Stable Models Semantics and Answer Set Programming

CSE 505 – Computing with Logic
Stony Brook University

http://www.cs.stonybrook.edu/~cse505
General Logic Programs

• A general program is a collection of rules of the form
  \[ a \leftarrow a_1, \ldots, a_n, \text{not } a_{n+1}, \ldots, \text{not } a_{n+k}. \]

• Let \( \Pi \) be a program and \( X \) be a set of atoms, by \( \Pi^X \)
  (Gelfond-Lifschitz transformation) we denote the positive
program obtained from \( \text{ground}(\Pi) \) by:
  • Deleting from \( \text{ground}(\Pi) \) any rule for that
    \( \{a_{n+1}, \ldots, a_{n+k}\} \cap X \neq \emptyset \), i.e., the body of the
    rule contains a naf-atom \text{not } a_1 \text{ and } a_1 \text{ belongs to } X; \text{ and}

  • Removing all of the naf-atoms from the remaining
    rules.

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General Logic Programs

- A set of atoms $X$ is called an *answer set* of a program $\Pi$ if $X$ is the minimal model of the program $\Pi^X$.

- Theorem: For every positive program $\Pi$, the minimal model of $\Pi$, $M_\Pi$, is also the unique answer set of $\Pi$.

- Example: Consider $\Pi_2 = \{a \leftarrow \text{not } b. \ b \leftarrow \text{not } a.\}$. We will show that it has two answer sets $\{a\}$ and $\{b\}$.

<table>
<thead>
<tr>
<th>$S_1 = \emptyset$</th>
<th>$S_2 = {a}$</th>
<th>$S_3 = {b}$</th>
<th>$S_4 = {a, b}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Pi_{S_1}^{S_2}$:</td>
<td>$\Pi_{S_2}^{S_2}$:</td>
<td>$\Pi_{S_3}^{S_3}$:</td>
<td>$\Pi_{S_4}^{S_4}$:</td>
</tr>
<tr>
<td>$a \leftarrow$</td>
<td>$a \leftarrow$</td>
<td>$b \leftarrow$</td>
<td></td>
</tr>
<tr>
<td>$b \leftarrow$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$M_{\Pi_{S_1}^{S_2}} = {a, b}$</td>
<td>$M_{\Pi_{S_2}^{S_2}} = {a}$</td>
<td>$M_{\Pi_{S_3}^{S_3}} = {b}$</td>
<td>$M_{\Pi_{S_4}^{S_4}} = \emptyset$</td>
</tr>
<tr>
<td>$M_{\Pi_{S_2}^{S_2}} \neq S_1$</td>
<td>$M_{\Pi_{S_2}^{S_2}} = S_2$</td>
<td>$M_{\Pi_{S_3}^{S_3}} = S_3$</td>
<td>$M_{\Pi_{S_4}^{S_4}} \neq S_4$</td>
</tr>
<tr>
<td>NO</td>
<td>YES</td>
<td>YES</td>
<td>NO</td>
</tr>
</tbody>
</table>
General Logic Programs

- $\Pi_5 = \{ p \leftarrow \text{not } p. \}$ does not have an answer set.
  - $S_1 = \emptyset$, then $\Pi^{S_1} = \{ p \leftarrow \}$ whose minimal model is $\{ p \}$. $\{ p \} \neq \emptyset$ implies that $S_1$ is not an answer set of $\Pi_5$.
  - $S_2 = \{ p \}$, then $\Pi^{S_2} = \emptyset$ whose minimal model is $\emptyset$. $\{ p \} \neq \emptyset$ implies that $S_2$ is not an answer set of $\Pi_5$.
  - This shows that this program does not have an answer set.

- A program may have zero, one, or more than one answer sets.
  - $\Pi_1 = \{ a \leftarrow \text{not } b. \}$ has a unique answer set $\{ a \}$.
  - $\Pi_2 = \{ a \leftarrow \text{not } b. b \leftarrow \text{not } a. \}$ has two answer sets: $\{ a \}$ and $\{ b \}$.
  - $\Pi_3 = \{ p \leftarrow a. a\leftarrow \text{not } b. b \leftarrow \text{not } a. \}$ has two answer sets: $\{ a, p \}$ and $\{ b \}$
  - $\Pi_4 = \{ a \leftarrow \text{not } b. b \leftarrow \text{not } c. d \leftarrow . \}$ has one answer set $\{ d, b \}$.
  - $\Pi_5 = \{ p \leftarrow \text{not } p. \}$ No answer set.
  - $\Pi_6 = \{ p \leftarrow \text{not } p, d. r \leftarrow \text{not } d. d \leftarrow \text{not } r. \}$ has one answer set $\{ r \}$.
Entailment w.r.t. Answer Set Semantics

- For a program $\Pi$ and an atom $a$, $\Pi$ entails $a$, denoted by $\Pi \models a$, if $a \in S$ for every answer set $S$ of $\Pi$.

- For a program $\Pi$ and an atom $a$, $\Pi$ entails $\neg a$, denoted by $\Pi \models \neg a$, if $a \notin S$ for every answer set $S$ of $\Pi$.

- If neither $\Pi \models a$ nor $\Pi \models \neg a$, then we say that $a$ is unknown with respect to $\Pi$.

Examples:

- $\Pi_1 = \{a \leftarrow \neg b.\}$ has a unique answer set $\{a\}$. $\Pi_1 \models a$, $\Pi_1 \models \neg b$.

- $\Pi_2 = \{a \leftarrow \neg b. b \leftarrow \neg a\}$ has two answer sets: $\{a\}$ and $\{b\}$. Both $a$ and $b$ are unknown w.r.t. $\Pi_2$.

- $\Pi_3 = \{p \leftarrow a. a \leftarrow \neg b. b \leftarrow \neg a.\}$ has two answer sets: $\{a, p\}$ and $\{b\}$. Everything is unknown.

- $\Pi_4 = \{p \leftarrow \neg p.\}$ No answer set. $p$ is unknown.
For a set of ground atoms $S$ and a **constraint** $c$

$$c \leftarrow a_1, \ldots, a_n, \text{not } a_{n+1}, \text{not } a_{n+k}$$

we say that $c$ is satisfied by $S$ if $\{a_1, \ldots, a_n\} \setminus S \neq \emptyset$ or $\{a_{n+1}, \ldots, a_{n+k}\} \cap S \neq \emptyset$.

Let $\Pi$ be a program with constraints.

Let $\Pi_O = \{r \mid r \in \Pi, \text{r has non-empty head}\}$ ($\Pi_O$ is the set of normal logic program rules in $\Pi$)

Let $\Pi_C = \Pi \setminus \Pi_O$ ($\Pi_C$ is the set of constraints in $\Pi$)

A set of atoms $S$ is an answer sets of a program $\Pi$ if it is an answer set of $\Pi_O$ and satisfies all the constraints in ground ($\Pi_C$).
Answer Sets of Programs with Constraints

- Example:
  - \( \Pi_1 = \{ a \leftarrow \neg b. \ b \leftarrow \neg a. \} \) has two answer sets \( \{a\} \) and \( \{b\} \).
  - But, \( \Pi_2 = \{ \begin{align*}
a & \leftarrow \neg b. \\
b & \leftarrow \neg a. \\
\leftarrow & \neg a \\
\leftarrow & \neg a
\end{align*} \} \)
    has only one answer set \( \{a\} \).
  - But, \( \Pi_3 = \{ \begin{align*}
a & \leftarrow \neg b. \\
b & \leftarrow \neg a. \\
\leftarrow & a
\end{align*} \} \)
    has only one answer set \( \{b\} \).
Computing Answer Sets

- Complexity: The problem of determining the existence of an answer set for finite propositional programs (programs without function symbols) is NP-complete.
- For programs with disjunctions, function symbols, etc. it is much higher.
- A consequence of this property is that there exists no polynomial-time algorithm for computing answer sets.
Answer set solvers

- Programs that compute answer sets of (finite and grounded) logic programs.
- Two main approaches:
  - Direct implementation: Due to the complexity of the problem, most solvers implement a variation of the generate-and-test algorithm.
    - DLV [http://www.dbai.tuwien.ac.at/proj/dlv/](http://www.dbai.tuwien.ac.at/proj/dlv/)
    - deres [http://www.cs.engr.uky.edu/ai/deres.html](http://www.cs.engr.uky.edu/ai/deres.html)
  - Using SAT solvers: A program $\Pi$ is translated into a satisfiability problem $F\Pi$ and a call to a SAT solver is made to compute solution of $F\Pi$. The main task of this approach is to write the program for the conversion from $\Pi$ to $F\Pi$.
Example: Graph Coloring

- Given a (bi-directed) graph and three colors red, green, and yellow. Find a color assignment for the nodes of the graph such that no edge of the graph connects two nodes of the same color.

- Graph representation:
  - The nodes: \texttt{node(1). … node(n)}.
  - The edges: \texttt{edge(i, j)}.

- Each node is assigned one color:
  - the weighted rule
    \[
    \text{l\{color(X, red), color(X, yellow), color(X, green)\}} \leftarrow \text{node(X)}. \]
  - or the three rules:
    \[
    \text{color(X, red)} \leftarrow \text{node(X), not color(X, green), not color(X, yellow)}. \\
    \text{color(X, green)} \leftarrow \text{node(X), not color(X, red), not color(X, yellow)}. \\
    \text{color(X, yellow)} \leftarrow \text{node(X), not color(X, green), not color(X, red)}. \\
    \]

- No edge connects two nodes of the same color:
  \[
  \leftarrow \text{edge(X, Y ), color(X, C), color(Y, C)}. \]
Example: Graph Coloring

%%% representing the graph
node(1). node(2). node(3). node(4). node(5).
edge(1,2). edge(1,3). edge(2,4). edge(2,5). edge(3,4).
edge(3,5).
%%% each node is assigned a color
color(X,red):- node(X), not color(X,green), not color(X, yellow).
color(X,green):- node(X), not color(X,red), not color(X, yellow).
color(X,yellow):- node(X), not color(X,green), not color(X, red).
%%% constraint checking
:- edge(X,Y), color(X,C), color(Y,C).

• Try with
  clingo -n 0 color.lp
and see the result.
Example: n-queens

- Place n queens on a $n \times n$ chess board so that no queen is attacked by another one.
- the chess board can be represented by a set of cells $\text{cell}(i, j)$ and the size $n$.
- Since two queens can not be on the same column, we know that each column has to have one and only one queen
  $$\{\text{cell}(I, J) : \text{row}(J)\} \leftarrow \text{col}(I).$$
- No two queens on the same row
  $$\leftarrow \text{cell}(I, J1), \text{cell}(I, J2), J1 \neq J2.$$
- No two queens on the same column (not really needed)
  $$\leftarrow \text{cell}(I1, J), \text{cell}(I2, J), I1 \neq I2.$$
- No two queens on the same diagonal
  $$\leftarrow \text{cell}(I1, J1), \text{cell}(I2, J2),$$
  $$|I1 - I2| = |J1 - J2|, I1 \neq I2.$$
Example: n-queens

%% representing the board, using n as a constant
col(1..n). % n column
row(1..n). % n row
%% generating solutions
1 {cell(I,J) : row(J)}:- col(I).
% two queens cannot be on the same row/column :- col(I), row(J1), row(J2), neq(J1,J2), cell(I,J1), cell(I,J2).
:- row(J), col(I1), col(I2), neq(I1,I2), cell(I1,J), cell(I2,J).
% two queens cannot be on a diagonal :- row(J1), row(J2), J1 > J2, col(I1), col(I2), I1 > I2, cell(I1,J1), cell(I2,J2), eq(I1 - I2, J1 - J2).
:- row(J1), row(J2), J1 > J2, col(I1), col(I2), I1 < I2, cell(I1,J1), cell(I2,J2), eq(I2 - I1, J1 - J2).