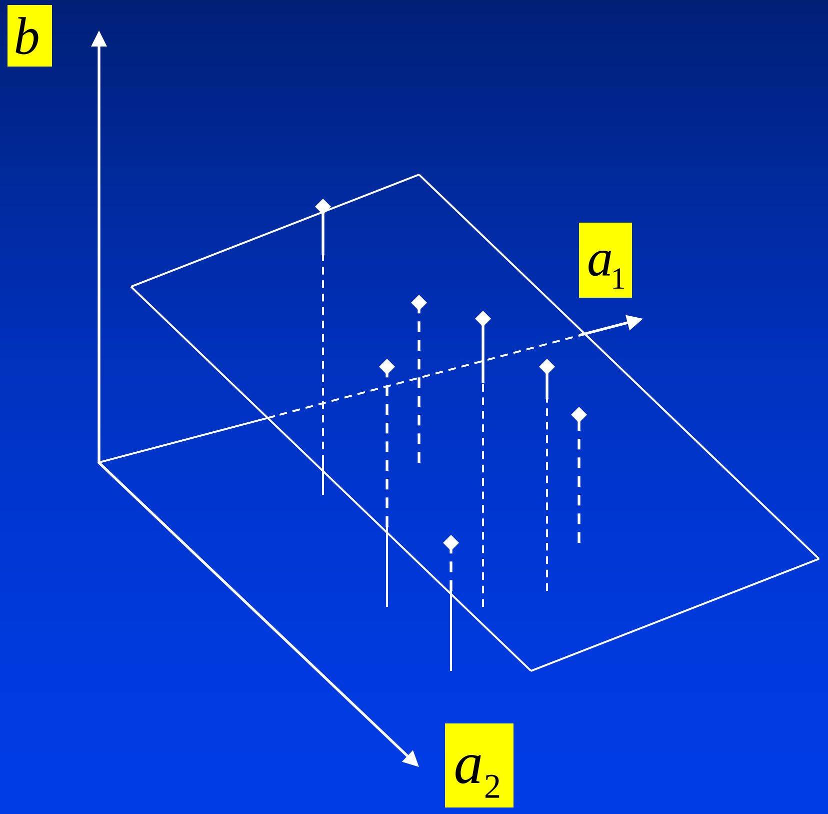


Under-constrained Problem



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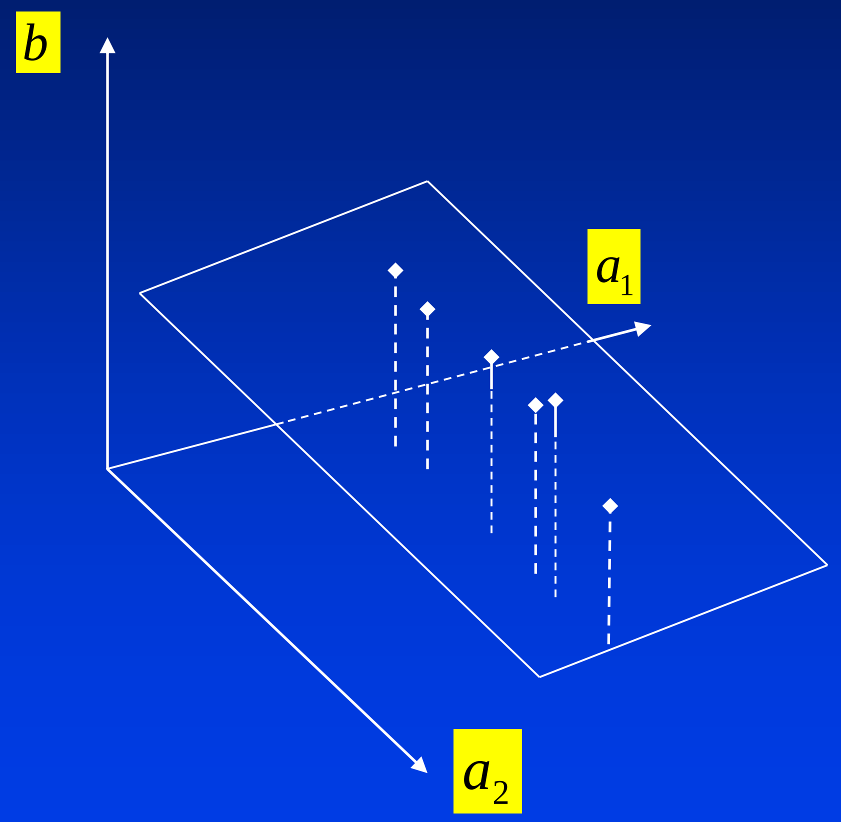
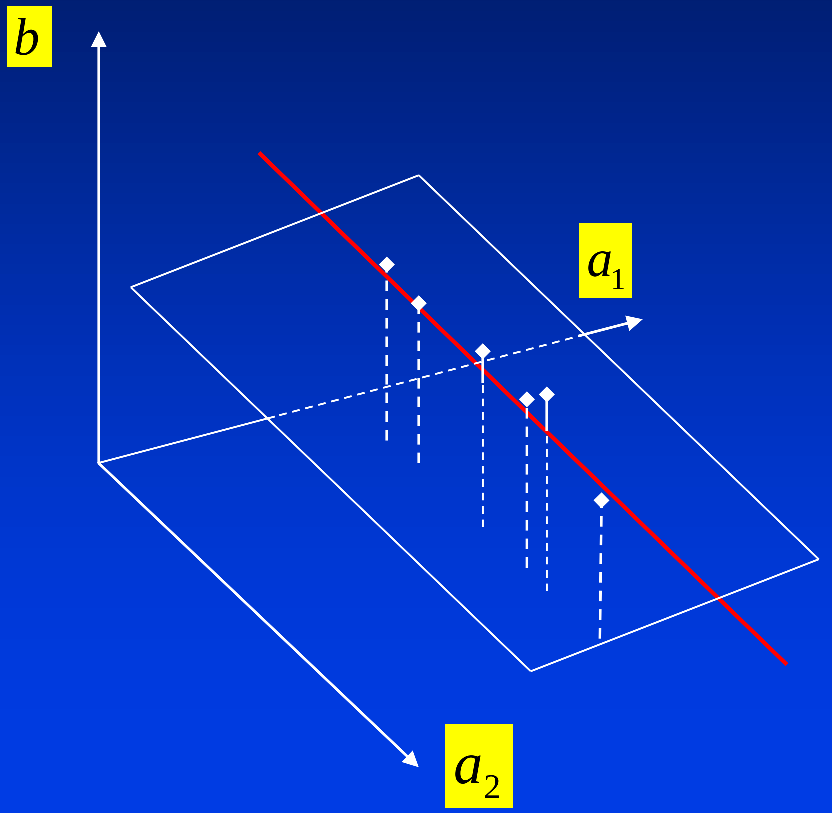
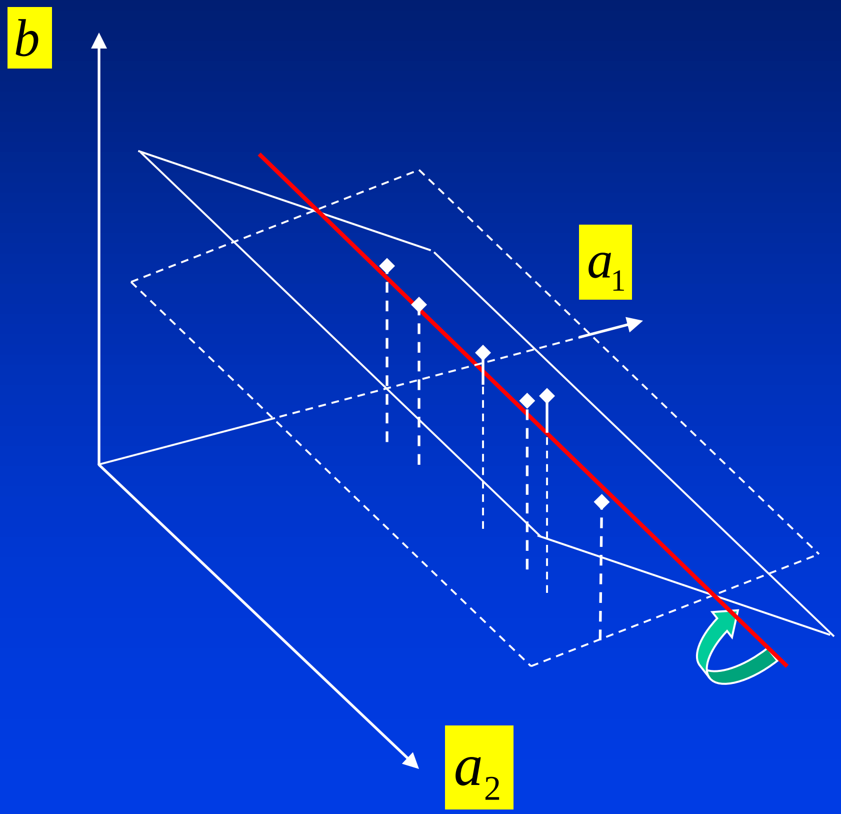


Under-constrained Problem



Under-constrained Problem

- Poorly selected data
- One or more of the parameters are redundant



Under-constrained Problem

- Poorly selected data
- One or more of the parameters are redundant

Add constraints

$$A^T A\mathbf{x} = A^T \mathbf{b} \quad \text{with} \min_{\mathbf{x}} \|\mathbf{x}\|$$

Minimizing $e(\boldsymbol{x})$

$\boldsymbol{x}_{\min}$ minimizes $e(\boldsymbol{x})$ if

Minimizing $e(\boldsymbol{x})$

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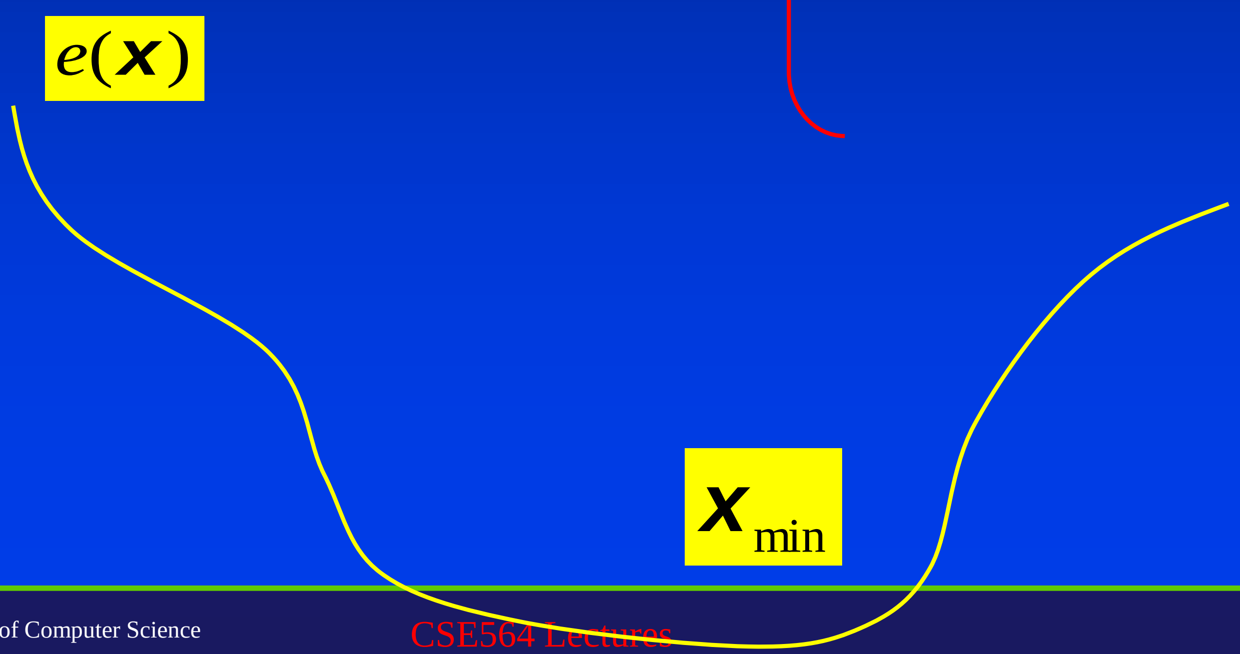
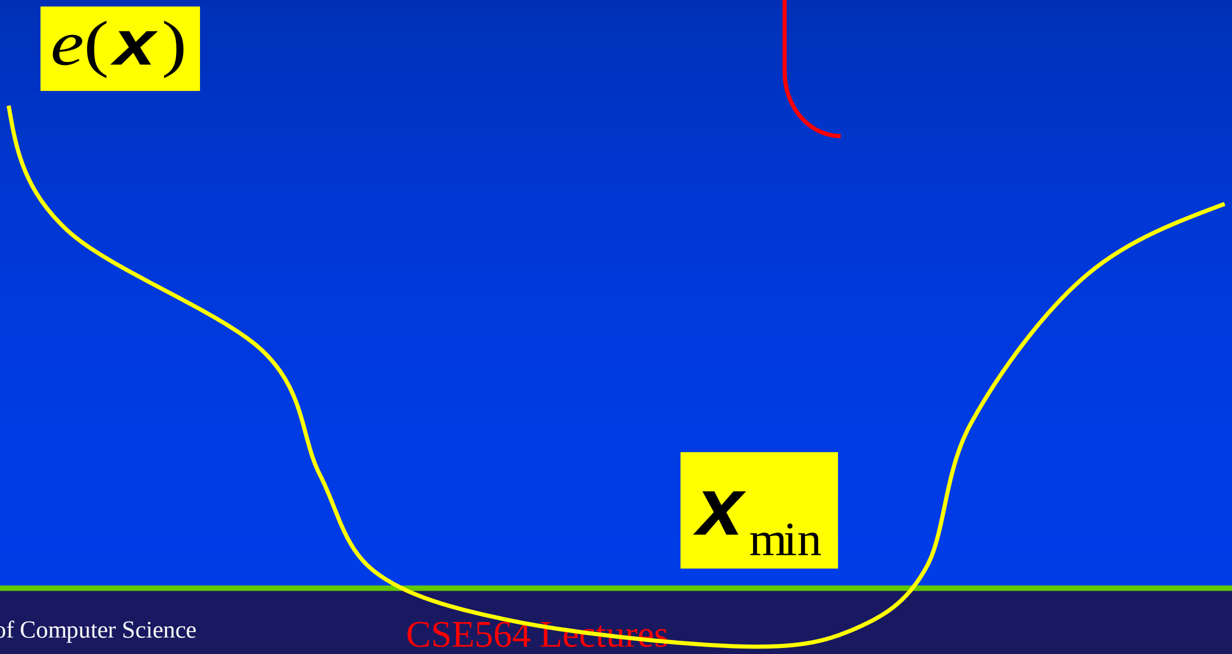


Minimizing $e(\boldsymbol{x})$

$\boldsymbol{x}_{\min}$ minimizes $e(\boldsymbol{x})$ if

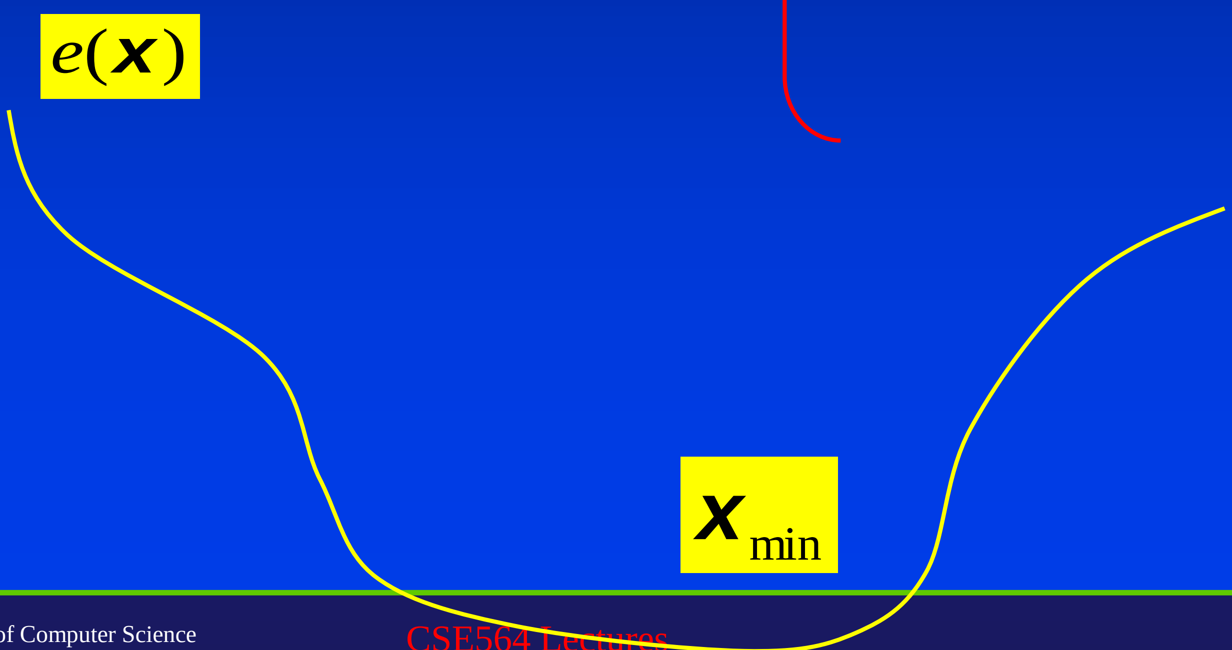
$e(\boldsymbol{x})$ is flat at $\boldsymbol{x}_{\min}$



Minimizing $e(\mathbf{x})$

$\mathbf{x}_{\min}$ minimizes $e(\mathbf{x})$ if

- $e(\mathbf{x})$ is flat at $\mathbf{x}_{\min}$
- $\nabla e(\mathbf{x}_{\min}) = \mathbf{0}$



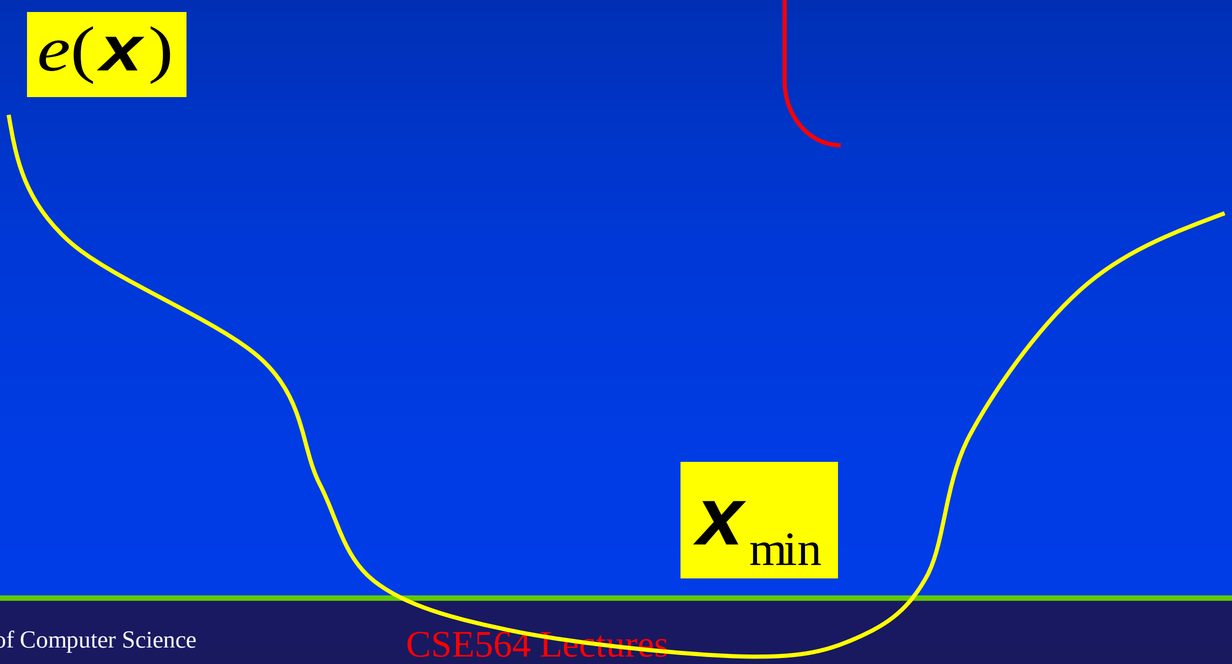
Minimizing $e(\boldsymbol{x})$

$\boldsymbol{x}_{\min}$ minimizes $e(\boldsymbol{x})$ if

- $e(\boldsymbol{x})$ is flat at $\boldsymbol{x}_{\min}$

$$\nabla e(\boldsymbol{x}_{\min}) = \boldsymbol{0}$$

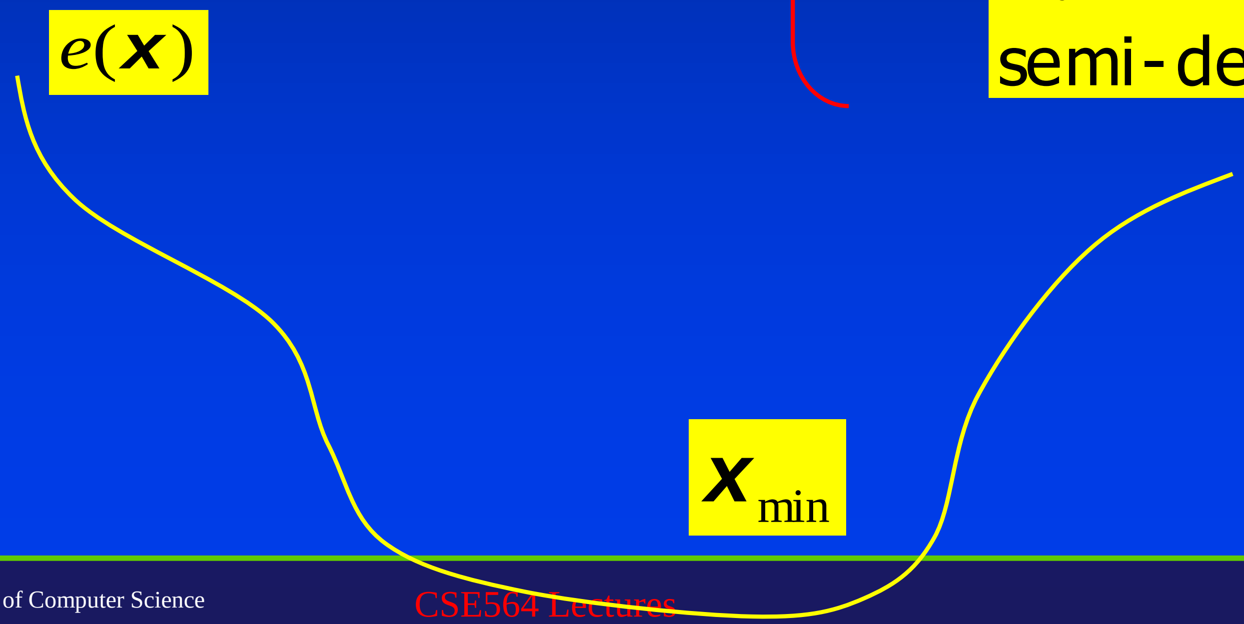
- $e(\boldsymbol{x})$ does not go down around $\boldsymbol{x}_{\min}$



Minimizing $e(\mathbf{x})$

$\mathbf{x}_{\min}$ minimizes $e(\mathbf{x})$ if

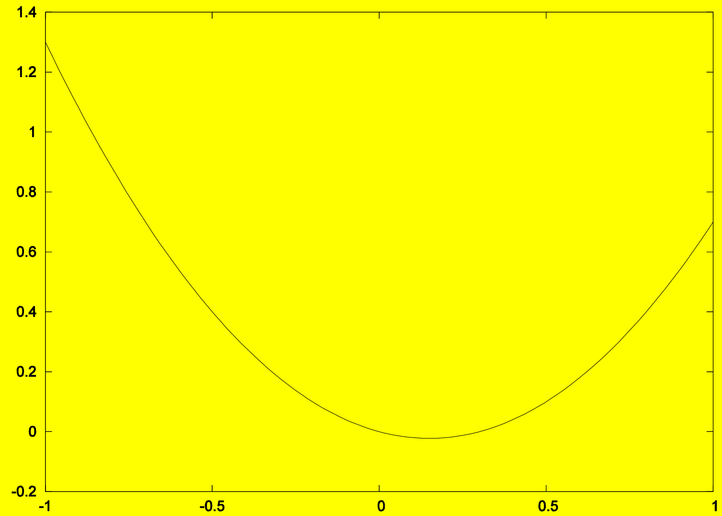
- $e(\mathbf{x})$ is flat at $\mathbf{x}_{\min}$
  - $\nabla e(\mathbf{x}_{\min}) = \mathbf{0}$
- $e(\mathbf{x})$ does not go down around $\mathbf{x}_{\min}$
  - $H_e(\mathbf{x}_{\min})$ is positive semi-definite



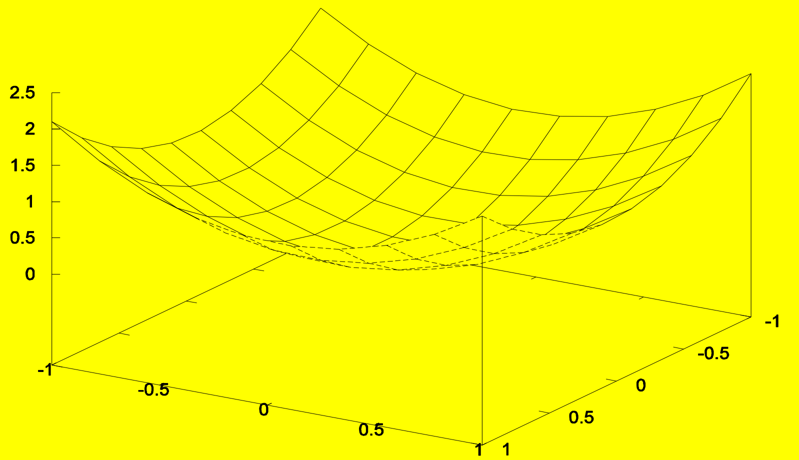
Positive Semi-definite

$$\boldsymbol{A} \text{ is positive semi-definite} \Leftrightarrow \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} \geq \boldsymbol{0}, \text{ for all } \boldsymbol{x}$$

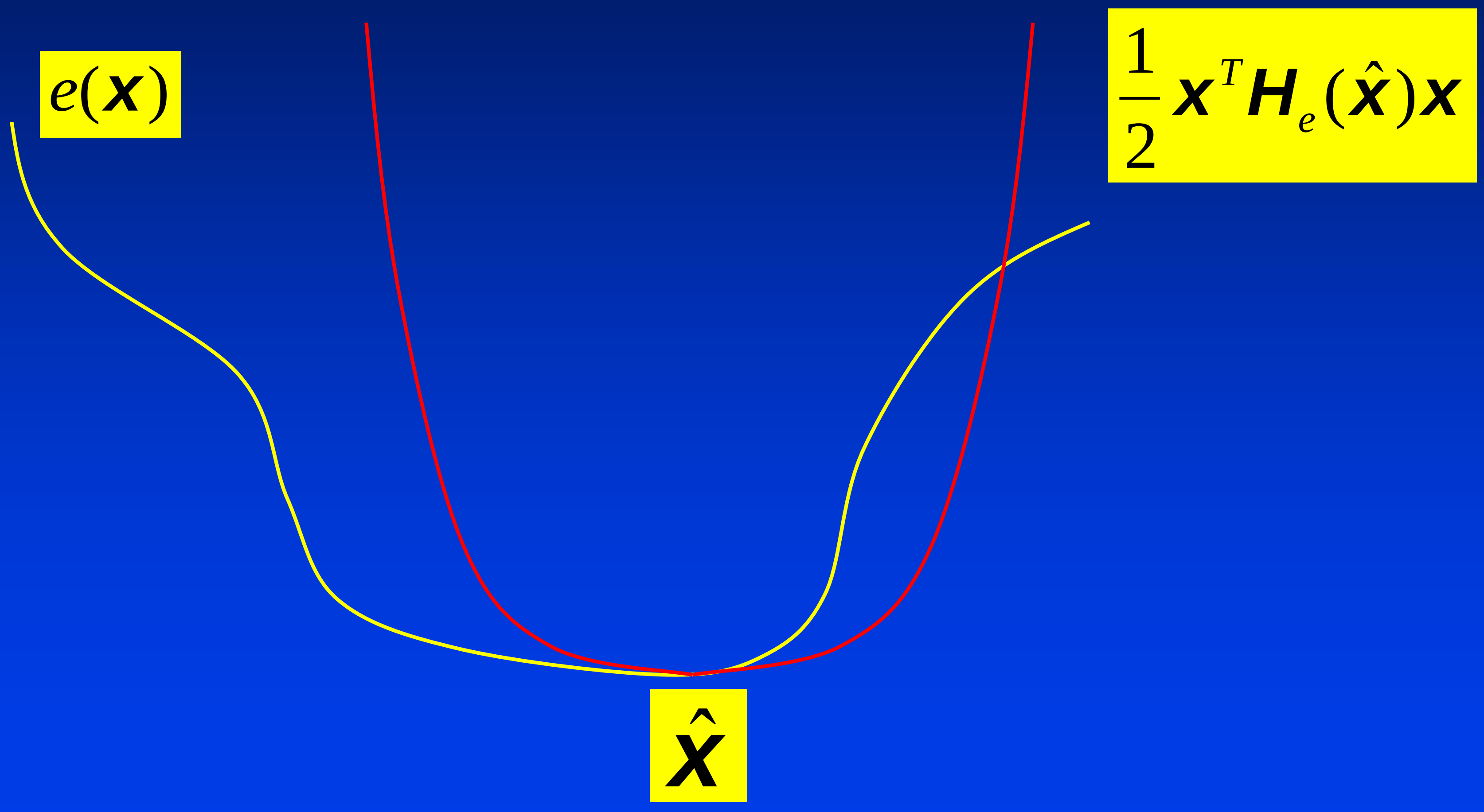
In 1-D



In 2-D

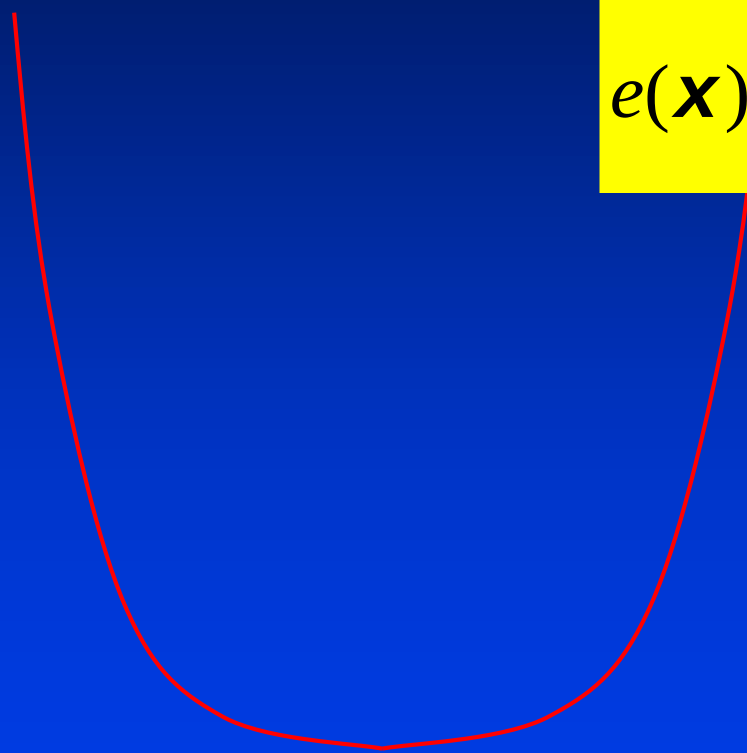


Minimizing $e(\boldsymbol{x})$



Minimizing $e(\mathbf{x}) = \|\mathbf{b} - \mathbf{A}\mathbf{x}\|^2$

$$e(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \mathbf{H}_e(\hat{\mathbf{x}})\mathbf{x}$$



$\hat{\mathbf{x}}$

Minimizing $e(\boldsymbol{x}) = \|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|^2$

$\hat{\boldsymbol{x}}$ minimizes $e(\boldsymbol{x})$ if

- $\boldsymbol{A}^T\boldsymbol{A}\hat{\boldsymbol{x}} = \boldsymbol{A}^T\boldsymbol{b}$
- $2\boldsymbol{A}^T\boldsymbol{A}$ is positive semi-definite

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Always true

Minimizing $e(\boldsymbol{x}) = \|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|^2$

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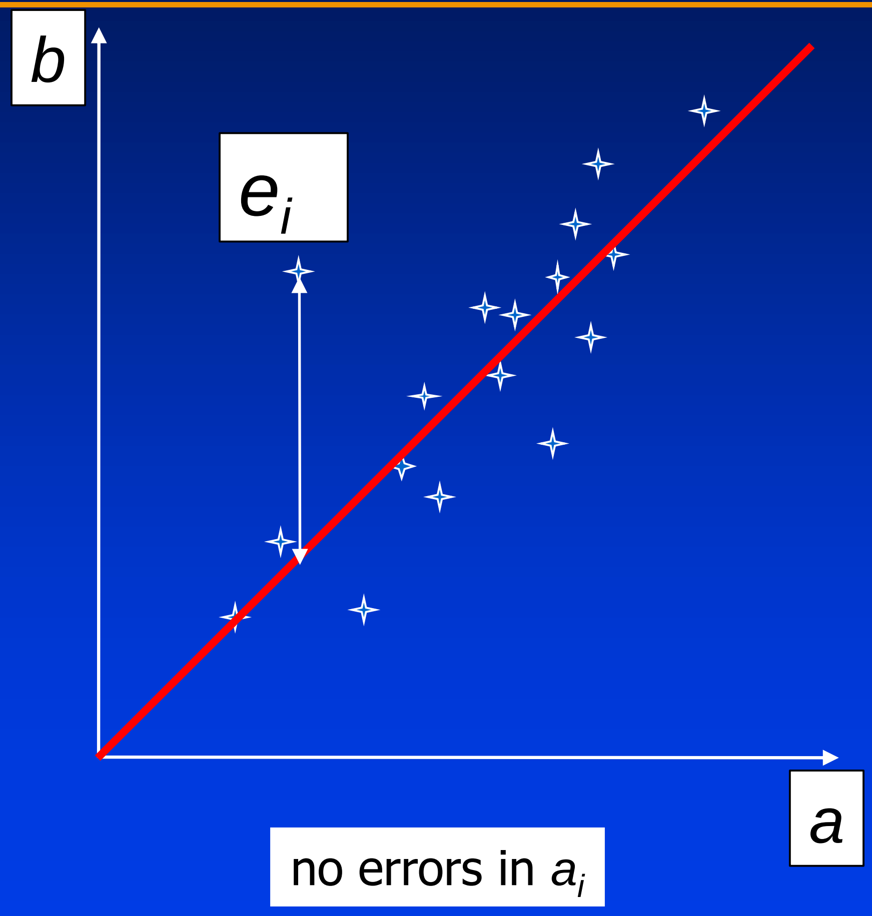
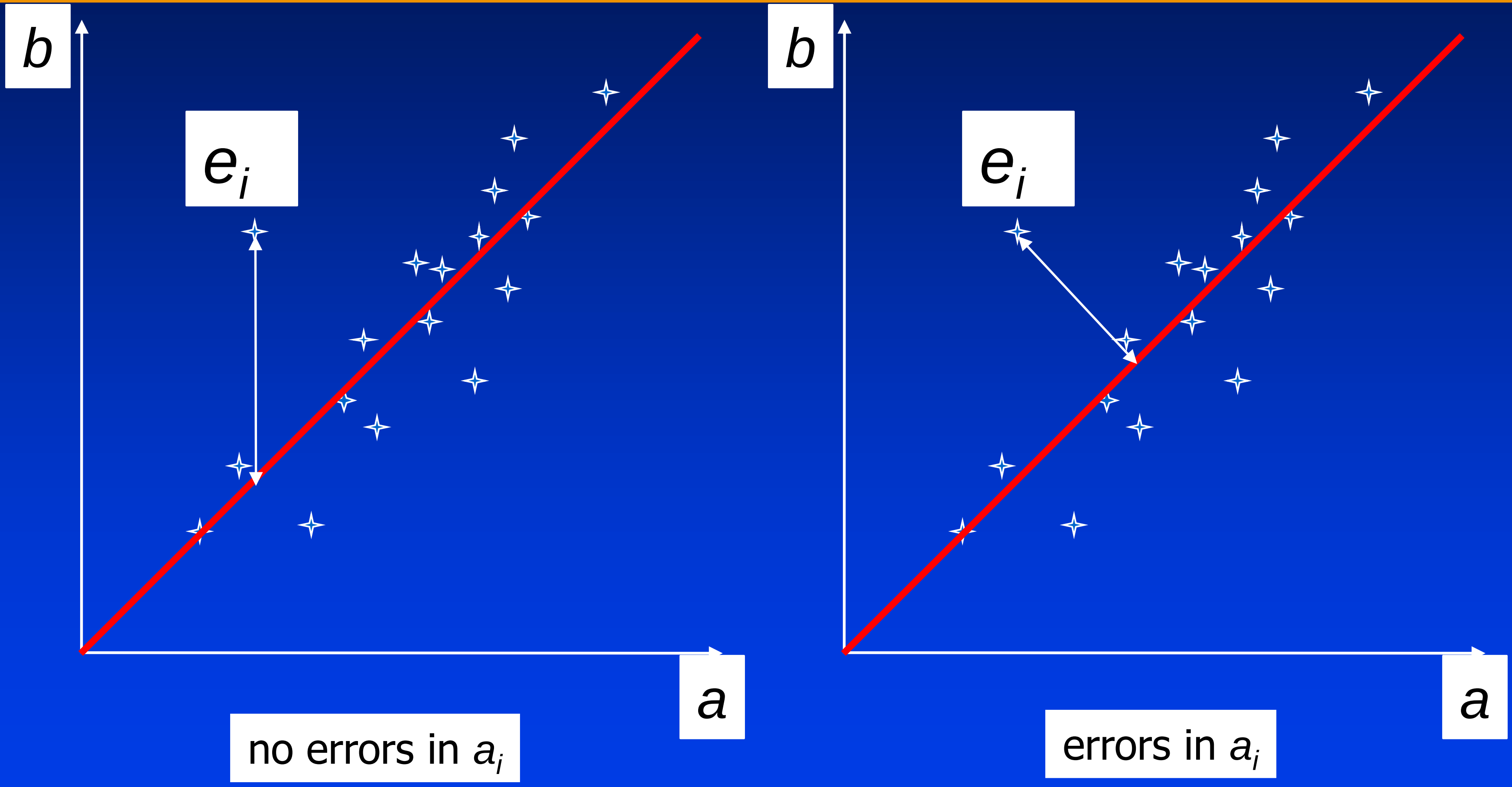
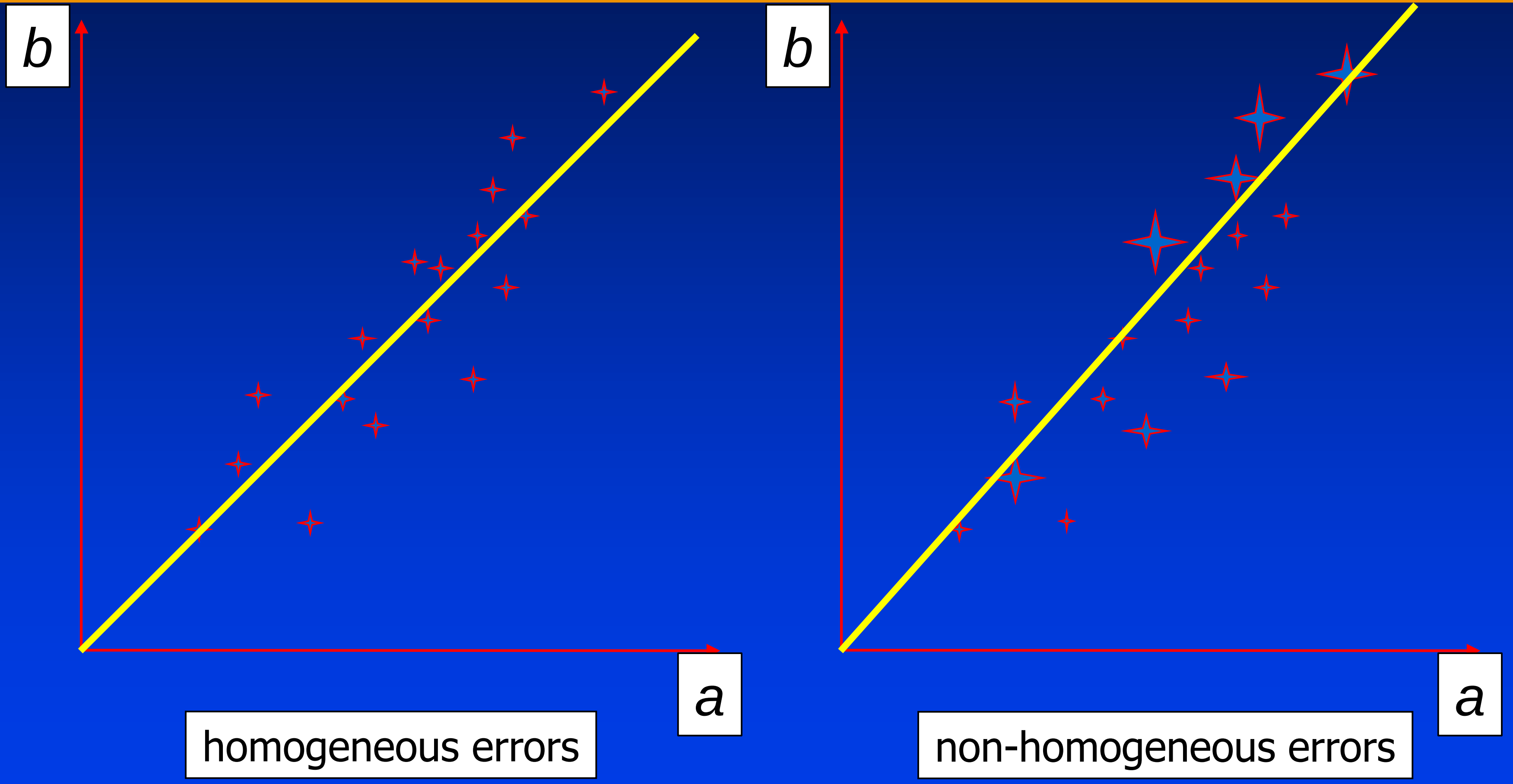
The ***Normal Equation***

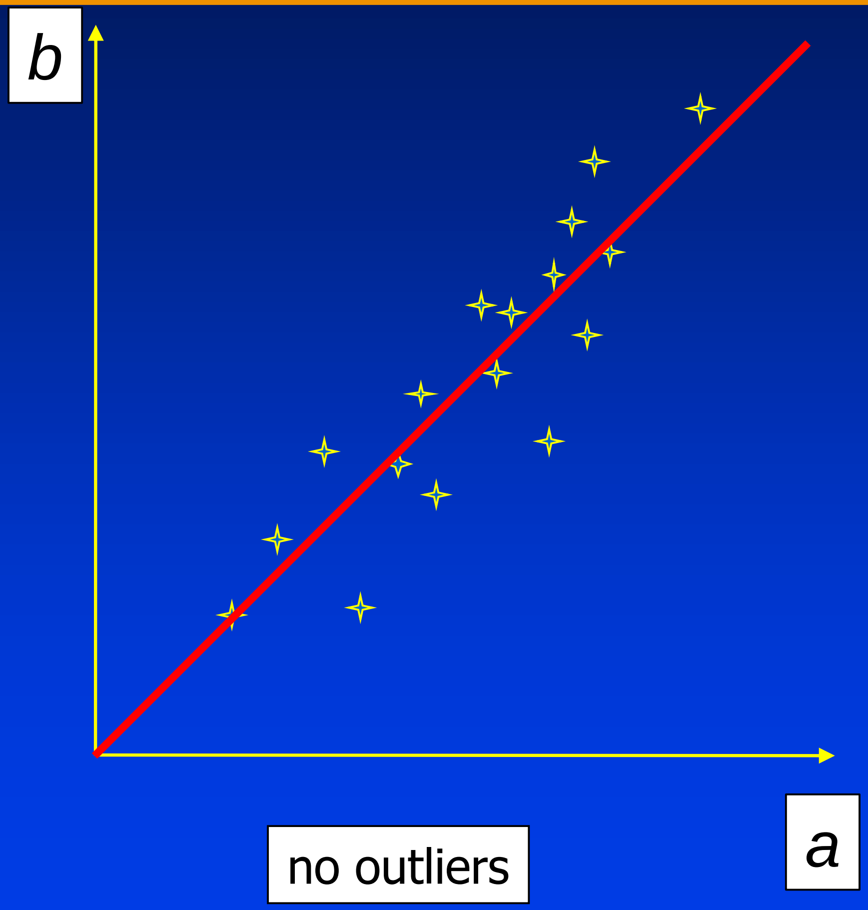
$2\boldsymbol{A}^T\boldsymbol{A}$ is positive semi-definite

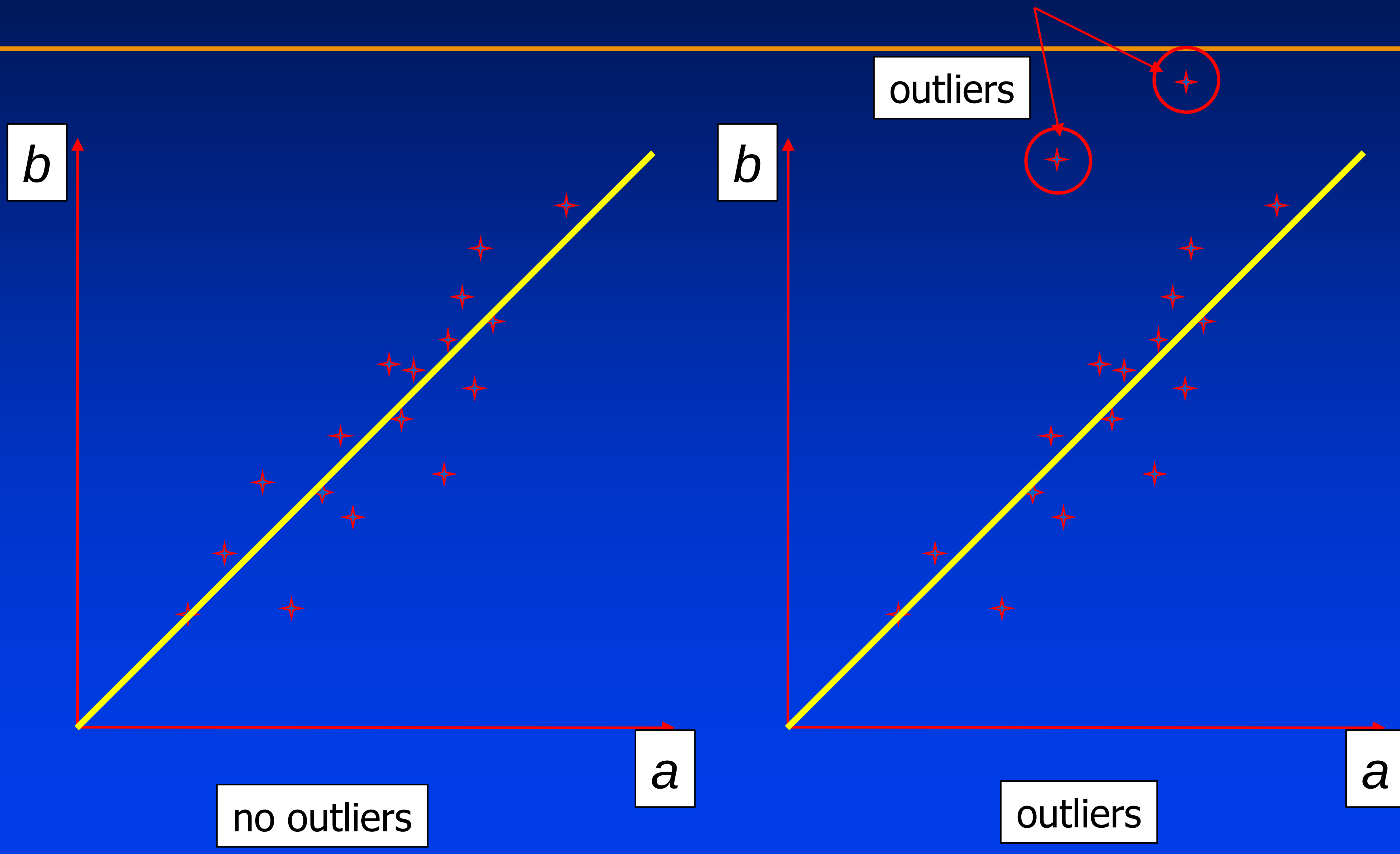
Always true


b
e_i
a
no errors in a_i


b
e_i
a
no errors in a_i
b
e_i
a
errors in a_i


b
a
homogeneous errors
b
a
non-homogeneous errors


b
a
no outliers



b
a
no outliers
b
a
outliers
outliers

Question

You should be able to prove that the equation above leads to the following expression for the best fit straight line:

$$y_p(x) = mx + b$$

$$m = \frac{\left(N \sum_i^N x_i y_i - \sum_i x_i \sum_i y_i\right)}{N \sum_i x_i^2 - \left(\sum_i x_i\right)^2}$$

$$b = \frac{\sum_i^N y_i - m \sum_i x_i}{N}$$

How good is the least-squares criteria?

- Optimality: the Gauss-Markov theorem

How good is the least-squares criteria?

- Optimality: the Gauss-Markov theorem

  Let $\{b_i\}$ and $\{x_j\}$ be two sets of random variables and define:

$$e_i = b_i - a_{i,1}x_1 - \ldots - a_{i,m}x_m$$

How good is the least-squares criteria?

- Optimality: the Gauss-Markov theorem

  Let $\{b_i\}$ and $\{x_j\}$ be two sets of random variables and define:

$$e_i = b_i - a_{i,1}x_1 - \ldots - a_{i,m}x_m$$

If

A1: $\{a_{i,j}\}$ are not random variables,

A2: $E(e_i) = 0$, for all i,

A3: $\mathrm{var}(e_i) = \sigma$, for all i,

A4: $\mathrm{cov}(e_i, e_j) = 0$, for all i and j,

How good is the least-squares criteria?

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A4: $\operatorname{cov}(e_i, e_j) = 0$, for all i and j,

Then $\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \sum e_i^2$ is the best unbiased linear estimator

Least Squares Interpolant

- We arrive at a system of equations through function minimization

$$2\mathbf{p}^T\mathbf{p}\mathbf{a} - 2\mathbf{p}^T\mathbf{y} = 0 \qquad \mathbf{a} = \left(\mathbf{p}^T\mathbf{p}\right)^{-1}\mathbf{p}^T\,\mathbf{y}^T$$

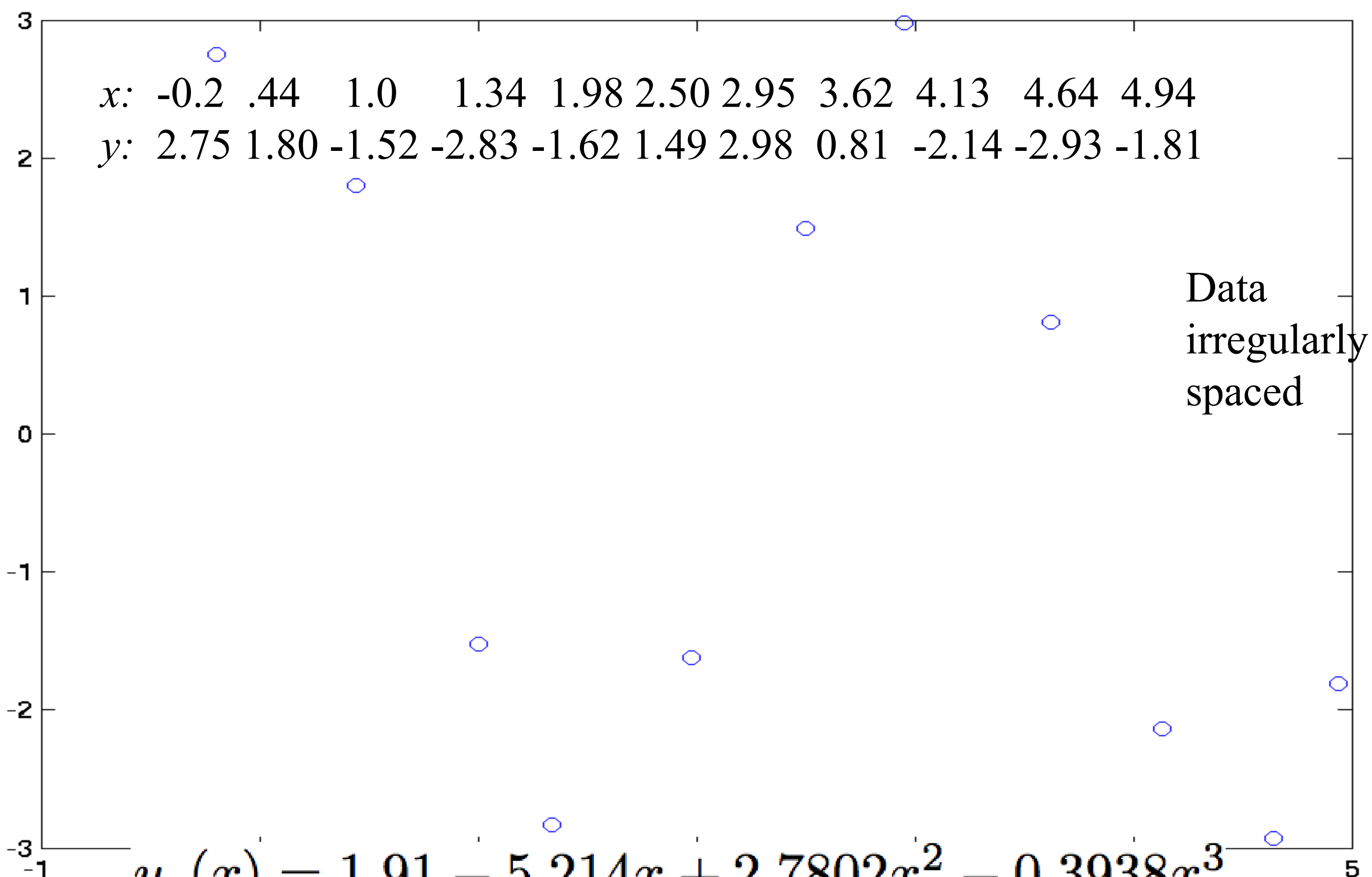
- We can introduce a pseudo-inverse

$$\mathbf{a} = \mathbf{p}^+\mathbf{y}^T$$

- For four points with a cubic polynomial

$$\mathbf{p} = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix}$$

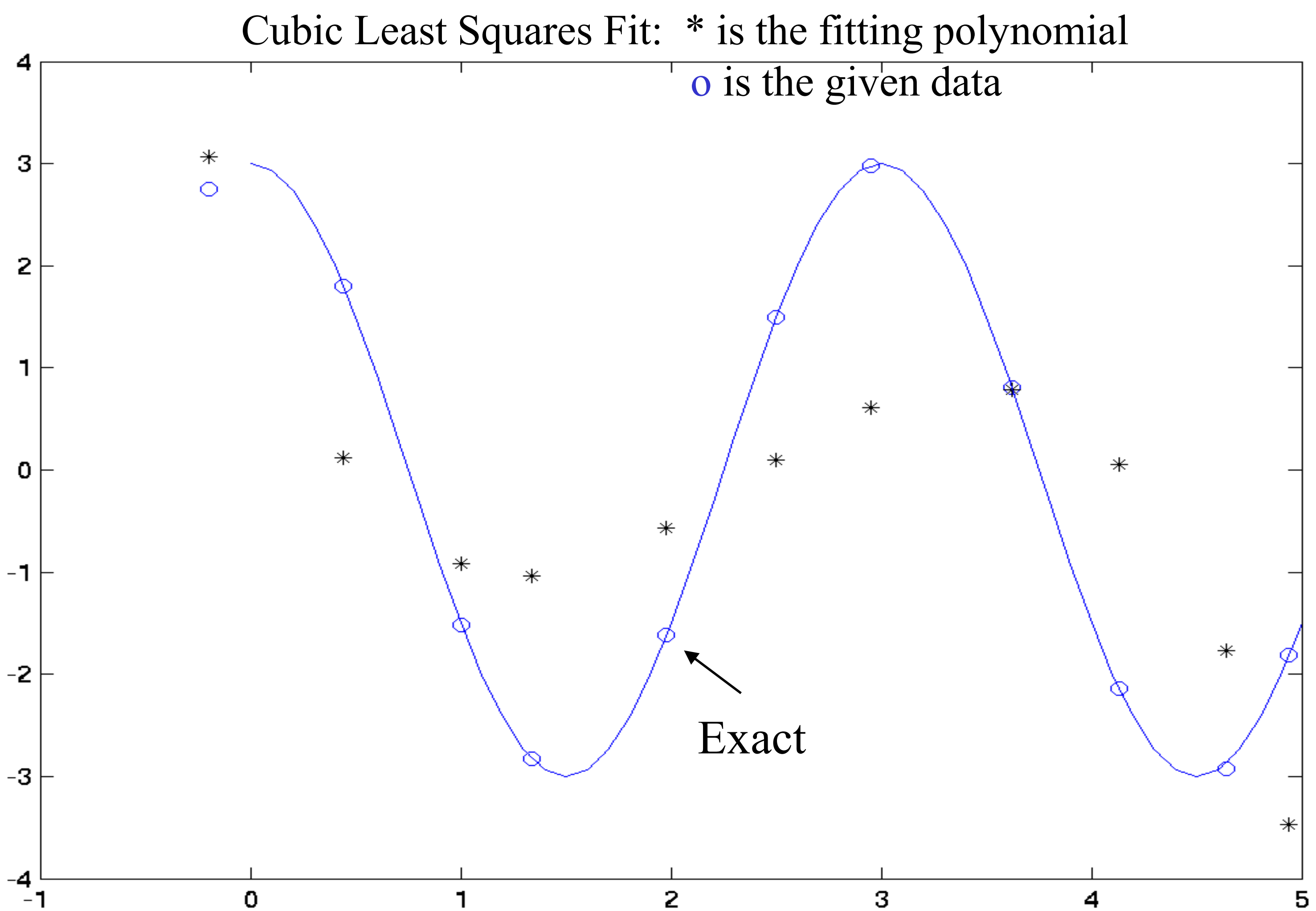
Cubic Least Squares Example



x:	-0.2	.44	1.0	1.34	1.98	2.50	2.95	3.62	4.13	4.64	4.94
y:	2.75	1.80	-1.52	-2.83	-1.62	1.49	2.98	0.81	-2.14	-2.93	-1.81

$$y_p(x) = 1.91 - 5.214x + 2.7802x^2 - 0.3938x^3$$

Least Squares Interpolant



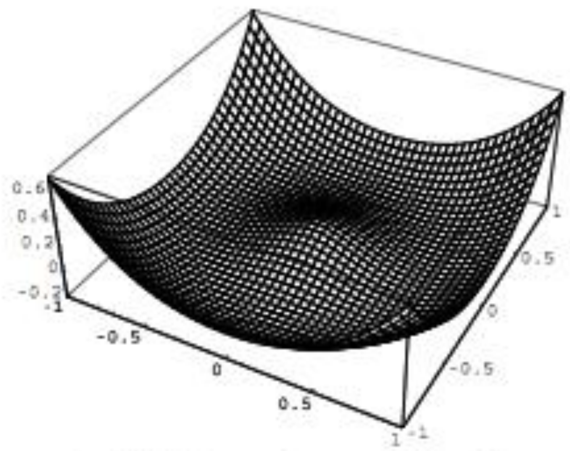
Piecewise Interpolation

- Piecewise polynomials: a collection of polynomials to fit all the data points
- Different choices: linear, quadratic, cubic
- Non-polynomials: radial basis functions (RBFs)

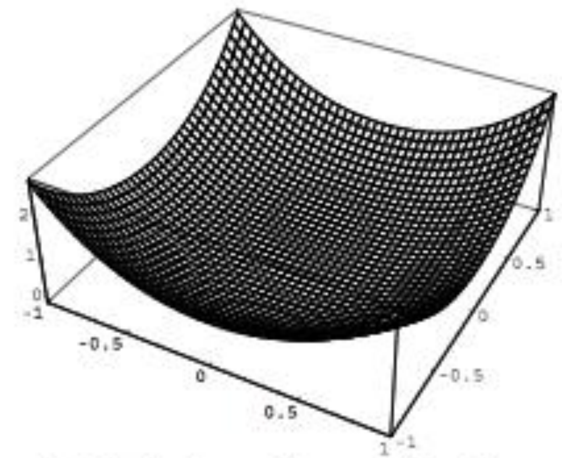
Radial Basis Functions

Developed to interpolate 2-D data: think bathymetry.
Given depths: $\mathbf{x}_i, i = 1, N$, interpolate to a rectangular grid.

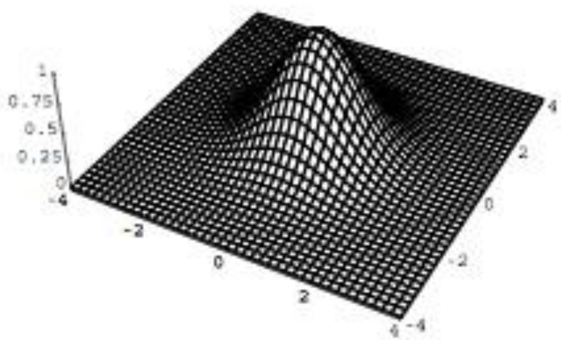
RBF



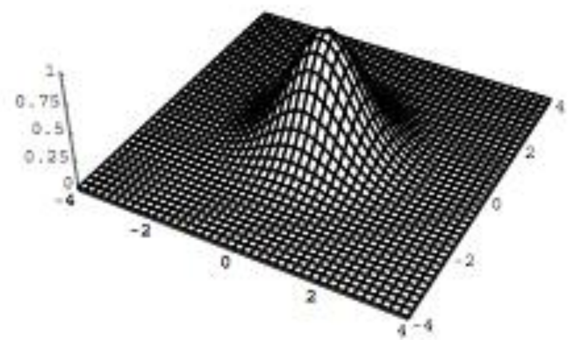
a) Thin-plate (2-d)
$\phi(r) = r^2 \log r$



b) Thin-plate (3-d)
$\phi(r) = r^3$



c) Gaussian
$\phi(r) = e^{r^2 \sigma^2}$



d) Compactly Supported
$\phi(r) = (1-r)_+^4 (4r+1)$

$r = |\mathbf{x} - \mathbf{x}_j|$

Radial Basis Functions

- Data points: $\mathbf{x}_i, i = 1, N$
- For each position, there is an associated value:

$$u_i, i = 1, N$$

- Radial basis function (located at each point):

$$g_j(\mathbf{x}) \equiv g(|\mathbf{x} - \mathbf{x}_j|), j = 1, N$$

$$u_p(\mathbf{x}) = \sum_{j=1}^{N} \alpha_j \, g_j(\mathbf{x})$$

Radial Basis Function for Data Fitting

- To find the unknown coefficients, we force the interpolant to go through all the data points:

$$\sum_{j=1}^{N} \alpha_j \, g_j(\mathbf{x}_i) = u_i, \quad i = 1, N$$

$$\mathbf{x}_i \equiv \left| \mathbf{x}_i - \mathbf{x}_j \right|$$

- We have n equations for the n unknown coefficients

Multiquadric RBF

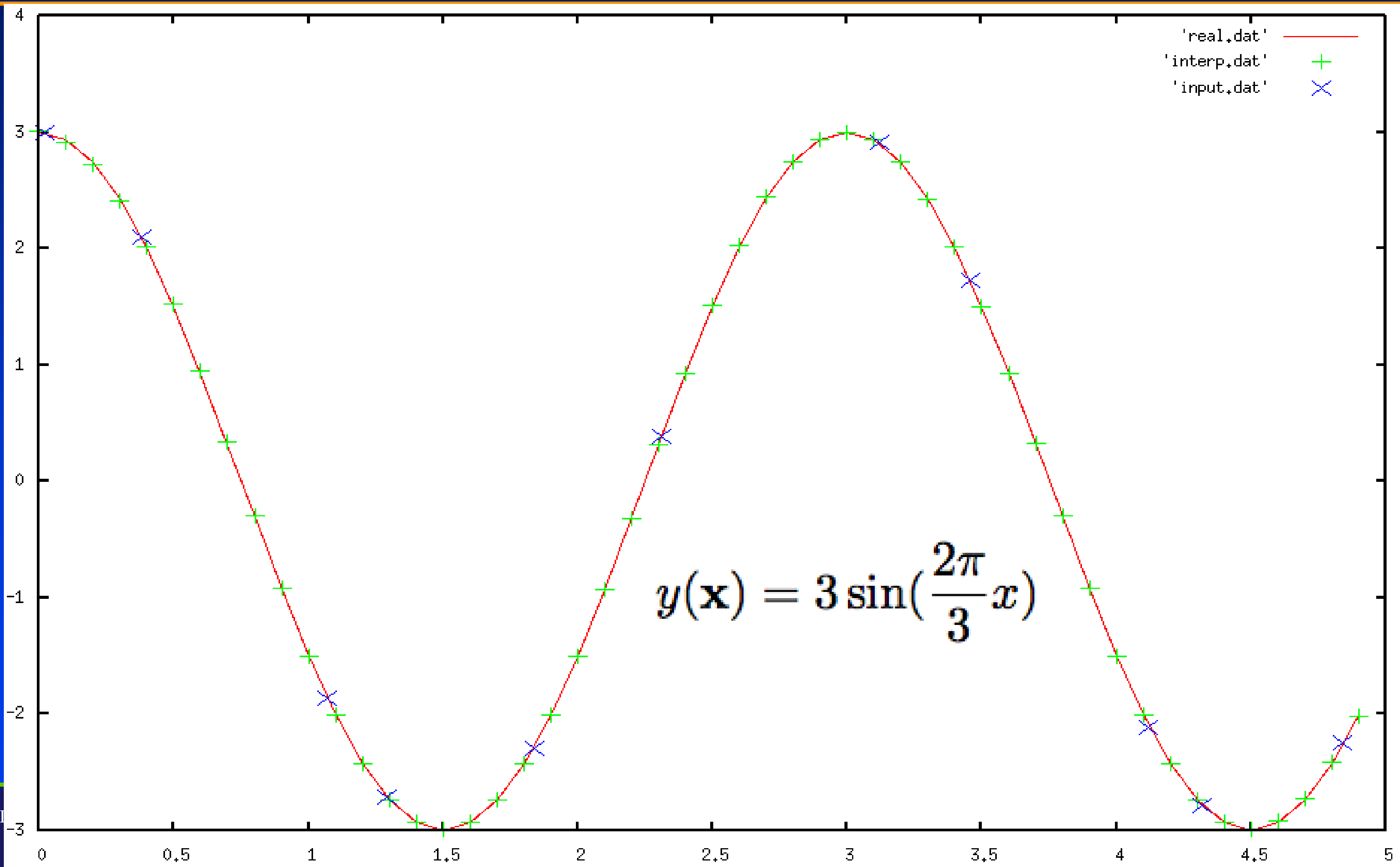
MQ: $$g_j(\mathbf{x}) = \sqrt{c_j^2 + r^2}$$

$$r = |\mathbf{x} - \mathbf{x}_j|$$

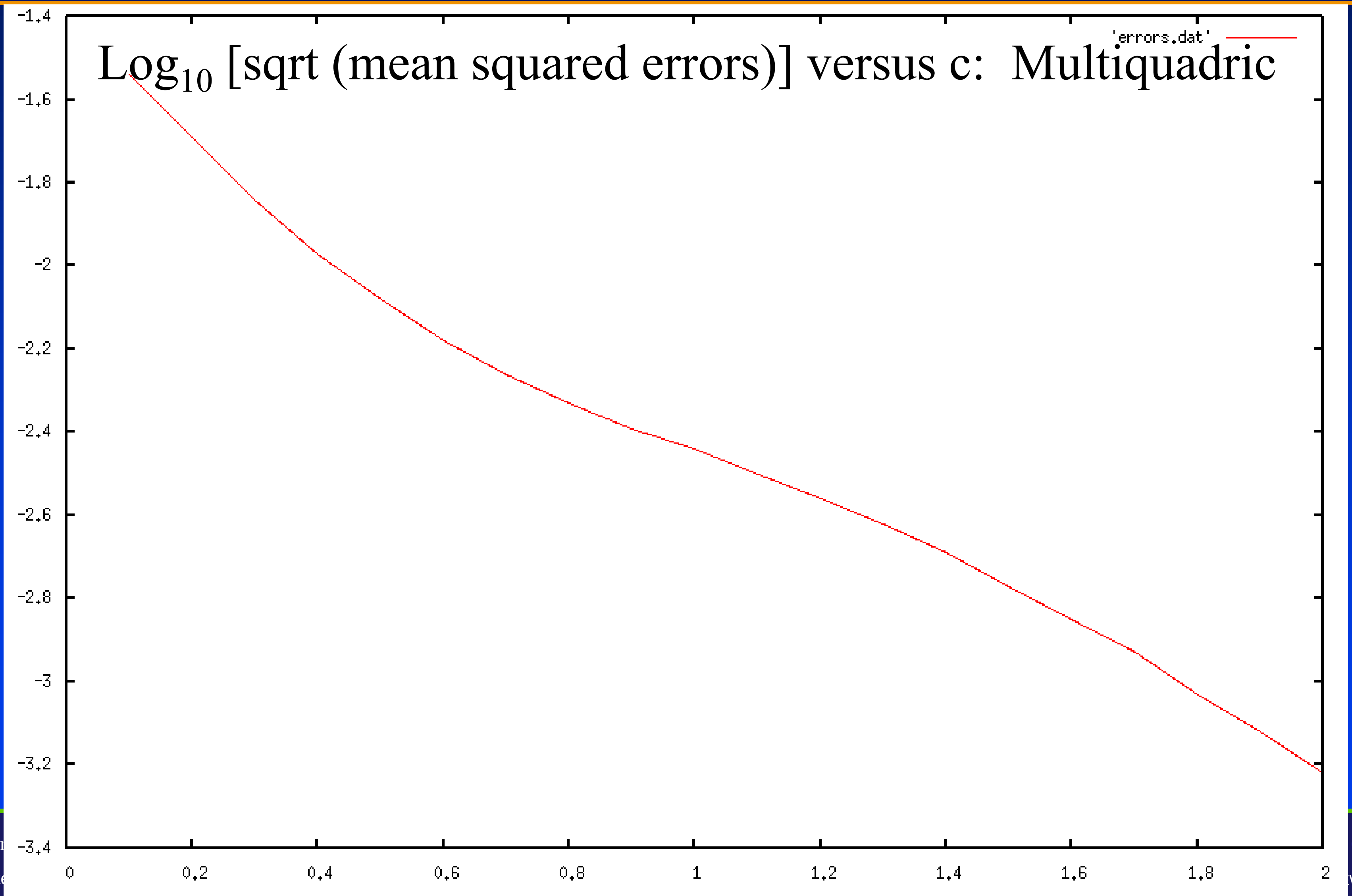
RMQ: $$g_j(\mathbf{x}) = \frac{1}{\sqrt{c_j^2 + r^2}}$$

Hardy, 1971; Kansa, 1990

11 (x,y) pairs: (0.2, 3.00), (0.38, 2.10), (1.07, -1.86), (1.29, -2.71), (1.84, -2.29), (2.31, 0.39), (3.12, 2.91), (3.46, 1.73), (4.12, -2.11), (4.32, -2.79), (4.84, -2.25) **SAME AS BEFORE**

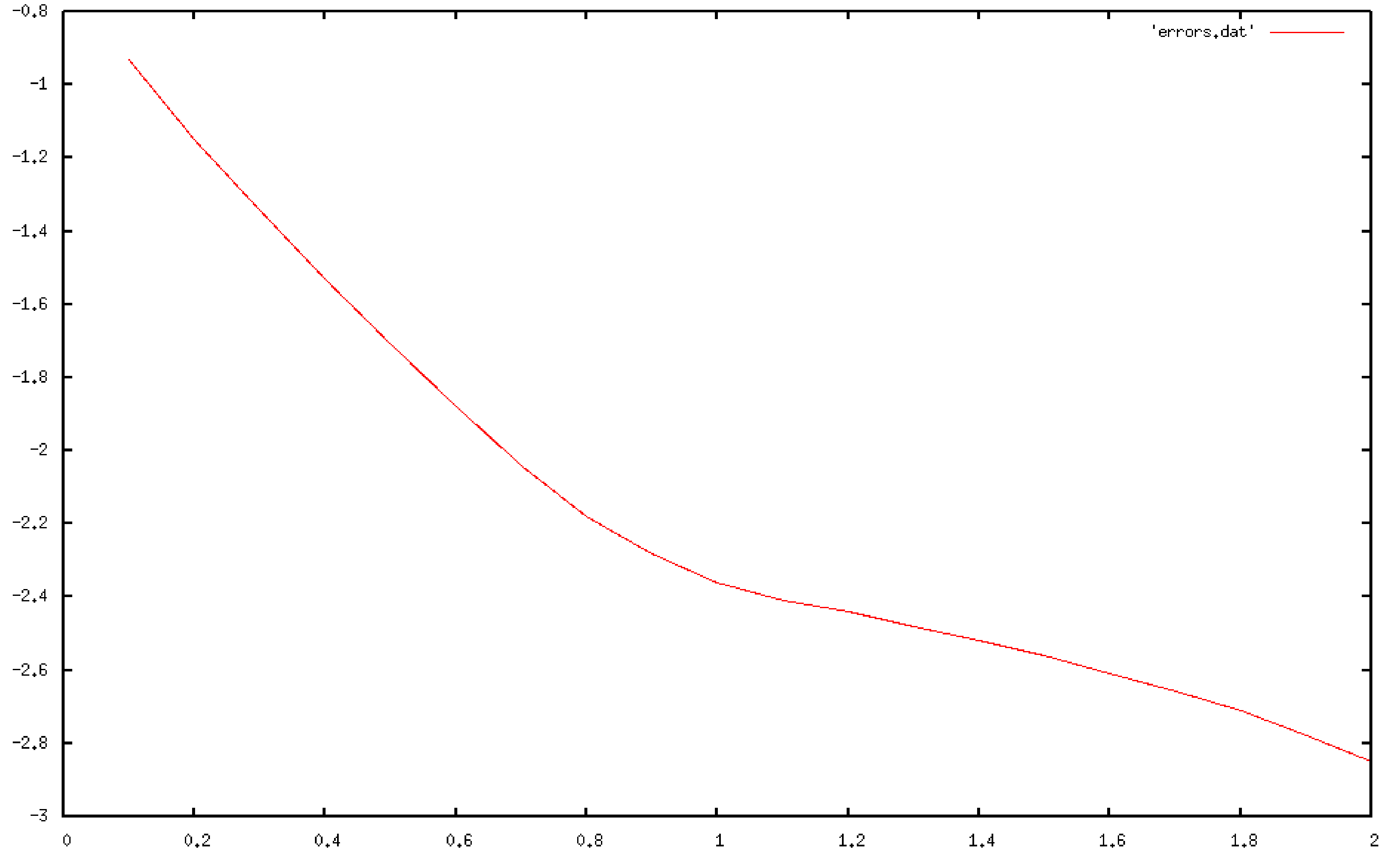


RBF Errors


Log10 [sqrt (mean squared errors)] versus c: Multiquadric
'errors.dat'
-1.4
-1.6
-1.8
-2
-2.2
-2.4
-2.6
-2.8
-3
-3.2
-3.4
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

RBF Errors

Log_{10} [sqrt (mean squared errors)] versus c: Reciprocal Multiquadric



Consistency (Property)

- Consistency is the ability of an interpolating function to reproduce a polynomial of a given order, the simplest consistency is constant consistency (reproduce unity)

$$\mathbf{x}_i \equiv \left|\mathbf{x}_i - \mathbf{x}_j\right|$$

$$\sum_{j=1}^{N} \alpha_j \, g_j(\mathbf{x}_i) = 1, \quad i = 1, N$$

If $g_j(0) = 1$, then a constraint results:

$$\sum_{j=1}^{N} \alpha_j = 1$$

Note: Not all RBFs have $g_j(0) = 1$

RBFs and PDEs

- Solve a boundary value problem: $\nabla^2 \phi(x, y) = 0$

$$\phi(x, y)\big|_{\text{on the boundary}} = f(x, y)$$

- We make use of RBFs as a possible solution

$$\phi_h(\mathbf{x}) = \sum_{j=1,N} \alpha_j \, g_j(\mathbf{x})$$

RBFs and PDEs

- The governing equation and boundary conditions

$$\phi_h(\mathbf{x}) = \sum_{j=1,N} \alpha_j \, g_j(\mathbf{x})$$

$$\sum_{j=1}^{N} \alpha_j \nabla^2 g_j(x_i) = 0 \text{ for all the interior points}$$

$$\sum_{j=1}^{N} \alpha_j g_j(x_i) = f_i \text{ for the boundary points}$$

These are N equations for the N unknown constants, α_j

RBFs and PDEs

- One common problem with many RBFs is that the n * n matrix is dense, one easy-fix is to use a RBF with compact support (matrix becomes sparse)

1D:

$$\begin{cases} (1-r/h)^3(3r/h+1) & \text{for } |r|<h \\ 0, & \text{otherwise} \end{cases}$$

3D:

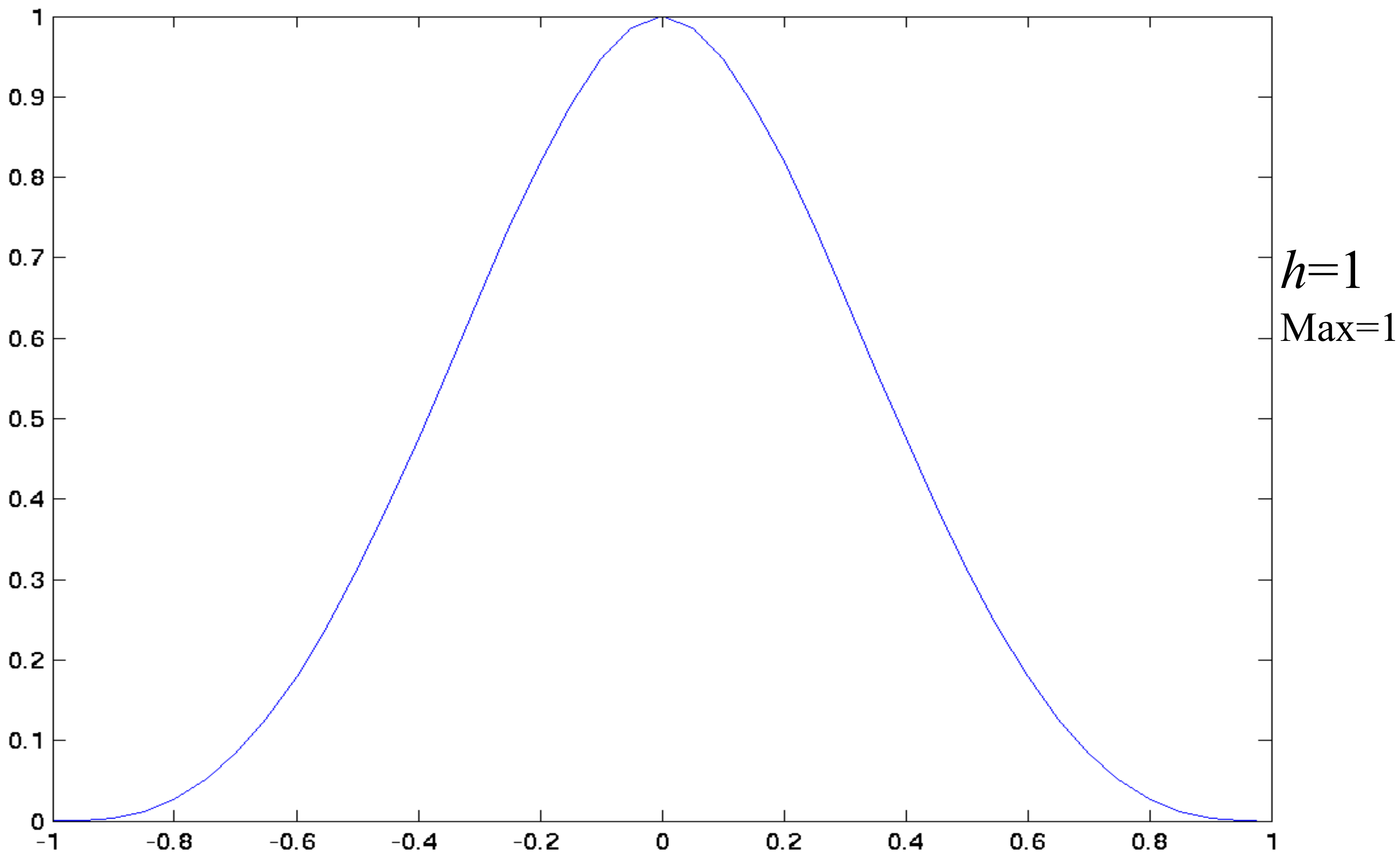
$$\begin{cases} (1-r/h)^4(4r/h+1) & \text{for } |r|<h \\ 0, & \text{otherwise} \end{cases}$$

$$(1-r/h)_+^4(4r/h+1)$$

RBFs with small 'footprints' (Wendland, 2005)

Advantages: matrix is sparse, but still $n * n$

Wendland 1-D RBF with Compact Support



Weighted Least Squares Approximation

- In the weighted least squares formulation, we will have to use a different error functional that now has a weighting function term inside the formulation

$$\min_{f \in P_m^d} \sum_i \theta(\|\bar{\mathbf{x}} - \mathbf{x}_i\|) \|(f(\mathbf{x}_i) - f_i)\|^2$$

Weighting Function Choices

- The weighting function should be locally defined

$$\theta(d) = e^{-\frac{d^2}{h^2}}$$

$$\theta(d) = (1 - d/h)^4 (4d/h + 1)$$

$$\theta(d) = \frac{1}{d^2 + \varepsilon^2}$$

Solution

- Once again, we take partial derivatives of the error functional
- Function minimization: the partial derivatives of the error functional must be set to zero
- We now obtain a linear system of equations

$$\frac{\partial E}{\partial \mathbf{c}(\bar{\mathbf{x}})} = \mathbf{0}$$

$$\sum_i \theta(d_i) 2 b_j(\mathbf{x}_i) \left[\mathbf{b}(\mathbf{x}_i)^T \mathbf{c}(\bar{\mathbf{x}}) - f_i \right] = 0$$

Solution

- The weighting functions participate in the solution
- Note that, this solution is actually locally meaningful, and it is applicable in a small neighborhood

$$\sum_i \left[\theta(d_i)\mathbf{b}(\mathbf{x}_i)\mathbf{b}(\mathbf{x}_i)^T \mathbf{c}(\bar{\mathbf{x}}) - \theta(d_i)\mathbf{b}(\mathbf{x}_i) f_i\right] = \mathbf{0}$$

$$\mathbf{c}(\bar{\mathbf{x}}) = \left[\theta(d_i)\sum_i \mathbf{b}(\mathbf{x}_i)\mathbf{b}(\mathbf{x}_i)^T\right]^{-1} \sum_i \theta(d_i)\mathbf{b}(\mathbf{x}_i) f_i$$

Global Approximation

- The concept of Partition-of-Unity (POU)

$$\varphi_j(x) = \frac{\theta_j(x)}{\sum_1^n \theta_i(x)}$$

$$f(\mathbf{x}) = \sum_j \varphi_j(\mathbf{x}) \mathbf{b}(\mathbf{x})^T \mathbf{c}(\bar{\mathbf{x}})$$

Moving Lease Squares

- Moving Least Squares Approximants

$$f(x) = \sum_i \phi_i(x) f_i = \sum_j b_j(x) c_j(x)$$

$$\min_c \sum_i \theta(\|\mathbf{x} - \mathbf{x}_i\|) \left\| (\mathbf{b}(\mathbf{x}_i)^T \mathbf{c}(\mathbf{x}) - f_i) \right\|^2$$

MLS Basis Functions

$$\phi_i(\mathbf{x}) = \mathbf{b}(\mathbf{x})^T \mathbf{A}(\mathbf{x})^{-1} \mathbf{B}_i(\mathbf{x})$$

$$\mathbf{A}(\mathbf{x}) = \sum_{i=1}^{n} \theta_i(\mathbf{x}) \mathbf{b}(\mathbf{x}_i) \mathbf{b}(\mathbf{x}_i)^T$$

$$\mathbf{B}(\mathbf{x}) = \left[\theta_1(\mathbf{x}) \mathbf{b}(\mathbf{x}_1) \quad \theta_2(\mathbf{x}) \mathbf{b}(\mathbf{x}_2) \quad \ldots\ldots \quad \theta_n(\mathbf{x}) \mathbf{b}(\mathbf{x}_n) \right]$$

Moving Least Squares Interpolant

$$u_p(\mathbf{x}) = \sum_{j}^{N} a_j(\mathbf{x}) p_j(\mathbf{x}) \equiv \mathbf{p}^T(\mathbf{x})\,\mathbf{a}(\mathbf{x})$$

$$p^T(\mathbf{x})$$

are monomials in x for 1D $(1, x, x^2, x^3)$
x,y in 2D, e.g. $(1, x, y, x^2, xy, y^2 \ldots)$

Note a_j are functions of $\boldsymbol{x}$

Moving Least Squares Interpolant

$$E(\mathbf{x}) = \sum_{i=1}^{N} W(\mathbf{x} - \mathbf{x}_i)\left(\mathbf{p}^T(\mathbf{x}_i)\,\mathbf{a}(\mathbf{x}) - u_i\right)^2$$

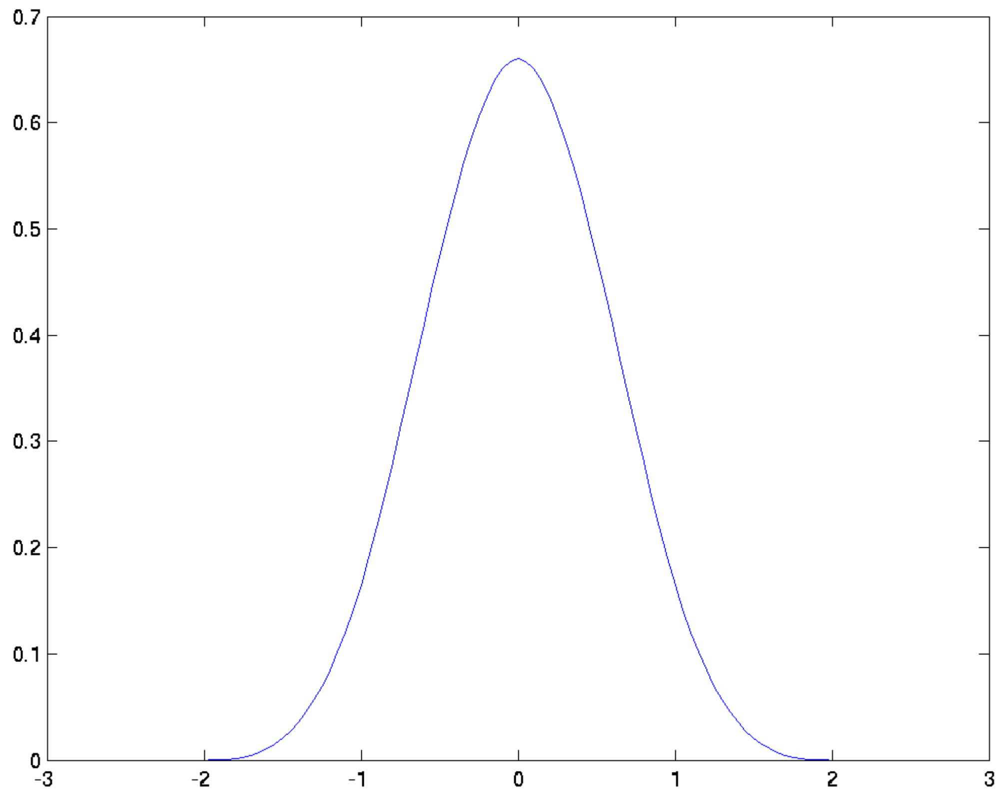
We define a weighted mean-squared error

where $W(\boldsymbol{x}-\boldsymbol{x}_i)$ is a weighting function that decays with increasing $\boldsymbol{x}-\boldsymbol{x}_i$.

Same as previous least squares approach, except for $W(\boldsymbol{x}-\boldsymbol{x}_i)$

Weighting Function

$$W(q) = \frac{2}{3h}\begin{cases} 1 - \frac{3}{2}q^2 + \frac{3}{4}q^3, & \text{for } q \le 1 \\ \frac{1}{4}(2-q)^3, & \text{for } 1 \le q \le 2 \\ 0, & \text{for } q > 2 \end{cases}$$



$q = x/h$

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STATE UNIVERSITY OF NEW YORK

Moving Least Squares Interpolant

Minimizing the weighted squared errors for the coefficients:

$$\frac{\partial E}{\partial \mathbf{a}} = \mathbf{A}(\mathbf{x})\mathbf{a}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\mathbf{u} = 0$$

where $\mathbf{u}^T = (u_1, u_2, ...u_n)$ $\mathbf{A} = \mathbf{P}^T\mathbf{W}(\mathbf{x})\mathbf{P}$, $\mathbf{B} = \mathbf{P}^T\mathbf{W}(\mathbf{x})$

$$\mathbf{P} = \begin{bmatrix} p_1(\mathbf{x}_1) & p_2(\mathbf{x}_1) & \dots & p_m(\mathbf{x}_1) \\ p_1(\mathbf{x}_2) & p_2(\mathbf{x}_2) & \dots & p_m(\mathbf{x}_2) \\ \dots & \dots & \dots & \dots \\ p_1(\mathbf{x}_n) & p_2(\mathbf{x}_n) & \dots & p_m(\mathbf{x}_n) \end{bmatrix}$$

$$\mathbf{W}(\mathbf{x}) = \begin{bmatrix} W(\mathbf{x}-\mathbf{x}_1) & 0 & \dots & 0 \\ 0 & W(\mathbf{x}-\mathbf{x}_2) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & W(\mathbf{x}-\mathbf{x}_n) \end{bmatrix}$$

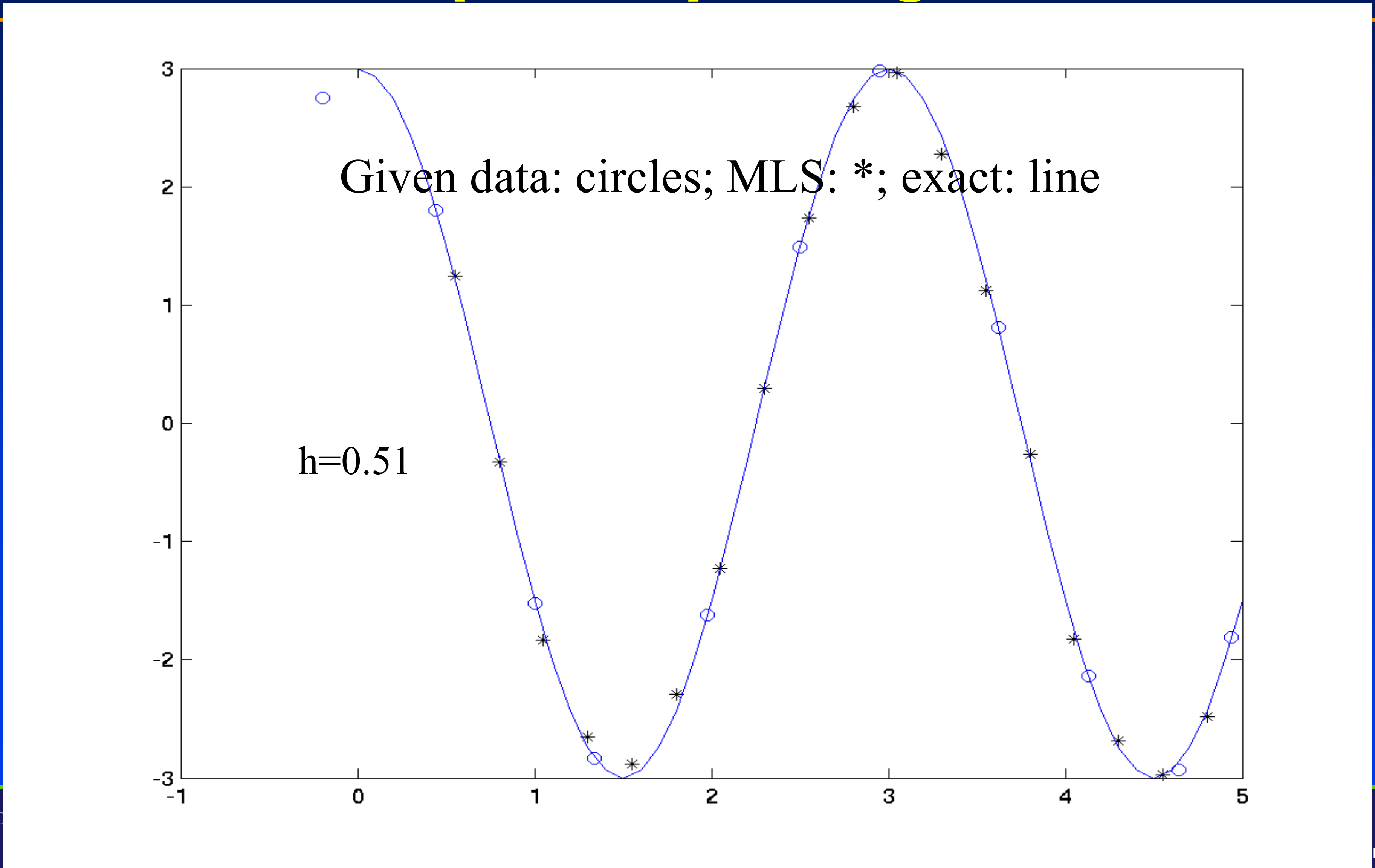
Moving Least Squares Interpolant

$$\mathbf{a}(\mathbf{x}) = \mathbf{A}^{-1}(\mathbf{x})\,\mathbf{B}(\mathbf{x})\,\mathbf{u}$$

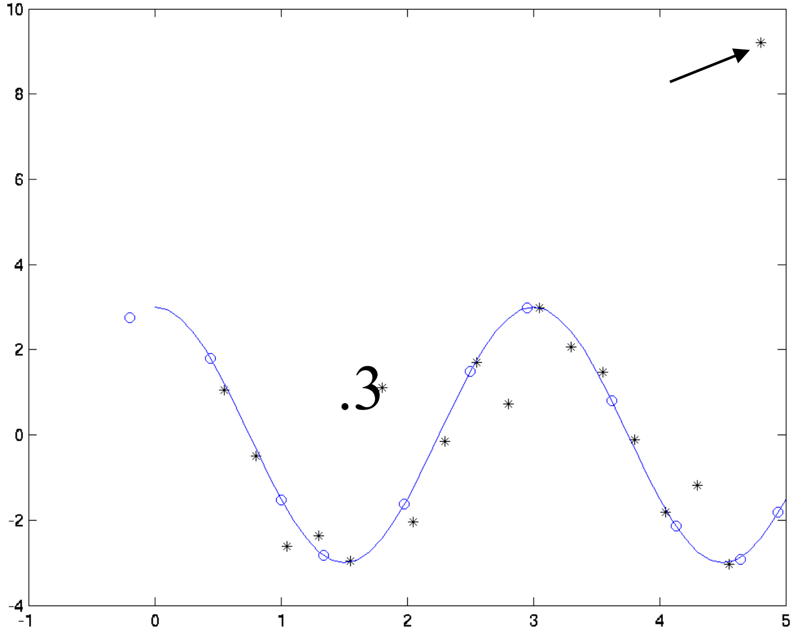
The final locally valid interpolant is:

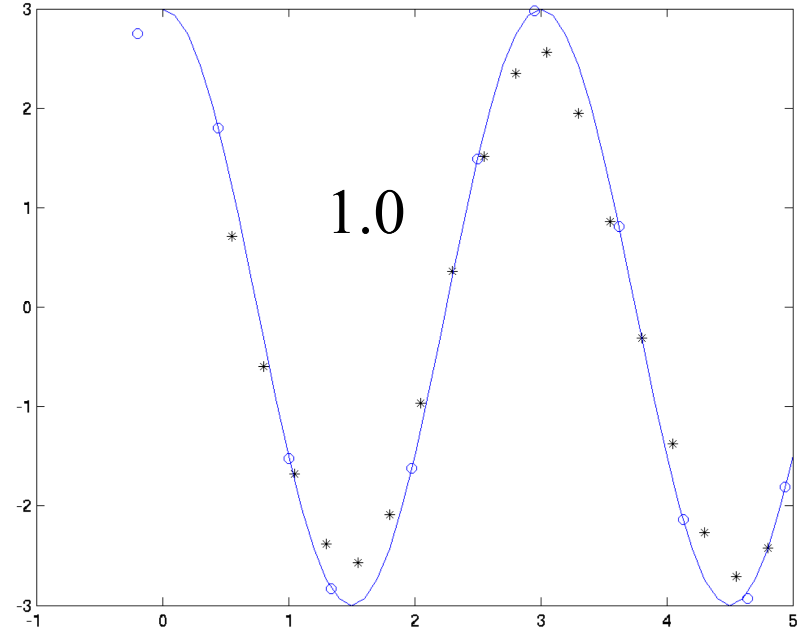
$$u_p(\mathbf{x}) = \sum_{j}^{N} a_j(\mathbf{x}) p_j(\mathbf{x}) \equiv \mathbf{p}^T(\mathbf{x})\,\mathbf{a}(\mathbf{x})$$

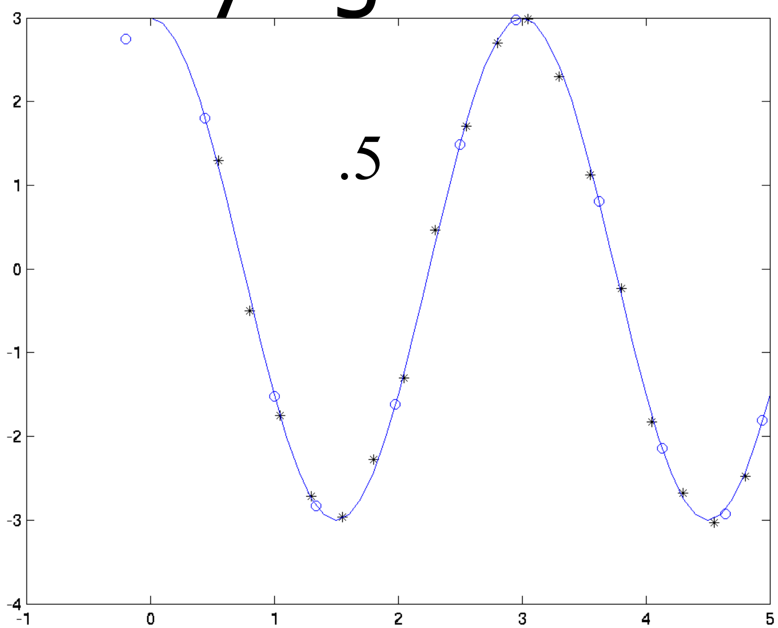
MLS Fit to (Same) Irregular Data

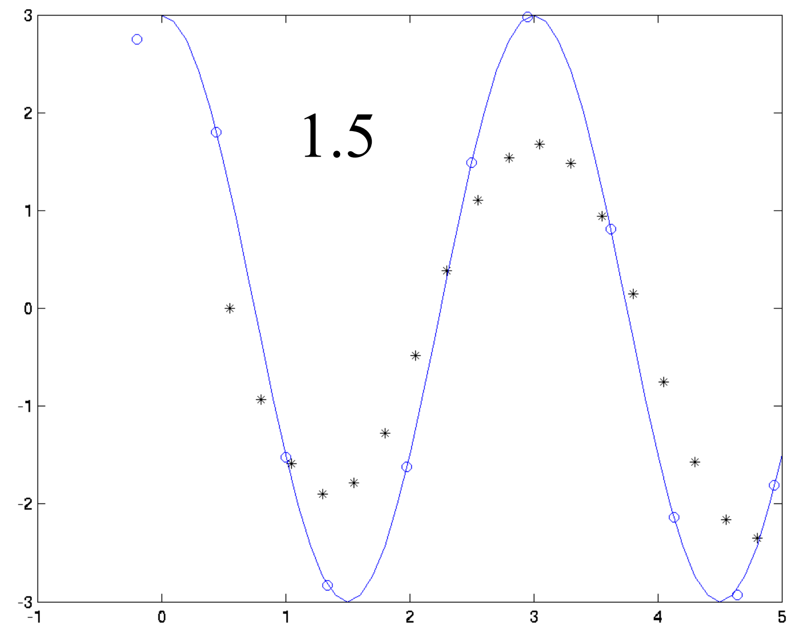


Varying h Values









Weighing Functions

- A cubic spline weight function is a good choice

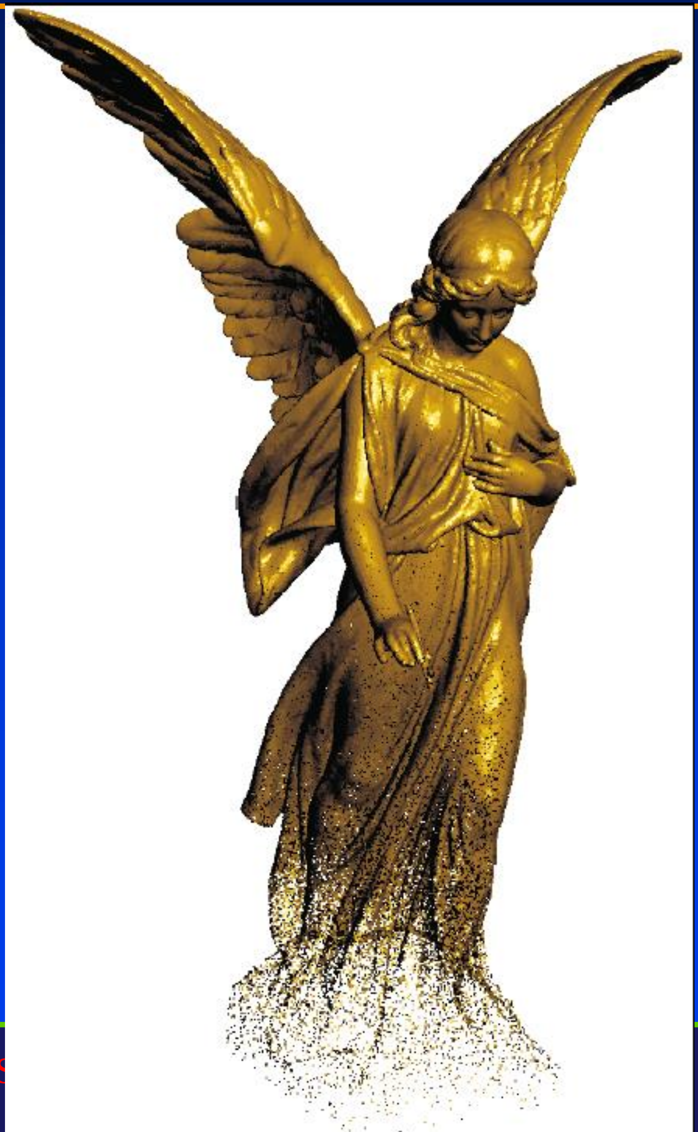
Partition of Unity

- When b is a constant term, MLS basis functions reduce to partition-of-unity basis functions for all the weighting functions

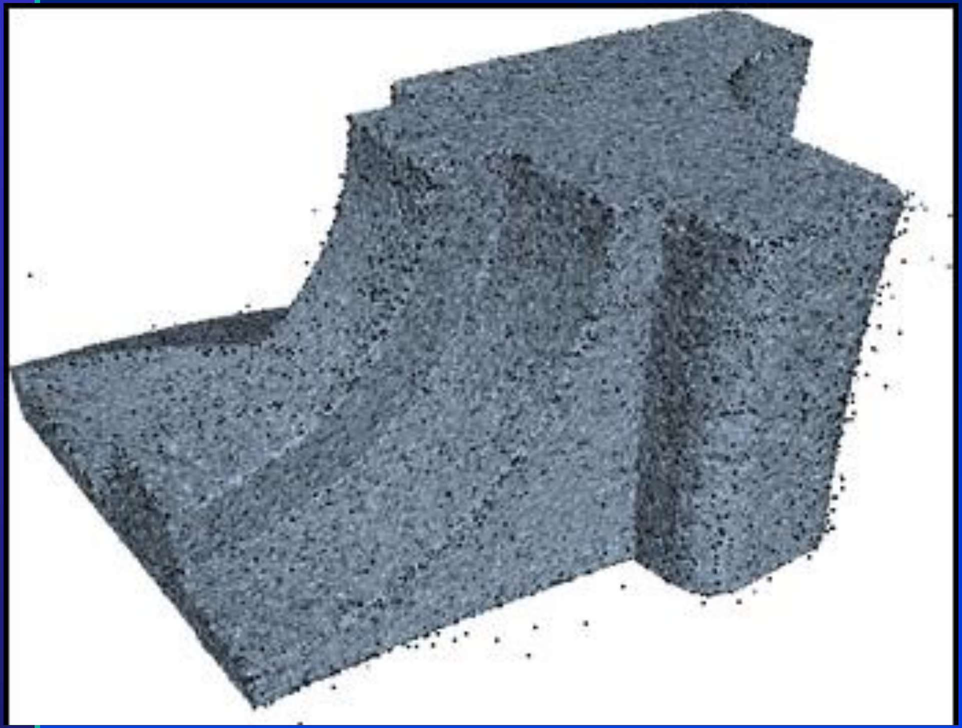
Applications

- A widespread and very powerful tool in Computer Graphics, with many applications
- Surface reconstruction from points
- Interpolating or approximating implicit surfaces
- Simulation
- Animation
- Partition of Unity

Surface Reconstruction



Sharp Feature Modeling



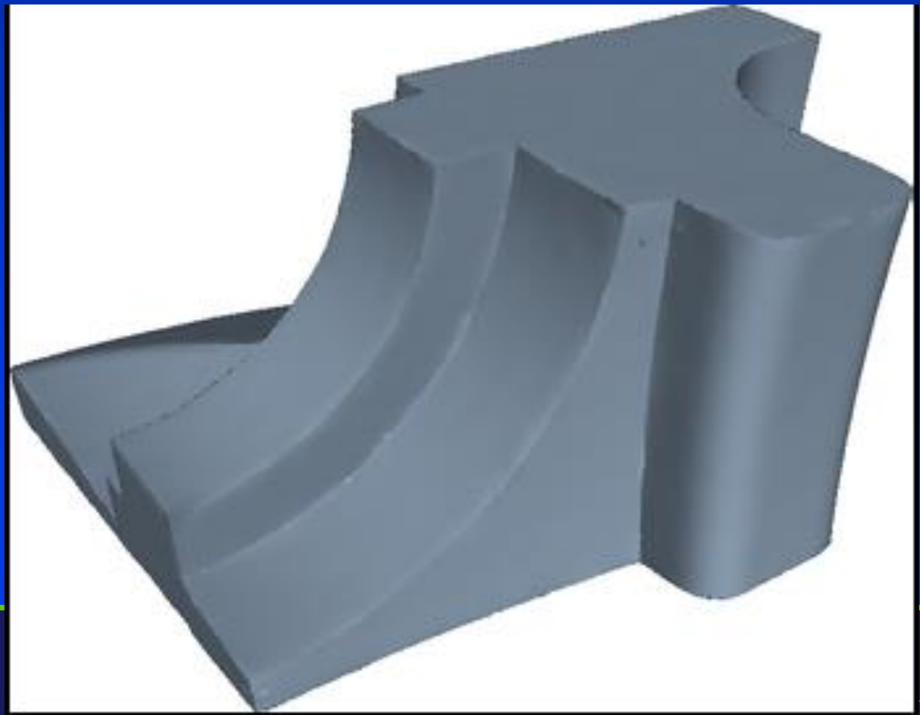
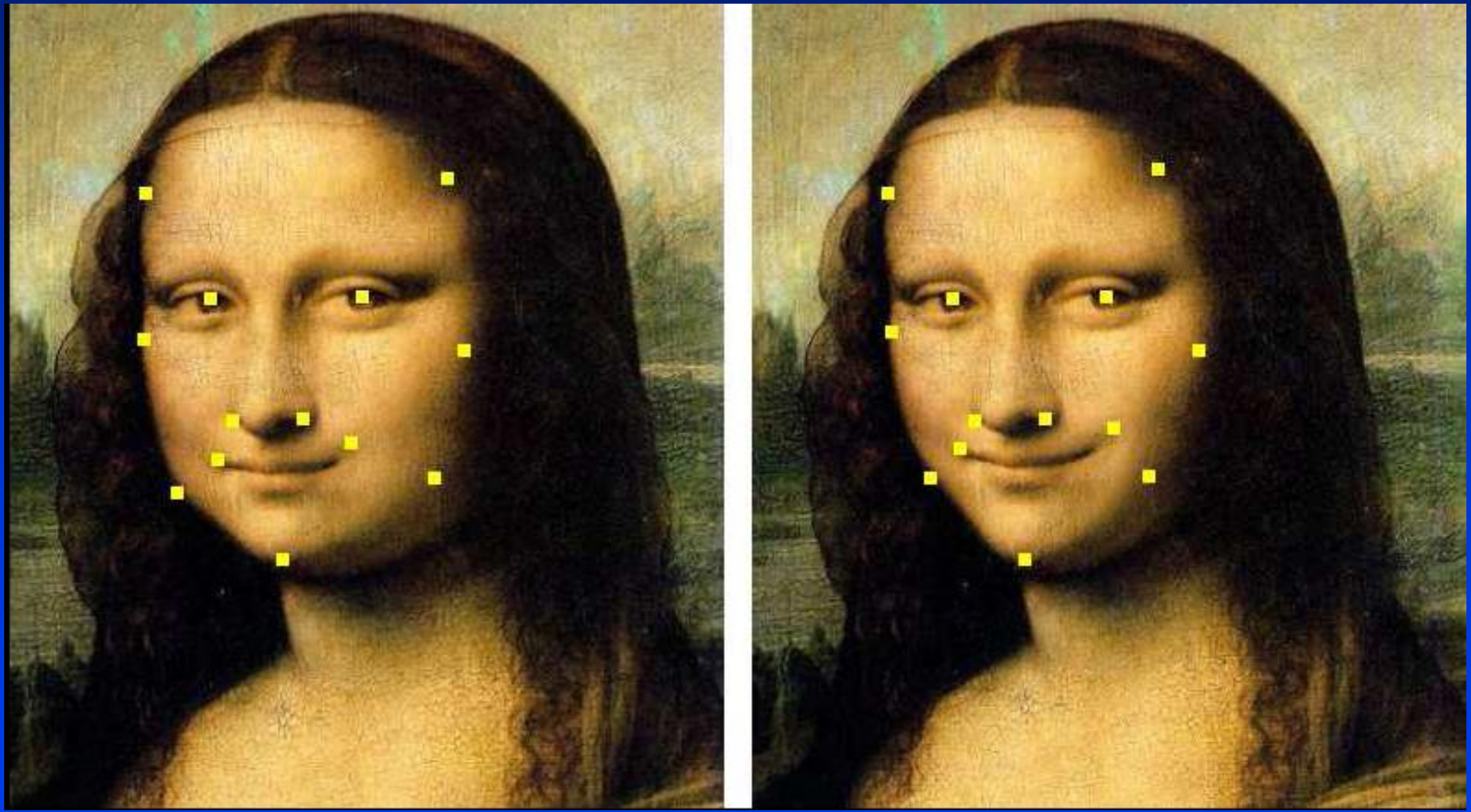


Image Editing



Conclusion

There are a variety of interpolation techniques for irregularly spaced data:

- Polynomial fits
- Best fit polynomials
- Piecewise polynomials
- Radial basis functions
- Moving least squares