CSE528 Computer Graphics: Theory, Algorithms, and Applications

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Differential Geometry for Curves and Surfaces (A Very Short Introduction)
Differential Geometry of a Curve

\( p(t) \)
Differential Geometry of a Curve

Point \( p \) on the curve at \( u_0 \)

\[ p = p(u_0) \]
Differential Geometry of a Curve

Tangent $T$ to the curve at $u_0$

$$p(u)$$

$$p_u = \frac{dp(u)}{du}$$

$$T = \frac{p_u}{\|p_u\|}$$
Differential Geometry of a Curve

Normal $N$ and Binormal $B$ to the curve at $u_0$

$$p_{uu} = \frac{d^2 p(u)}{du^2}$$

$$p(u)$$

$$p(u)$$

$$B = \frac{p_u \times p_{uu}}{\|p_u \times p_{uu}\|}$$

$$N = \frac{p_{uu} - (T \cdot p_{uu})T}{\|p_{uu} - (T \cdot p_{uu})T\|} = B \times T = \frac{(p_u \times p_{uu}) \times p_u}{\|p_u \times p_{uu}\| \|p_u\|}$$
Differential Geometry of a Curve

Curvature $\kappa$ at $u_0$ and the radius $\rho$ osculating circle

$$\kappa = \frac{1}{\rho}$$
Differential Geometry of a Curve

\[ p(u) \]

\[ p(u_0) \]
\[ p(u_1) \]

\[ N(u_0) \]
\[ N(u_1) \]

\[ N_T \]
Intrinsic Properties of Curves

- Different representations for the SAME curve

\[ p(t) = (\cos(t), \sin(t)) \]

\[ q(t) = \left( \frac{1-t^2}{1+t^2}, \frac{2t^2}{1+t^2} \right) \]
Intrinsic Properties of Curves

\[ p(t) = (\cos(t), \sin(t)) \]

\[ q(t) = \left( \frac{1 - t^2}{1 + t^2}, \frac{2t^2}{1 + t^2} \right) \]

\[ p(0) = q(0) = (1, 0) \]
Intrinsic Properties of Curves

\[ p(t) = (\cos(t), \sin(t)) \]

\[ q(t) = \left( \frac{1-t^2}{1+t^2}, \frac{2t^2}{1+t^2} \right) \]

\[ p(0) = q(0) = (1,0) \]

\[ p'(0) = (0,1) \neq (0,2) = q'(0) \]

Identical curves but different derivatives!!!
Arc Length: The Basic Concept

\[ s(t) = \int_{a}^{t} \left\| p'(t) \right\| dt \]

- \( s(t) = t \) implies arc-length parameterization
- Independent under any parameterization!!!
Parametric Curves

- **A curve:**
  - A set of points moving (along a curve) with one degree of freedom

- **Torsion:**
  - How much a spatial curve deviates from a plane – how much it attempts to “escape” the osculating plane

- **Arc length:**
  - The real length that is measured along a curve

- **Characterization of all planar curves:**
  - torsion = 0

- **Characterization of all straight lines:**
  - curvature = 0
Frenet Frame

- **Unit-length tangent**

\[
T(t) = \frac{p'(t)}{\|p'(t)\|}
\]

\(T(t)\) and \(p(t)\) are shown in the image.
Frenet Frame

- **Unit-length tangent**

\[ T(t) = \frac{p'(t)}{\|p'(t)\|} \]

- **Unit-length normal**

\[ N(t) = \frac{T''(t)}{\|T''(t)\|} \]
Frenet Frame

- **Unit-length tangent**

- **Unit-length normal**

- **Binormal**

\[
T(t) = \frac{p'(t)}{\|p'(t)\|}
\]

\[
N(t) = \frac{T'(t)}{\|T'(t)\|}
\]

\[
B(t) = T(t) \times N(t)
\]
Frenet Frame

\[ T(t) = \frac{p'(t)}{\|p'(t)\|} \]
\[ N(t) = \frac{T'(t)}{\|T'(t)\|} \]
\[ B(t) = T(t) \times N(t) \]

- Provides an orthogonal frame anywhere on curve

\[ B(t) \cdot T(t) = B(t) \cdot N(t) = T(t) \cdot N(t) = 0 \]
Frenet Frame

\[ T(t) = \frac{p'(t)}{\|p'(t)\|} \quad N(t) = \frac{T'(t)}{\|T'(t)\|} \quad B(t) = T(t) \times N(t) \]

- Provides an orthogonal frame anywhere on curve

\[ B(t) \cdot T(t) = B(t) \cdot N(t) = T(t) \cdot N(t) = 0 \]

Trivial due to the cross-product computation
Frenet Frame

\[ T(t) = \frac{p'(t)}{\|p'(t)\|} \]

\[ N(t) = \frac{T'(t)}{\|T'(t)\|} \]

\[ B(t) = T(t) \times N(t) \]

- Provides an orthogonal frame anywhere on curve

\[ B(t) \cdot T(t) = B(t) \cdot N(t) = T(t) \cdot N(t) = 0 \]

\[ T(t) \cdot T(t) = 1 \]
Frenet Frame

\[ T(t) = \frac{p'(t)}{\|p'(t)\|} \quad N(t) = \frac{T'(t)}{\|T'(t)\|} \quad B(t) = T(t) \times N(t) \]

- Provides an orthogonal frame anywhere on curve

\[ B(t) \cdot T(t) = B(t) \cdot N(t) = T(t) \cdot N(t) = 0 \]
\[ T(t) \cdot T(t) = 1 \]
\[ T'(t) \cdot T(t) + T(t) \cdot T'(t) = 0 \]
Frenet Frame

\[
T(t) = \frac{p'(t)}{\|p'(t)\|} \quad N(t) = \frac{T'(t)}{\|T'(t)\|} \quad B(t) = T(t) \times N(t)
\]

- Provides an orthogonal frame anywhere on curve

\[
B(t) \cdot T(t) = B(t) \cdot N(t) = T(t) \cdot N(t) = 0
\]

\[
T(t) \cdot T(t) = 1
\]

\[
T'(t) \cdot T(t) + T(t) \cdot T'(t) = 0
\]

\[
T(t) \cdot N(t) = 0
\]
Frenet Frames: Applications

- Camera motion animation
- Extruding a cylinder along a path (generalized cylinders)
Frenet Frames

- Problems: The Frenet frame becomes unstable at inflection points or even undefined when

\[
T'(t) = 0
\]

\[
T(t) = \frac{p'(t)}{\|p'(t)\|}
\]

\[
N(t) = \frac{T'(t)}{\|T'(t)\|}
\]

\[
B(t) = T(t) \times N(t)
\]
Osculating Plane

- Plane defined by the point $p(t)$ and the vectors $T(t)$ and $N(t)$
- Locally the curve resides in this plane
Curvature

- Measure how much the curve bends
Curvature

- Measure how much the curve bends

\[ \kappa = \left| \frac{\partial T}{\partial s} \right| \]
Curvature

- Measure of how much the curve bends

\[ \kappa = \left| \frac{\partial T}{\partial s} \right| \]

\[ \frac{\partial T}{\partial t} = \frac{\partial T}{\partial s} \frac{\partial s}{\partial t} \]
Curvature

• Measure how much the curve bends

\[ \kappa(t) = \frac{\|T'(t)\|}{\|p'(t)\|} \]
Curvature

• **Measure how much the curve bends**

\[
\kappa(t) = \frac{\left\| T'(t) \right\|}{\left\| p'(t) \right\|} = \frac{\left\| p'(t) \times p''(t) \right\|}{\left\| p'(t) \right\|^3}
\]
Curvature

• Measure how much the curve bends

\[ \kappa(t) = \frac{\|T'(t)\|}{\|p'(t)\|} = \frac{\|p'(t) \times p''(t)\|}{\|p'(t)\|^3} \]

\[ r = \frac{1}{\kappa(t)} \]
Torsion

- Measure how much the curve twists or how quickly the curve leaves the osculating plane

\[ \tau(s) = \| B'(s) \| \]
Frenet Equations

- $T'(s) = \kappa(s)N(s)$
- $N'(s) = \tau(s)B(s) - \kappa(s)T(s)$
- $B'(s) = -\tau(s)N(s)$
Differential Geometry of Surfaces
Differential Geometry of a Surface

\( s(u, v) \)
Differential Geometry of a Surface

Point \( p \) on the surface at \( p(u_0, v_0) \)
Differential Geometry of a Surface

Tangent $p_u$ in the $u$ direction

$$p_u = \frac{\partial p(u,v)}{\partial u}$$
Tangent $p_v$ in the $v$ direction

$$p_v = \frac{\partial p(u, v)}{\partial v}$$
Differential Geometry of a Surface

Plane of tangents $T$

$$T = up_u + vp_v$$

$p(u,v)$
Differential Geometry of a Surface

Normal $N$

$$N = \frac{s_u \times s_v}{\|s_u \times s_v\|}$$

$p(u,v)$
Differential Geometry of a Surface

Normal section

$p(u, v)$

$p_u$

$p_v$

$T$

$N$
Differential Geometry of a Surface

Curvature

\[ \rho_T = \frac{1}{\kappa_T} \]
Differential Geometry of a Surface

Curvature

\[ \kappa_T = -T \cdot N_T \]

\[ \rho_T = \frac{1}{\kappa_T} \]
Bi-variate Parametric Surfaces

- Consider a curve \( r(t) = (u(t), v(t)) \)
Bi-variate Surfaces

- Consider a curve \( r(t) = (u(t), v(t)) \)
- \( p(r(t)) \) is a curve on the surface
Parametric Surfaces

- **Consider a curve** $r(t) = (u(t), v(t))$
- **$p(r(t))$ is a curve on the surface**

\[
s(t) = \int_{t_0}^{t} \left\| p'(r(t)) \right\| dt
\]
Parametric Surfaces

• Consider a curve \( r(t) = (u(t), v(t)) \)
• \( p(r(t)) \) is a curve on the surface

\[
\begin{align*}
\|p'(r(t))\| &= \left\| \frac{\partial p}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial p}{\partial v} \frac{\partial v}{\partial t} \right\| \\
s(t) &= \int_{t_0}^{t} \|p'(r(t))\| \, dt
\end{align*}
\]
Surfaces

- Consider a curve \( r(t) = (u(t), v(t)) \)
- \( p(r(t)) \) is a curve on the surface

\[
\|p'(r(t))\| = \sqrt{p_u \cdot p_u \left( \frac{\partial u}{\partial t} \right)^2 + 2p_u \cdot p_v \frac{\partial u}{\partial t} \frac{\partial v}{\partial t} + p_v \cdot p_v \left( \frac{\partial v}{\partial t} \right)^2}
\]
Surfaces

- Consider a curve \( r(t) = (u(t), v(t)) \)
- \( p(r(t)) \) is a curve on the surface

\[
s(t) = \int_{t_0}^{t} \left\| p'(r(t)) \right\| dt
\]

\[
\left\| p'(r(t)) \right\| = \sqrt{p_u \cdot p_u \left( \frac{\partial u}{\partial t} \right)^2 + 2p_u \cdot p_v \frac{\partial u}{\partial t} \frac{\partial v}{\partial t} + p_v \cdot p_v \left( \frac{\partial v}{\partial t} \right)^2}
\]

\[
E = p_u \cdot p_u
\]
\[
F = p_u \cdot p_v
\]
\[
G = p_v \cdot p_v
\]

First fundamental form
First Fundamental Form

\[
E = p_u \cdot p_u \quad F = p_u \cdot p_v \quad G = p_v \cdot p_v
\]

• **Given any curve** in parameter space 
  \( r(t) = (u(t), v(t)) \), arc length of curve on surface is

\[
s(t) = \int_{t_0}^{t} \sqrt{E \frac{\partial u}{\partial t}^2 + 2F \frac{\partial u}{\partial t} \frac{\partial v}{\partial t} + G \frac{\partial v}{\partial t}^2} \, dt
\]
First Fundamental Form

\[ E = p_u \cdot p_u \quad F = p_u \cdot p_v \quad G = p_v \cdot p_v \]

- The infinitesimal surface area at \( u, v \) is given by

\[ \| p_u \times p_v \| \]
First Fundamental Form

\[
E = p_u \cdot p_u \quad F = p_u \cdot p_v \quad G = p_v \cdot p_v
\]

- The infinitesimal surface area at \( u, v \) is given by

\[
\|p_u \times p_v\| = \|a\|^2 \|b\|^2 \sin(\theta)^2
\]
First Fundamental Form

\[ E = p_u \cdot p_u \quad F = p_u \cdot p_v \quad G = p_v \cdot p_v \]

- The infinitesimal surface area at \( u, v \) is given by

\[
\| p_u \times p_v \| = \left\| a \times b \right\|^2 = \| a \|^2 \| b \|^2 \left( 1 - \cos(\theta)^2 \right)
\]
First Fundamental Form

\[ E = p_u \cdot p_u \quad F = p_u \cdot p_v \quad G = p_v \cdot p_v \]

- The infinitesimal surface area at \( u, v \) is given by

\[ \| p_u \times p_v \| \]

\[ \| a \times b \|^2 = \| a \|^2 \| b \|^2 - (a \cdot b)^2 \]
First Fundamental Form

\[ E = p_u \cdot p_u \quad F = p_u \cdot p_v \quad G = p_v \cdot p_v \]

- The infinitesimal surface area at \( u, v \) is given by

\[ \| p_u \times p_v \| \]

\[ \| a \times b \| ^2 = \| a \| ^2 \| b \| ^2 - (a \cdot b)^2 \]

\[ \| p_u \times p_v \| = \sqrt{\| p_u \| ^2 \| p_v \| ^2 - (p_u \cdot p_v)^2} \]
First Fundamental Form

\[ E = p_u \cdot p_u \quad F = p_u \cdot p_v \quad G = p_v \cdot p_v \]

- The infinitesimal surface area at \( u, v \) is given by

\[ \|p_u \times p_v\| \]

\[ \|a \times b\|^2 = \|a\|^2 \|b\|^2 - (a \cdot b)^2 \]

\[ \|p_u \times p_v\| = \sqrt{EG - F^2} \]
First Fundamental Form

- Surface area at \( u, v \) is given by

\[
\iint_U \sqrt{EG - F^2}
\]
Second Fundamental Form

- Consider a curve $p(r(s))$ parameterized with respect to arc-length where $r(s) = (u(s), v(s))$
- Curvature is given by $T'(s) = p''(r(s)) = \kappa(s)N(s)$
Second Fundamental Form

- Consider a curve $p(r(s))$ parameterized with respect to arc-length where $r(s) = (u(s), v(s))$
- Curvature is given by $T'(s) = p''(r(s)) = \kappa(s)N(s)$

$$T'(s) = \kappa(s)N(s) = p_{uu} \frac{\partial u}{\partial s}^2 + 2 p_{uv} \frac{\partial u}{\partial s} \frac{\partial v}{\partial s} + p_{vv} \frac{\partial v}{\partial s}^2 + p_v \frac{\partial^2 v}{\partial s^2} + p_u \frac{\partial^2 u}{\partial s^2}$$

$p(r(s))$
Second Fundamental Form

- Consider a curve \( p(r(s)) \) parameterized with respect to arc-length where \( r(s) = (u(s), v(s)) \)
- Curvature is given by \( T'(s) = p''(r(s)) = \kappa(s)M(s) \)
- \( T'(s) = \kappa(s)N(s) = p_{uu} \frac{\partial u}{\partial s}^2 + 2p_{uv} \frac{\partial u}{\partial s} \frac{\partial v}{\partial s} + p_{vv} \frac{\partial v}{\partial s}^2 + p_v \frac{\partial^2 v}{\partial s^2} + p_u \frac{\partial^2 u}{\partial s^2} \)
- Let \( n \) be the normal of \( p(u,v) \)
  \( n \cdot N(s) = \cos(\phi) \)
Second Fundamental Form

- Consider a curve $p(r(s))$ parameterized with respect to arc-length where $r(s) = (u(s), v(s))$
- Curvature is given by $T'(s) = p''(r(s)) = \kappa(s)N(s)$

\[ T'(s) = \kappa(s)N(s) = p_{uu} \frac{\partial u}{\partial s}^2 + 2 p_{uv} \frac{\partial u}{\partial s} \frac{\partial v}{\partial s} + p_{vv} \frac{\partial v}{\partial s}^2 + p_v \frac{\partial^2 v}{\partial s^2} + p_u \frac{\partial^2 u}{\partial s^2} \]

- Let $n$ be the normal of $p(u,v)$

\[ n \cdot N(s) = \cos(\phi) \quad n \cdot T'(s) = \kappa(s) \cos(\phi) \]

\[ \kappa(s) \cos(\phi) = n \cdot p_{uu} \frac{\partial u}{\partial s}^2 + 2n \cdot p_{uv} \frac{\partial u}{\partial s} \frac{\partial v}{\partial s} + n \cdot p_{vv} \frac{\partial v}{\partial s}^2 \]
Second Fundamental Form

- Consider a curve $p(r(s))$ parameterized with respect to arc-length where $r(s) = (u(s), v(s))$

- Curvature is given by
  $$ T'(s) = p''(r(s)) = \kappa(s)N(s) $$

- Let $n$ be the normal of $p(u,v)$
  $$ n \cdot N(s) = \cos(\phi) \quad n \cdot T'(s) = \kappa(s) \cos(\phi) $$
  $$ \kappa(s) \cos(\phi) = n \cdot p_{uu} \frac{\partial u}{\partial s}^2 + 2n \cdot p_{uv} \frac{\partial u}{\partial s} \frac{\partial v}{\partial s} + n \cdot p_{vv} \frac{\partial v}{\partial s}^2 $$

- Let $L = n \cdot p_{uu}$
- Let $M = n \cdot p_{uv}$
- Let $N = n \cdot p_{vv}$
Meusnier’s Theorem

- **Assume**: \( n \cdot N(s) = 1 \), \( \kappa(s) \) is called the *normal curvature*

- **Meusnier’s Theorem** states that all curves on \( p(u,v) \) passing through a point \( x \) having the same tangent, have the same normal curvature
Lines of Curvature

• We can parameterize all tangents through $x$ using a single parameter $\lambda$

$$\kappa(\lambda) = \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2}$$
Principle Curvatures

\[ \kappa_1 = \min \kappa(\lambda) \quad \kappa_2 = \max \kappa(\lambda) \]
Principle Curvatures

\[ \kappa_1 = \min \kappa(\lambda) \quad \kappa_2 = \max \kappa(\lambda) \]

\[ \kappa'(\lambda) = 0 = \frac{(E+2F\lambda+G\lambda^2)(M+N\lambda)-(L+M\lambda+N\lambda^2)(F\lambda+G\lambda)}{(E+2F\lambda+G\lambda^2)^2} \]
Principle Curvatures

\[ \kappa_1 = \min \kappa(\lambda) \quad \kappa_2 = \max \kappa(\lambda) \]

\[ \kappa'(\lambda) = 0 = \frac{(E+2F\lambda+G\lambda^2)(M+N\lambda)-(L+M\lambda+N\lambda^2)(F\lambda+G\lambda)}{(E+2F\lambda+G\lambda^2)^2} \]

\[ EM - FL + (EN - GL)\lambda + (FN - GM)\lambda^2 = 0 \]
Gaussian and Mean Curvature

- **Gaussian Curvature:**
  \[ K = \kappa_1 \kappa_2 = \frac{LN - M^2}{EG - F^2} \]

- **Mean Curvature:**
  \[ H = \kappa_1 + \kappa_2 = \frac{NE - 2MF + LG}{EG - F^2} \]
Gaussian and Mean Curvature

- **Gaussian Curvature:**
  \[ K = \kappa_1 \kappa_2 = \frac{LN - M^2}{EG - F^2} \]

- **Mean Curvature:**
  \[ H = \kappa_1 + \kappa_2 = \frac{NE - 2MF + LG}{EG - F^2} \]

- \( K > 0 \): elliptic
- \( K < 0 \): hyperbolic
- \( \kappa_1 = 0 \lor \kappa_2 = 0 \): parabolic
- \( \kappa_1 = 0 \land \kappa_2 = 0 \): flat
Energy Formulation for Surfaces

\[ \kappa_G = \kappa_1 \kappa_2 \]
\[ \kappa_M = \frac{\kappa_1 + \kappa_2}{2} \]
\[ \kappa_B = \frac{\kappa_1^2 + \kappa_2^2}{4} - 2\kappa_1 \kappa_2 \]
\[ = 4 \left( \frac{\kappa_1^2 + 2\kappa_1 \kappa_2 + \kappa_2^2}{4} \right) - 2\kappa_1 \kappa_2 \]
\[ = 4 \left( \frac{\kappa_1 + \kappa_2}{2} \right)^2 - 2\kappa_1 \kappa_2 \]
\[ = 4\kappa_M^2 - 2\kappa_G \]

\[ E_B = \int_S \kappa_1^2 + \kappa_2^2 \, \partial A \]
\[ = \int_S 4\kappa_M^2 - 2\kappa_G \, \partial A \]
\[ = 4\int_S \kappa_M^2 \, \partial A - 2\int_S \kappa_G \, \partial A \]
\[ = 4\int_S \kappa_M^2 \, \partial A - 2(2\pi \chi(S)) \]
\[ = 4\int_S \kappa_M^2 \, \partial A - 4\pi(2 - 2G) \]
Bending Energy

\[ \kappa_G = \kappa_1 \kappa_2 \]

\[ \kappa_M = \frac{\kappa_1 + \kappa_2}{2} \]

\[ \kappa_B = \kappa_1^2 + \kappa_2^2 \]

\[ = \left( \kappa_1^2 + 2\kappa_1 \kappa_2 + \kappa_2^2 \right) - 2\kappa_1 \kappa_2 \]

\[ = 4 \left( \frac{\kappa_1^2 + 2\kappa_1 \kappa_2 + \kappa_2^2}{4} \right) - 2\kappa_1 \kappa_2 \]

\[ = 4 \left( \frac{\kappa_1 + \kappa_2}{2} \right)^2 - 2\kappa_1 \kappa_2 \]

\[ = 4\kappa_M^2 - 2\kappa_G \]

\[ E_B = \int_S \kappa_1^2 + \kappa_2^2 \, \partial A \]

\[ = \int_S 4\kappa_M^2 - 2\kappa_G \, \partial A \]

\[ = 4 \int_S \kappa_M^2 \, \partial A - 2 \int_S \kappa_G \, \partial A \]

\[ = 4 \int_S \kappa_M^2 \, \partial A - 2(2\pi \chi(S)) \]

\[ = 4 \int_S \kappa_M^2 \, \partial A - 4\pi(2 - 2G) \]

Minimizing \[ \int_S \kappa_1^2 + \kappa_2^2 \, \partial A \] = Minimizing \[ \int_S \kappa_M^2 \, \partial A \]