Chapter 10: Introduction to Intuitionistic Logic

PART 1: INTRODUCTION

The intuitionistic logic has developed as a result of certain philosophical views on the foundation of mathematics, known as *intuitionism*.

Intuitionism was originated by L. E. J. Brouwer in 1908.

The first Hilbert proof system (Hilber style formalization) of the intuitionistic logic is due to A. Heyting (1930).

We present here a Hilbert style proof system developed by Rasiowa in 1959 that is equivalent to the Heyting's original formalization.

We discuss the relationship between intuitionistic and classical logic.

We also present the original version of Gentzen work (1935).

Gentzen was the first who formulated a first syntactically decidable formalization for classical and intuitionistic logic and proved its equivalence with the Heyting's original Hilbert style formalization (famous Gentzen's Hauptsatz).

We present first, as it has happened historically, the intutionistic proof systems called also formalizations of the intuitionistic logic.

The semantics for the intuitionistic logic will be presented in a seperate chapter.

Intuitionistic semantics was fist defined by Tarski in 1937, and Tarski and Stone in 1938 in terms of pseudo-boolean algebras, called also Heyting algebras to memorize Heying first proof system.

An intuitionistic tautology is a formula that is true in all pseudo-boolean algebras.

Pseudo-boolean algebras are called algebraic models for the intuitionistic logic.

An uniform theory and presentation of algebraic models for classical, intuitionistic, modal and many other logics was given by Rasiowa and Sikorski in 1964, and Rasiowa in 1978.

Alternative semantics is given in terms of Kripke models.

Kripke models were invented by Kripke in 1964. They provide semantics for not only the intuitionistic logic, but also for all known modal logics, believe logics, and many others.

Both semantics algebraic and Kripke models are equivalent for the intuitionistic logic.

Motivation for intuitionistic approach.

The basic difference between classical and intuitionists perspective lies in the interpretation of the word *exists*.

For example, let $A(x)$ be a statement in the arithmetic of natural numbers. For the mathematicians the sentence

$$\exists x A(x)$$

is true if it is a theorem of arithmetic, i.e. if it can be *deduced* from the axioms of arithmetic by means of classical logic.

When a mathematician proves sentence $\exists x A(x)$, this does not mean that he/she is able to indicate a *method of construction* of a natural number n such that $A(n)$ holds.

For an intuitionist the sentence

$$\exists x A(x)$$

is true only he is able to provide a constructive method of finding a number n such that $A(n)$ is true.

Moreover, mathematicians often obtain a proof of existential sentence

$$\exists x A(x)$$

by proving a logically equivalent sentence

$$\neg \forall x \ \neg A(x).$$

Next they use the classical logical equivalence

$$\neg \forall x \ \neg A(x)) \equiv \exists x A(x)$$

(and Modus Ponens twice) and say that they have proved $\exists x A(x))$.

For the intuitionist such method is not acceptable, for it does not give any *method of constructing* a number n such that $A(n)$ holds.

The same argument applies to the following proof by contradiction.

To prove a statement

$$\exists x A(x)$$

we assume, $\neg \exists x A(x)$, and hence, by de-Morgan Law, we have assumed

$$\forall x \neg A(x).$$

If a contradiction follows,

$$\exists x A(x))$$

has been proven.

For these reasons the intuitionist do not accept the classical tautologies

$$(\neg \forall x \ \neg A(x) \Rightarrow \exists x A(x)),$$

$$(\forall x \neg A(x) \Rightarrow \neg \exists x A(x))$$

as as intuitionistically provable sentences, or consequently by intuitionistic Completeness Theorem, as intuitionistic tautologies.

The intuitionists interpret differently then clas- sicists not only quantifiers but also the propositional connectives.

Intuitive ideas are as follows.

Intuitionistic implication $(A \Rightarrow B)$ is considered by to be true if there exists a method by which a *proof of* B can be deduced from the proof of A.

In the case of the classical implication

$$(\neg\forall x\ \neg A(x) \Rightarrow \exists x A(x))$$

there is no general method which, from a proof of the sentence

$$\neg\forall x \neg A(x),$$

permits is to obtain proof of the sentence

$$\exists x A(x).$$

Hence, the intuitionists can't accept it as an intuitionistically provable formula, or intuitionistic tautology.

Intuitionistic negation of a statement A, $\neg A$, is considered intuitionistically true if the acceptance of the sentence A leads to absurdity.

As a result intuitionistic understanding of negation and implication we have that in the intuitionistic proof system I, called intuitionistic logic I

$$\vdash_I (A \Rightarrow \neg\neg A),$$

but

$$\nvdash_I (\neg\neg A \Rightarrow A).$$

Consequently, any intuitionistic semantics I must be such that,

$$\models_I (A \Rightarrow \neg\neg A)$$

and

$$\not\models_I (\neg\neg A \Rightarrow A).$$

Intuitionistic disjunction $(A \cup B)$ is true if one of the sentences A, B is true and there is a method by which it is possible to find out which of them is true.

As a consequence classical law of excluded middle

$$(A \cup \neg A)$$

is not acceptable by the intuitionists since there is no general method of finding out, for any given sentence A, whether A or $\neg A$ is true.

Hence, the intuitionistic proof system I, or logic for short, must be such that

$$\not\vdash_I (A \cup \neg A).$$

The intutionistic semantics I must be such that

$$\models_I (A \cup \neg A).$$

Intuitionists' view of the concept of infinite set also differs from that which is generally accepted in mathematics.

Intuitionists reject the idea of infinite set as a closed whole.

They look upon an infinite set as something which is constantly in a state of formation.

For example, the set of all natural numbers is infinite in the sense that for any given finite set of natural numbers it is always possible to add one more natural number.

The notion of the set of all subsets of the set of all natural numbers is not regarded meaningful.

Intuitionists reject the general idea of a set as defined by a modern set theory.

An exact, formal exposition of the basic ideas of intuitionism is outside the range of our investigations.

Our goal is to give, in this chapter, a presentation of the intuitionistic logic formulated as a proof system and discuss the relationship between classical and intuitionistic logics.

PART 2: Hilbert Proof System for propositional intuitionistic logic.

Language is a propositional language

$$\mathcal{L} = \mathcal{L}_{\{\cup, \cap, \Rightarrow, \neg\}}$$

with the set of formulas denoted by $\mathcal{F}$.

Axioms

A1 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$,

A2 $(A \Rightarrow (A \cup B))$,

A3 $(B \Rightarrow (A \cup B))$,

A4 $((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))$,

A5 $((A \cap B) \Rightarrow A)$,

A6 $((A \cap B) \Rightarrow B)$,

A7 $((C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (A \cap B)))$,

A8 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))$,

A9 $(((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))$,

A10 $(A \cap \neg A) \Rightarrow B)$,

A11 $((A \Rightarrow (A \cap \neg A)) \Rightarrow \neg A)$,

where A, B, C are any formulas in $\mathcal{L}$.

Rules of inference: we adopt a Modus Ponens rule

$$\text{(MP)}\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

as the only inference rule.

A proof system I

$$I = (\ \mathcal{L},\ \mathcal{F}, \mathbf{A1} - \mathbf{A11},\ \textbf{(MP)}\ ),$$

for $\mathcal{L}$, **A1 - A11** defined above, is called Hilbert Style Formalization for Intuitionistic Propositional Logic.

This set of axioms is due to Rasiowa (1959). It differs from Heyting original set of axioms but they are equivalent.

We introduce, as usual, the notion of a formal proof in I and denote by

$$\vdash_I A$$

the fact that A has a formal proof in I, or that that A is *intuitionistically provable*.

We write

$$\models_I A$$

to denote that the formula A is intuitionistic tautology.

Completeness Theorem for I For any formula $A \in \mathcal{F}$,

$$\vdash_I A \ \imath \ and\ only\ if \ \models_I A.$$

The Completeness Theorem gives us the right to replace the notion of a theorem (provable formula) of a given intuitionistic proof system by an independent of the proof system and more intuitive (as we all have some notion of truthfulness) notion of the intuitionistic tautology.

The intuitionistic logic has been created as a rival to the classical one. So a question about the relationship between these two is a natural one.

The following classical tautologies are provable in I and hence, by the Completeness Theorem, are also intuitionistic tautologies.

1. $(A \Rightarrow A)$,

2. $(A \Rightarrow (B \Rightarrow A))$,

3. $(A \Rightarrow (B \Rightarrow (A \cap B)))$,

4. $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

5. $(A \Rightarrow \neg\neg A)$,

6. $\neg(A \cap \neg A)$,

7. $((\neg A \cup B) \Rightarrow (A \Rightarrow B))$,

8. $(\neg(A \cup B) \Rightarrow (\neg A \cap \neg B))$,

9. $((\neg A \cap \neg B) \Rightarrow (\neg(A \cup B))$,

10. $((\neg A \cup \neg B) \Rightarrow (\neg(A \cap B))$,

11. $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$,

12. $((A \Rightarrow \neg B) \Rightarrow (B \Rightarrow \neg A))$,

13. $(\neg\neg\neg A \Rightarrow \neg A)$,

14. $(\neg A \Rightarrow \neg\neg\neg A)$,

15. $(\neg\neg(A \Rightarrow B) \Rightarrow (A \Rightarrow \neg\neg B))$,

16. $((C \Rightarrow A) \Rightarrow ((C \Rightarrow (A \Rightarrow B)) \Rightarrow (C \Rightarrow B))$.

Examples of classical tautologies that are not intuitionistic tautologies

17. $(A \cup \neg A)$,

18. $(\neg\neg A \Rightarrow A)$,

19. $((A \Rightarrow B) \Rightarrow (\neg A \cup B))$,

20. $(\neg(A \cap B) \Rightarrow (\neg A \cup \neg B))$,

21. $((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow A))$,

22. $((\neg A \Rightarrow \neg B) \Rightarrow (B \Rightarrow A))$,

23. $((A \Rightarrow B) \Rightarrow A) \Rightarrow A)$.

Connections between Classical and Intuitionistic logics.

The first connection is quite obvious. Let us observe that if we add the axiom

A12 $(A \cup \neg A)$

to the set of axioms of the system I we obtain a complete Hilbert proof system C for the classical logic.

This proves the following.

Theorem 1 Every formula that is derivable intuitionistically is classically derivable, i.e.

$$if \quad \vdash_I A, \quad then \quad \vdash A$$

where we use symbol $\vdash$ for classical (complete classical proof system) provability.

By the Completeness Theorem we get the following.

Theorem 2 For any formula $A \in \mathcal{F}$,

$$if \quad \models_I A, \quad then \quad \models \ A.$$

The next relationship shows how to obtain intuitionistic tautologies from the classical tautologies and vice versa.

The following relationships were proved by Glivenko in 1929 and independently, in a semantic form by Tarski in 1938.

Theorem 3 (Glivenko) For any formula $A \in \mathcal{F}$, A is a classically provable if and only if $\neg\neg A$ is an intuitionistically provable, i.e.

$$\vdash_I A \quad iff \quad \vdash \neg\neg A$$

where we use symbol $\vdash$ for classical (complete classical proof system) provability.

Theorem 4 (Tarski) For any formula $A \in \mathcal{F}$, A is a classical tautology if and only if $\neg\neg A$ is an intuitionistic tautology, i.e.

$$\models\ A \ \ if\ and\ only\ if\ \ \models_I \neg\neg A.$$

The following relationships were proved by Gödel in 1331.

Theorem 5 (Gödel) For any $A, B \in \mathcal{F}$, a formula $(A \Rightarrow \neg B)$ is a classically provable if and only if it is an intuitionistically provable, i.e.

$$\vdash \ (A \Rightarrow \neg B) \ \ if\ and\ only\ if \ \ \vdash_I (A \Rightarrow \neg B).$$

Theorem 6 (Gödel) If a formula A contains no connectives except $\cap$ and $\neg$, then A is a classically provable if and only if it is an intuitionistically provable tautology.

By the Completeness Theorems for classical and intuitionisctic logics we get the following equivalent semantic form of theorems 5 and 6.

Theorem 7 For any $A, B \in \mathcal{F}$, a formula $(A \Rightarrow \neg B)$ is a classical tautology if and only if it is an intuitionistic tautology, i.e.

$$\models (A \Rightarrow \neg B) \;\; if\ and\ only\ if \;\; \models_I (A \Rightarrow \neg B).$$

Theorem 8 If a formula A contains no connectives except $\cap$ and $\neg$, then A is a classical tautology if and only if it is an intuitionistic tautology.

On intuitionistically derivable disjunctions.

In a classical logic it is possible for the disjunction $(A \cup B)$ to be a tautology when neither A nor B is a tautology. The tautology $(A \cup \neg A)$ is the simplest example. This does not hold for the intuitionistic logic.

Theorem 9 (stated without the proof by Gödel in 1910)

For any $A, B \in \mathcal{F}$, a formula $(A \cup B)$ is intuitionistically provable if and only if A is intuitionistically provable or B is intuitionistically provable i.e.

$$\vdash_I (A \cup B) \;\; if\ and\ only\ if \;\; \vdash_I A, \;\; or \;\; \vdash_I B.$$

Theorem 9 was proved by Gentzen in 1935 via his proof system **LI** which is presented and discussed in the next chapter.

We obtain, via the Completeness Theorem the following equivalent semantic version of the above.

Theorem 10 (Tarski) For any $A, B \in \mathcal{F}$, a disjunction $(A \cup B)$ is intuitionistic tautology if and only if either A or B is intuitionistic tautology, i.e.

$$\models_I (A \cup B) \quad if\ and\ only\ if \quad \models_I A \ \ or \ \ \models_I B.$$