Predicate Logic

CSE 505 – Computing with Logic

Stony Brook University

http://www.cs.stonybrook.edu/~cse505

The *alphabet* of predicate logic

- Variables
- Constants (identifiers, numbers etc.)
- Functors (identifiers with arity **`>0`**; e.g. **`date/3,tree/3`**)
- Predicate symbols (identifiers with arity **`>=0`**; e.g. **`append/3`**)
- Connectives:
  - $\wedge$ (conjunction)
  - $\vee$ (disjunction)
  - $\neg$ (negation)
  - $\leftrightarrow$ (logical equivalence)
  - $\rightarrow$ (implication)
- Quantifiers: $\forall$ (universal), $\exists$ (existential)
- Auxiliary symbols such as parentheses and comma

Predicate Logic Formulas

- ***Terms*** (T) over an alphabet A is the smallest set such that:
  - Every constant **c** ∈ A is also **c** ∈ T
  - Every variable **X** ∈ A is also **X** ∈ T
  - If **f/n** ∈ A and t_1, t_2, …, t_n ∈ T then $f(t_1, t_2, \ldots, t_n)$ ∈ T
- ***Well-formed formulas*** (*wffs*, denoted by ***F***) over alphabet A is the smallest set such that:
  - If **p/n** is a predicate symbol in **A** and t_1, t_2,…, t_n ∈ T then $p(t_1, t_2, \ldots, t_n)$ ∈ ***F*** (called ***atomic formula***)

Note: variable-free atomic formulas are called *ground atomic formulas*

  - If F,G ∈ ***F*** then so are (¬F), (F ∧ G), (F ∨ G), (F → G) and (F ↔ G)
  - If F ∈ ***F*** and **X** is a variable in A then (∀**X** F) and (∃**X** F) ∈ ***F***

Bound and Free Variables

- A variable **X** is ***bound*** in formula F if (∀**X** G) or (∃**X** G) is a sub-formula of F
- A variable that occurs in F, but is not bound in F is said to be ***free*** in F
- A formula F is ***closed*** if it has no free variables
- Let $\mathbf{X_1}, \mathbf{X_2}, \ldots, \mathbf{X_n}$ be all the free variables in F. Then:
  - $(\forall \mathbf{X_1} (\ldots (\forall \mathbf{X_n}\ F) \ldots))$ is the *universal closure* of F, and is denoted by ∀F
  - $(\exists \mathbf{X_1} (\ldots (\exists \mathbf{X_n}\ F) \ldots))$ is the *existential closure* of F, and is denoted by ∃F

Interpretation

- An ***interpretation*** I of an alphabet is:
  - a non-empty domain D, and
  - a mapping that associates:
    - each constant **`c`** ∈ A with an element $\mathbf{c}_\mathbf{I}$ ∈ D
    - each n-ary functor **`f`** ∈ A with an function $\mathbf{f}_\mathbf{I} : D^n \rightarrow D$
    - each n-ary predicate symbol **`p`** ∈ A with an relation $\mathbf{p}_\mathbf{I} \subseteq D^n$
- For instance, one interpretation of the symbols in our "relations" program is that **`'bob'`**, **`'pam'`** et. al. are **people** in some set, and **`parent/2`** is the parent-of relation, etc.
- Another interpretation could be that **`'bob'`**, **`'pam'`**, etc. are natural numbers, **`parent/2`** is the greater-than relation, etc.

Valuation

- Given an interpretation I, the semantics of a variable-free (a.k.a. ***ground***) term is clear from I itself:

$$\mathtt{I(f(t_1, t_2, \ldots, t_n)) = f_I(I(t_1), I(t_2), \ldots, I(t_n))}$$

- But to attach a meaning to terms with variables, we must first **<u>give a meaning to its variables</u>**!
  - This is done by a ***valuation***: which is a **<u>mapping</u>** from variables to the domain D of an interpretation:

$$\varphi = \{\mathbf{X_1} \to \mathbf{d_1}, \mathbf{X_2} \to \mathbf{d_2}, \ \ldots, \ \mathbf{X_n} \to \mathbf{d_n}\}$$

  - $\varphi[\mathbf{X} \to \mathbf{d}]$ is identical to φ except that it maps **X** to **d**

Semantics of terms

- Terms are given a meaning with respect to a **valuation:**
  - Given an interpretation I and valuation φ, the ***meaning*** of a term **t**, denoted by $\varphi_{\mathtt{I}}(\mathtt{t})$ is defined as:
    - if **t** is a constant **c** then $\varphi_{\mathtt{I}}(\mathtt{t}) = \mathtt{c}_{\mathtt{I}}$
    - if **t** is a variable **X** then $\varphi_{\mathtt{I}}(\mathtt{t}) = \varphi\mathtt{X}$
    - if **t** is a structure $\mathtt{f(t_1,\ t_2,...,\ t_n)}$ then

$$\varphi_{\mathtt{I}}(\mathtt{t}) = \mathtt{f}_{\mathtt{I}}(\varphi_{\mathtt{I}}(\mathtt{t}_1), \varphi_{\mathtt{I}}(\mathtt{t}_2), \ldots, \varphi_{\mathtt{I}}(\mathtt{t}_n))$$

Example

- Let **A** be an alphabet containing constant **zero**, a unary functor **s/1** and a binary functor **plus/2**
- **I**, defined as follows, is an interpretation with **N** (the set of natural numbers) as its domain, such as:
  - $\mathtt{zero}_{\mathtt{I}} = 0$
  - $\mathtt{s}_{\mathtt{I}}(\mathtt{x}) = 1 + \mathtt{x}_{\mathtt{I}}$
  - $\mathtt{plus}_{\mathtt{I}}(\mathtt{x},\ \mathtt{y}) = \mathtt{x}_{\mathtt{I}} + \mathtt{y}_{\mathtt{I}}$
- Now, if $\varphi = \{\mathtt{X} \rightarrow \mathtt{1}\}$, then

$$\varphi_{\mathtt{I}}(\mathtt{plus(s(zero),X)}) = \varphi_{\mathtt{I}}(\mathtt{s(zero)}) + \varphi_{\mathtt{I}}(\mathtt{X})$$

$$= (\mathtt{1} + \varphi_{\mathtt{I}}(\mathtt{zero})) + \varphi(\mathtt{X})$$

$$= (\mathtt{1} + \mathtt{0}) + \mathtt{1} = \mathtt{2}$$

Semantics of Well-Formed Formulae

- A formula's meaning is given w.r.t. an interpretation I and valuation φ
  - $I \vDash \varphi\ p(t_1, t_2, \ldots, t_n)$ iff $(\varphi_I(t_1), \varphi_I(t_2), \ldots, \varphi_I(t_n)) \in p_I$
  - $I \vDash \varphi\ \neg F$ iff $I \nvDash \varphi\ F$
  - $I \vDash \varphi\ F \land G$ iff $I \vDash \varphi\ F$ and $I \vDash \varphi\ G$
  - $I \vDash \varphi\ F \lor G$ iff $I \vDash \varphi\ F$ or $I \vDash \varphi\ G$ (or both)
  - $I \vDash \varphi\ F \rightarrow G$ iff $I \vDash \varphi\ G$ whenever $I \vDash \varphi\ F$
  - $I \vDash \varphi\ F \leftrightarrow G$ iff $I \vDash \varphi\ F \rightarrow G$ and $I \vDash \varphi\ G \rightarrow F$
  - $I \vDash \varphi\ \forall X\ F$ iff $I[X \rightarrow d] \vDash \varphi\ F$ for every $d \in |I|$ (the domain D of I)
  - $I \vDash \varphi\ \exists X\ F$ iff $I[X \rightarrow d] \vDash \varphi\ F$ for some $d \in |I|$

Semantics of Well-Formed Formulae

- Given a set of closed formulas P, an interpretation I is said to be a ***model*** of P iff every formula of P is true in I

Example 1.

- Consider the language with **`zero`** as the lone constant, **`s/1`** as the only functor symbol, and a predicate symbol **`p/1`**
- Consider an interpretation **I** with **|I| = N** (the set of natural numbers), $\mathbf{zero_I = 0}$ and $\mathbf{s_I(x) = 1 + x_I}$
- Now consider the formula:

$$\mathbf{F1 = p(zero) \land (\forall X\ p(s(s(X))) \leftrightarrow p(X))}$$

- Find interpretations for **`p/1`** such that $\mathbf{I \models F1}$
  - $\mathbf{p_{I1} = \{0, 2, 4, 6, 8, 10, \ldots\}}$
  - $\mathbf{p_{I2} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots\}}$

Example 2.

- Recall Example 1:

F1 = p(zero) ∧ (∀X p(s(s(X))) ↔ p(X))

- Extend the previous example with another predicate symbol **q/1**, and consider the formula:

F2 = q(s(zero)) ∧ (∀X q(s(s(X)))↔q(X))

- Now extend the previous interpretations such that:

I ⊨ F1 ∧ F2

Example 2.

- p_{I1} = {0, 2, 4, 6, 8, 10, …}
- q_{I1} = {1, 3, 5, 7, 9, 11, …}

- p_{I2} = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, …}
- q_{I2} = {1, 3, 5, 7, 9, 11, …}

- p_{I3} = {0, 2, 4, 6, 8, 10, …}
- q_{I3} = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, …}

- p_{I4} = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, …}
- q_{I4} = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, …}

Example 3.

- Recall Example 2:

$$F1 = p(zero) \land (\forall X\ p(s(s(X))) \leftrightarrow p(X))$$

$$F2 = q(s(zero)) \land (\forall X\ q(s(s(X))) \leftrightarrow q(X))$$

- Consider a new formula:

$$F3 = (\forall X\ q(s(X)) \leftrightarrow p(X))$$

- Now extend the previous interpretations such that:

$$I \models F1 \land F2 \land F3$$

Example 3.

- p_{I1} = {0, 2, 4, 6, 8, 10, …}
- q_{I1} = {1, 3, 5, 7, 9, 11, …}

- p_{I4} = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, …}
- q_{I4} = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, …}

Interpretations and Consequences

- So, are there any interpretations **I** such that $\mathbf{I} \models \mathbf{F1} \wedge \mathbf{F2}$, but $\mathbf{I} \not\Vdash \mathbf{F3}$?
  - Yes:

p_{I2} = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, …}
q_{I2} = {1, 3, 5, 7, 9, 11, …}

p_{I3} = {0, 2, 4, 6, 8, 10, …}
q_{I3} = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, …}

Logical Consequence

- Let P and F be closed formulas.
  - F is a ***logical consequence*** of P (denoted by P ⊨ F) iff
  - F is true in every model of P.

Logical Consequence: An Example

1) $(\forall X\ (\forall Y\ (mother(X) \wedge child(Y,X)) \rightarrow loves(X,Y)))$

2) $mother(mary) \wedge child(tom, mary)$

- Is **loves(mary, tom)** a logical consequence of the above two statements?

  Yes. Proof:

  - For 1) to be true in some interpretation **I**:

    $I \models \varphi(mother(X) \wedge child(Y,X)) \rightarrow loves(X,Y)$

    must hold for any valuation **φ**.

  - Specifically, for $\varphi = [X \rightarrow mary,\ Y \rightarrow tom]$

    $I \models \varphi(mother(mary) \wedge child(tom,mary)) \rightarrow loves(mary,tom)$

  - Hence **loves(mary,tom)** is true in **I** if 2) above is true in **I**.