Radial Basis Functions for Computer Graphics (A Brief Introduction)
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Radial Basis Functions and Their Applications in Graphics
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• Radial Basis Functions (RBFs) are a powerful solution to the Problem of *Scattered Data Fitting*
  – N point samples are given as data inputs, we want to interpolate, extrapolate, and/or approximate

• This problem occurs in many areas:
  – Mesh repair and model completion
  – Surface reconstruction
    • Range scanning, geographic surveys, medical data
  – Field visualization (2D and 3D)
  – Image warping, morphing, registration
  – Artificial intelligence
  – Etc.
2D Radial Basis Functions

- Implicit Curve
- Parametric Height Field
A Very Brief History

- Discovered by Duchon in 1977
- Applications to Computer Graphics:
  - Savchenko, Pasko, Okunev, Kunii – 1995
    - Basic RBF, complicated topology bits
  - Turk & O’Brien – 1999
    - ‘Variational implicit surfaces’
    - Interactive modeling, shape transformation
  - Carr et al
    - 1997 – Medical Imaging
    - 2001 – Fast Reconstruction
3D Radial Basis Functions

- Implicit Surface
- Scalar Field

\[ RBF(x) = 0 \text{ iso-surface} \]
\[ RBF(x) < 0 \]
\[ RBF(x) > 0 \]
Extrapolation (Hole-Filling Capability)

- **Mesh repair and model completion**
  - Fit surface to vertices of mesh
  - RBF will fill holes
    \textit{if it minimizes curvature}!!!
Smoothing

- Smooth out noisy range scan data
- Repair the rough segmentation
Now a bit of math...

(don’t panic)
The Scattered Data Fitting Problem

- We wish to reconstruct a function $S(x)$, given $N$ samples $(x_i, f_i)$, such that $S(x_i) = f_i$
  - $x_i$ are the points from measurement
  - Reconstructed function is denoted $S(x)$
- Infinite number of solutions
- We have specific constraints:
  - $S(x)$ should be continuous over the entire domain
  - We want a ‘smooth’ surface
The Generic Form of RBF Solution

\[ s(x) = \sum_{i=1}^{N} \lambda_i \phi(\|x - x_i\|) + P(x) \]

- \( \lambda_i \) is the weight of center \( x_i \)
- \( \phi(r) \) is the basic function
- \( P(x) \) is a low-degree polynomial component
- \( \|x\| \) is the Euclidean norm
Terminology: Support

- **Support** is the ‘footprint’ of the function.

- **Two types of support matters most:**
  - *Compact* or *Finite* support: function value is zero outside of a certain interval.
  - *Non-Compact* or *Infinite* support: not compact (no interval, goes all the way to \( \infty \)).
Basic Functions \( (\phi) \)

- Essentially, they can be of any functional type
  - However, very difficult to define properties of the RBFs for an arbitrary basic function

- Support of function has major implications
  - A non-compactly supported basic function implies a global solution, dependent on all points!
    - Allows extrapolation (hole-filling)
Standard (Commonly-Used) Basic Functions

- **Polyharmonics (C^n continuity)**
  - 2D: \( \phi(r) = r^{2n} \log(r) \)
  - 3D: \( \phi(r) = r^{2n-1} \)

- **Multiquadric:**
  \( \phi(r) = \sqrt{r^2 + c^2} \)

- **Gaussian:**
  - compact support, used in artificial intelligence field
  \( \phi(r) = e^{-cr^2} \)
Polyharmonics

- **2D Biharmonic:**
  - Thin-Plate Spline
  \[ \phi(r) = r^2 \log(r) \]

- **3D Biharmonic:**
  - \( C^1 \) continuity, Polynomial is degree 1
  - Node Restriction: nodes not colinear
  \[ \phi(r) = r \]

- **3D Triharmonic:**
  - \( C^2 \) continuity, Polynomial is degree 2
  \[ \phi(r) = r^3 \]

- **Important Bit:** Can provide \( C^n \) continuity
Guaranteeing Smoothness

- RBF’s are members of $BL^{(2)}(R^3)$, the Beppo-Levi space of distributions on $R^3$ with square integrable second derivatives

- $BL^{(2)}(R^3)$ has a rotation-invariant semi-norm:

$$\|s\| = \int \left( s_{xx}^2 + s_{yy}^2 + s_{zz}^2 + 2s_{xy}^2 + 2s_{xz}^2 + 2s_{yz}^2 \right) dx$$

- Semi-norm is a measure of energy of $s(x)$
  - Functions with smaller semi-norm are ‘smoother’
  - Smoothest function is the RBF (Duchon proved this)
What about $P(x)$?

- $P(x)$ ensures minimization of the curvature
- 3D Biharmonic: $P(x) = a + bx + cy + dz$
- Must solve for coefficients $a, b, c, d$
  - Adds 4 equations and 4 variables to the linear system
- Additional solution constraints:

\[
\sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} \lambda_i x_i = \sum_{i=1}^{N} \lambda_i y_i = \sum_{i=1}^{N} \lambda_i z_i = 0
\]
Finding an RBF Solution

- The weights and polynomial coefficients are unknowns
- We know $N$ values of $s(x)$:
  
  $$s(x_j) = \sum_{i=0}^{N} \lambda_i \phi\left(\|x_j - x_i\|\right) + P(x_j)$$

  $$f_j = \lambda_1 \phi\left(\|x_j - x_1\|\right) + \ldots + \lambda_N \phi\left(\|x_j - x_N\|\right) + a + bx_j + cy_j + dz_j$$

- We also have 4 side conditions:
  
  $$\sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} \lambda_i x_i = \sum_{i=1}^{N} \lambda_i y_i = \sum_{i=1}^{N} \lambda_i z_i = 0$$
The Linear System $Ax = b$

$$\sum \lambda_i \phi_{ji} + P(x_j) = f_i$$

$$\sum \lambda_i = 0$$

$$\sum \lambda_i x_i = 0$$

$$\sum \lambda_i y_i = 0$$

$$\sum \lambda_i z_i = 0$$

$$\phi_{11} \cdots \phi_{1N} \begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{N1} \cdots \phi_{NN} & 1 & x_N & y_N & z_N \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix}$$

$$1 \cdots 1 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 \cdots x_N$$

$$y_1 \cdots y_N$$

$$z_1 \cdots z_N$$

where $\phi_{ji} = \phi(||x_j - x_i||)$
Properties of the Matrix

• Depends heavily on the basic function

• Polyharmonics:
  – Diagonal elements are zero – not diagonally dominant
  – Matrix is symmetric and positive semi-definite
  – Ill-conditioned if there are near-coincident centers

• Compactly-supported basic functions have a sparse matrix
  – Introduce surface artifacts
  – Can be numerically unstable
Analytic Gradients

- Easy to calculate
- Continuous depending on basic function
- Partial derivatives for biharmonic gradient can be calculated in parallel:

\[
\frac{\partial s}{\partial x} = \sum_{i=1}^{N} \frac{c_i (x - x_i)}{||x - x_i||} + b \\
\frac{\partial s}{\partial y} = \sum_{i=1}^{N} \frac{c_i (y - y_i)}{||x - x_i||} + c \\
\frac{\partial s}{\partial z} = \sum_{i=1}^{N} \frac{c_i (z - z_i)}{||x - x_i||} + d
\]
Fitting 3D RBF Surfaces

(it’s tricky)
Basic Procedure

1. Acquire N surface points
2. Assign them all the value 0
   (This will be the iso-value for the surface)
3. Solve the system, polygonize, and render:
Off-Surface Points

Why did we get a blank screen?
- Matrix was $Ax = 0$
- Trivial solution is $s(x) = 0$
- We need to constrain the system

Solution: Off-Surface Points
- Points inside and outside of surface
  - Project new centers along point normals
  - Assign values: $<0$ inside; $>0$ outside
  - Projection distance has a large effect on smoothness
Invalid Off-Surface Points

• Have to make sure that off-surface points stay inside/outside surface!
  – Nearest-Neighbor test
... Point Normals?

- Easy to get from polygonal meshes
- Difficult to get from anything else

- Can guess normal by fitting a plane to local neighborhood of points
  - Need outward-pointing vector to determine orientation
    - Range scanner position, black pixels
  - For ambiguous cases, don’t generate off-surface point
Computational Complexity

• How long will it take to fit 1,000,000 centers?
  – Forever (more or less)
    • 3.6 TB of memory to hold matrix
    • O(N^3) to solve the matrix
    • O(N) to evaluate a point
  – Infeasible for more than a few thousand centers

• Fast Multipole Methods make it feasible
  – O(N) storage, O(NlogN) fitting and O(1) evaluation
  – Mathematically complex
Center Reduction

- Remove redundant centers
- Greedy algorithm
- Buddha Statue:
  - 543,652 surface points
  - 80,518 centers
  - $5 \times 10^{-4}$ accuracy
FastRBF

- **FarFieldTechnology (com)**

- **Commercial implementation**
  - 3D biharmonic fitter with Fast Multipole Methods
  - Adaptive Polygonizer that generates optimized triangles
  - Grid and Point-Set evaluation

- **Expensive**
  - They have a free demo limited to 30k centers
    - Use iterative reduction to fit surfaces with more points
Applications

(and eye candy)
Cranioplasty (Carr 97)
Molded Cranial Implant
Morphing

- Turk99 (SIGGRAPH)
- 4D Interpolation between two surfaces
Morphing With Influence Shapes
Statue of Liberty

- 3,360,300 data points
- 402,118 centers
- 0.1m accuracy
Credits

• Pictures copied from:
  – Papers by J.C. Carr and Greg Turk
  – FastRBF.com

• References:


Questions?