Splines and Applications
Functions

- Functions are the basic mathematical tool for describing and analyzing many physical processes of interest.

- Frequently in applications, we do not have the function itself and have to construct an approximation to it based on limited information about the underlying process.

- Such approximation problems are central part of applied Mathematics.
Two Major Categories of Approximation Problems

• Data fitting problems

It is required to construct an approximation to unknown function based on finite amount of data (often measurements) on the function.

For example, consider a weather map where data are collected at a set of weather stations, but a continuous model of temperature, pressure, etc., is desired.
Another Type of Problems

- Operator-equation problems

These simply arise when you have a model for the physical process which involves an equation that either cannot be solved explicitly or cannot be solved at all.
Common Approach to Find Approximation to the Function

• Choose a reasonable *class of functions* in which to look for an approximation

• Devise an appropriate *selection scheme*
Polynomials

- The most convenient class of functions to work with is a class of polynomials:
  - Easy to store on a digital computer
  - Smooth
  - Approximate any continuous function as close as we like (celebrated Weierstrass theorem)
Huge Drawback

- Polynomials are very inflexible

The more points we want to interpolate, the higher degree polynomial we have to use. But high degree polynomials tend to wiggle a lot.
Runge’s Phenomenon

- The red curve is the Runge function, the blue curve is a 5th-order polynomial, while the green curve is a 9th-order polynomial. The approximation only gets worse.
- Consider the function:

\[ f(x) = \frac{1}{1 + 25x^2} \]

- Runge found that if you interpolate this function at equidistant points between -1 and 1 with a polynomial which has a degree smaller or equal with n, the resulting interpolation would oscillate toward the end of the interval, i.e. close to -1 and 1. It can even be proved that the interpolation error tends toward infinity when the degree of the polynomial increases:

\[ \lim_{n \to \infty} \left( \max_{-1 \leq x \leq 1} |f(x) - P_n(x)| \right) = \infty \]

- Runge’s phenomenon demonstrates that lower-order polynomials are generally to be preferred instead of raising the degree of the interpolation polynomial.

red curve - Runge function;
blue curve - 5th order polynomial;
green curve - 9th order polynomial.
The approximation only gets worse.
Idea to Resolve This Problem

• Break the big interval into small ones and consider polynomials (different ones!) on each separate interval.

• So we came to the concept of splines. Simply speaking, **spline is a piecewise polynomial curve** that you all are familiar with.
Draftsman’s Spline

Duck
The Beginning

I. J. Schoenberg

April 21, 1903 – Feb. 21, 1990

The Natural Cubic Splines

1) $s$ is a piecewise cubic polynomial
2) $s \in C^2[a, b]$
3) $s''(a) = s''(b) = 0$

The Great Property

Minimize Energy $\approx \int_a^b \kappa(t)^2 \approx \int_a^b [f''(t)]^2 \, dt$
Spline Theory and Applications

• Approximation theory
• Numerical analysis
• Computer science
• Application areas
  Engineering
  Biosciences, Chemistry, Physics, Geophysics, Meteorology
  Medicine
  Business and Social Sciences
  Imaging and Visualization
  Computer-aided design and Manufacture
  Computer Vision and Robotics
Polynomial Splines

Finite dimensional space

Stable, local basis (B-splines)
Two Main Categories of Splines

- **Interpolating spline** (passes through all of the control points)

- **Approximating spline** (passes near all of the control points)
Parametric Splines with Control Polygon
Cubic Spline Smoothness Theorem

If $S$ is the natural cubic spline function that interpolates a twice-continuously differentiable function $f$ at knots $a = t_0 < t_1 < \ldots < t_n = b$

Then

$$\int_a^b [S''(x)]^2 \, dx \leq \int_a^b [f''(x)]^2 \, dx$$
Tensor-product Splines
Areas

• The study of various classes of approximating functions is precisely the problem of **APPROXIMATION THEORY**

• The design and analysis of effective algorithms utilizing these approximation classes are a major part of **NUMERICAL ANALYSIS**.
What Do the Followings Have in Common?

- What do CSE528 Lectures have in common?

- What do the images have in common?

- What do the diagrams have in common?
Answer

- All these 2D and 3D models are created with the help of splines.
Computer Aided Geometric Design

- Whenever free-form curves and surfaces are represented mathematically, as they are in CAGD, analysis and manufacturing, B-splines (basis splines) are the foundation of an efficient implementations.
• B-splines are especially important in the aircraft and automotive industries, where shape is all important. Designers may not see B-splines—they may manipulate a handle on the end of curve to control its curvature instead—but B-splines are likely to be the hidden bearings on which the design engine runs.
Car Design
Another Application at Boeing

- “Flight test data, like take of distance as a function of temperature, altitude, weight, and many other variables, used to be stored in a succession of look-up tables in a bulky flight manual. Now the 777 pilot enters the flight parameters into a computer, and a spline-based interpolation of that multivariate function immediately returns the take-off distance.

The look-up process has been completely abandoned”.
• Another important application of splines at Boeing is in the computation of optimal orbit and flight trajectories. The continuous variables representing the physics of the vehicle and the control mechanism – flaps and thrusters, for example, - are replaced by spline approximation.
Quadrilateral Grid

Quasi-isometric grid around an airfoil

SiGMA
Date: 12/1/97
Size: 70x40
CAM Design

\[ h(\theta) \]

\[ h'(\theta) \]

\[ C_h \]

<table>
<thead>
<tr>
<th>Function</th>
<th>X start</th>
<th>Y start</th>
<th>Y' start</th>
<th>Y'' start</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Synchron</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>2</td>
<td>Automatic</td>
<td>50.000000</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>3</td>
<td>Automatic</td>
<td>150.000000</td>
<td>45.000000</td>
<td>-0.400000</td>
</tr>
<tr>
<td>4</td>
<td>Synchron</td>
<td>270.000000</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
</tbody>
</table>

Graphs showing functional relationships and data points.
Font Design
TrueType Fonts

- TrueType fonts are outline fonts which means that they can deliver good quality output at any resolution or size:
Applications to Medicine

- “Fly through” models
- Edge detection in ultra sound images
- Modeling of molecule
- Etc.
Molecule Design
Image Segmentation
“Fly through” Models

Goal: 3D reconstruction from 2D slices obtained from CT scan.

- The data starts out as slices (images) taken at regular intervals throughout a portion of the body.
- Then slices are segmented to separate the various tissues.
- Using an algorithm based on bivariate B-spline 3D model is created:

Image interpolation creates a number of new slices between known slices in order to obtain an isotropic volume image.
3D Reconstruction from Slices
CT Skull Fly Through
CT Lung Fly Through
Image Compression
Spline Wavelets

Cubic spline orthogonal scaling function

Fourier transform of the scaling function

Orthogonal cubic spline wavelet

Fourier transform of the wavelet
Another Applications of Image Interpolation

• Examples of situations when reconstruction of 3D object from 2D needed:
  - prosthesis design
  - surgery planning
Practical Problem

• The number of scans must be limited in order to protect patients from the risks related to X-ray absorption.

• 3D reconstruction accuracy depends on which set of image slices are used.

• The main goal is to maximize the density of information minimizing the X-ray absorption.
Iso-Surface Extraction from Volume Data
Image Morphing

The cost between the two images is: 2884.551543
Image Morphing

- Image morphing is the construction of an image sequence depicting a gradual transition between two images.
Hole Filling
Reverse Engineering
Blending & Hole Filling
Smooth Shape (locally / globally)
Bivariate Splines on Triangulations

\[ S^r_d(\triangle) := \{ s \in C^r(\Omega) : s|_T \in \mathbb{P}_d, \text{ for all } T \in \triangle \}, \]
Splines defined on the Sphere and Other Manifolds
Arbitrary Control Meshes
Interactive Multi-Res. Modeling
• Triangular Manifold Splines

Xianfeng David Gu, Ying He, Hong Qin

•

SMI 2005, “Manifold Splines”
GMP 2006, “Manifold T-Splines” (Kexiang Wang, Hongyu Wang)
General Ideas and Pipeline

- **Input**: reasonably dense, manifold triangular mesh
- **Construct a global affine parameterization**
  - Affine transformations between groups of triangles
  - *(Except for a few points)*
- **Construct charts on groups of triangles**
  - Transition functions trivially affine
- **Embedding functions are triangular splines**
  - Splines with triangular mesh as “knot vector”
  - “Fix” holes by patching
Global Parameterization

- Conformal parameterization
- Checkerboard, except for octagon
- Cut by removing center point
- Can now “unfold” locally into plane
Embedding Functions

- **Triangular splines**
  - Built on 2D triangular mesh
  - Affine invariant
Avoid Functions’ Blending

- **Affine invariant embedding functions + affine transition functions** means chart functions agree where they overlap
  - **Use same control points**
Other Types of Spline Functions

- Use T-splines, Powell-Sabin splines
- Can also use T-splines for embedding function
  - Globally parameterize to a square
Advantages

- Simple, affine transformations for transition functions
- $C^k$
- Triangular splines can handle sharp features
Disadvantages

• Need to fix holes in the parameterization
• Triangular splines require optimization
  – Also expensive to compute
• Limited control over parameterization