Chapter 9
Completeness Theorem Proofs

We consider a sound proof system (under classical semantics)

\[ S = ( \mathcal{L}_{\Rightarrow, \neg}, \mathcal{AL}, \text{MP} ), \]

such that the formulas listed below are provable in \( S \).

1. \((A \Rightarrow (B \Rightarrow A))\),

2. \(((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))\),

3. \(((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))\),
We present here two proofs of the following theorem.
Completeness Theorem For any formula $A$ of $S$, 

$$\models A \text{ if and only if } \vdash_S A.$$ 

Observation 1 All the above formulas have proofs in the system $H_2$ and the system $H_2$ is sound, hence the Completeness Theorem for the system $S$ implies the completeness of the system $H_2$.

Observation 2 We have assumed that the system $S$ is sound, i.e. that the following theorem holds for $S$.

Soundness Theorem

For any formula $A$ of $S$, 

if $\vdash_S A$, then $\models A$. 
It means that in order to prove the Completeness Theorem we need to prove only the following implication.

For any formula $A$ of $S$,

If $\models A$, then $\vdash_S A$.

Both proofs of the Completeness Theorem rely strongly of the Deduction Theorem, as discussed and proved in the previous chapter.
Deduction theorem was proved for the system $H_1$ that is different that $S$, but all formulas that were used in its proof are provable in $S$, so it is valid for $S$ as well, as it was for the system $H_2$, i.e. the following theorem holds.

**Deduction Theorem for $S$**

For any formulas $A$, $B$ of $S$ and $\Gamma$ be any subset of formulas of $S$,

$$\Gamma, A \vdash_S B \text{ if and only if } \Gamma \vdash_S (A \Rightarrow B).$$
It is possible to prove the Completeness Theorem independently from the Deduction Theorem and we will present two of such a proof in later chapters.

The first proof presented here is similar in its structure to the proof of the deduction theorem and is due to Kalmar, 1935.

It shows how one can use the assumption that a formula $A$ is a tautology in order to construct its formal proof. It is hence called a proof-construction method.

The second proof is a proof of the equivalent opposite implication to the Completeness part, i.e. we show how one can deduce that a formula $A$ is not a tautology from the fact that it does not have a proof. It is hence called a counter-model construction method.
Completeness Theorem

A Proof - Construction Method

We first present one definition and prove one lemma.

We write $\vdash A$ instead of $\vdash_S A$, as the system $S$ is fixed.

Definition Let $A$ be a formula and $b_1, b_2, \ldots, b_n$ be all propositional variables that occur in $A$.

Let $v$ be variable assignment $v : VAR \rightarrow \{T, F\}$. 
DEFINITION 1

We define, for $A, b_1, b_2, ..., b_n$ and $v$ a corresponding formulas $A', B_1, B_2, ..., B_n$ as follows:

$$A' = \begin{cases} A & \text{if } v^*(A) = T \\ \neg A & \text{if } v^*(A) = F \end{cases}$$

$$B_i = \begin{cases} b_i & \text{if } v(b_i) = T \\ \neg b_i & \text{if } v(b_i) = F \end{cases}$$

for $i = 1, 2, ..., n$. 
Example 1: let $A$ be a formula $(a \Rightarrow \neg b)$.

Let $v$ be such that

$$v(a) = T, \quad v(b) = F.$$  

In this case: $b_1 = a, \ b_2 = b$, and $v^*(A) = v^*(a \Rightarrow \neg b) = v(a) \Rightarrow \neg v(b) = T \Rightarrow \neg F = T$.

The corresponding $A', B_1, B_2$ are:

$A' = A \quad (\text{as } v^*(A) = T),$

$B_1 = a \quad (\text{as } v(a) = T),$

$B_2 = \neg b \quad (\text{as } v(b) = F)$. 
Example 2

Let $A$ be a formula

$$((\neg a \Rightarrow \neg b) \Rightarrow c)$$

and let $v$ be such that

$$v(a) = T, \ v(b) = F, v(c) = F.$$ 

**Evaluate** $A'$, $B_1, \ldots B_n$ as defined by the definition 1.
In this case $n = 3$ and

\[ b_1 = a, \; b_2 = b, \; b_3 = c, \]

and we evaluate

\[ v^*(A) = v^*(((\neg a \Rightarrow \neg b) \Rightarrow c) = \]

\[ (((\neg v(a) \Rightarrow \neg v(b)) \Rightarrow v(c)) = \]

\[ (((\neg T \Rightarrow \neg F') \Rightarrow F') = (T \Rightarrow F') = F. \]
The corresponding $A'$, $B_1, B_2, B_2$ are:

$$A' = \neg A = \neg((\neg a \Rightarrow \neg b) \Rightarrow c)$$

as $v^*(A) = F$,

$B_1 = a$ (as $v(a) = T$),

$B_2 = \neg b$ (as $v(b) = F$).

$B_3 = \neg c$ (as $v(c) = F$).
The lemma stated below describes a method of transforming a semantic notion of a tautology into a syntactic notion of provability. It defines, for any formula $A$ and a variable assignment $v$ a corresponding deducibility relation.

**MAIN LEMMA** For any formula $A$ and a variable assignment $v$, if $A', B_1, B_2, ..., B_n$ are corresponding formulas defined by our definition, then

$$B_1, B_2, ..., B_n \vdash A'.$$
Example 3 Let $A, v$ be as defined by the Example 1, then the Lemma asserts that

$$a, \neg b \vdash (a \Rightarrow \neg b).$$

Example 4 Let $A, v$ be as defined in Example 2, then the lemma asserts that

$$a, \neg b, \neg c \vdash \neg((\neg a \Rightarrow \neg b) \Rightarrow c)$$
Proof of the MAIN LEMMA  The proof is by induction on the degree of $A$ i.e. a number $n$ of logical connectives in $A$.

Case:  $n = 0$

In the case that $n = 0$ $A$ is atomic and so consists of a single propositional variable, say $a$.

Clearly, if $v^*(A) = T$ then we $A' = A = a$, $B_1 = a$.

We obtain that $a \vdash a$

by the Deduction Theorem and the fact that $\vdash (A \Rightarrow A)$, i.e. also $\vdash (a \Rightarrow a)$.
In case when \( v^*(A) = F \) we have that
\[
A' = \neg A = \neg a,
\]
\[
B_1 = \neg a,.
\]

We obtain that
\[
\neg a \vdash \neg a
\]
also by the Deduction Theorem and assumption \( \vdash (A \Rightarrow A) \) in \( S \).

This proves that Lemma holds for \( n = 0 \)
Now assume that the lemma holds for any $A$ with $j < n$ connectives.

Prove: lemma holds for $A$ with $n$ connectives.

There are several subcases to deal with.
**Case: A is ¬A_1**

If $A$ is of the form $¬A_1$ then $A_1$ has less then $n$ connectives.

**By the inductive assumption** we have the formulas

$$A'_1, \quad B_1, B_2, ..., B_n$$

corresponding to the $A_1$ and the propositional variables $b_1, b_2, ..., b_n$ in $A_1$, such that

$$B_1, B_2, ..., B_n \vdash A'_1$$

**Observe**, that the formulas $A$ and $¬A_1$ have the same propositional variables.

**So the corresponding** formulas $B_1, B_2, ..., B_n$ are the same for both of them.
We are going to show that the inductive assumption allows us to prove that the lemma holds for $A$, ie. that

$$B_1, B_2, ..., B_n \vdash A'.$$

There two cases to consider.

Case: $v^*(A_1) = T$

If $v^*(A_1) = T$ then by definition

$$A_1' = A_1$$

and by the inductive assumption

$$B_1, B_2, ..., B_n \vdash A_1.$$
In this case: $v^*(A) = v^*(\neg A_1) = \neg v^*(A_1) = \neg T = F$

So we have that $A' = \neg A = \neg\neg A_1$.

Since we have assumed about $S$ that

$\vdash (A_1 \Rightarrow \neg\neg A_1)$

we obtain by the monotonicity that also

$B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow \neg\neg A_1)$. 
By inductive assumption and Modus Ponens we have that also

\[ B_1, B_2, ..., B_n \vdash \neg
A_1, \]

and as \( A' = \neg A = \neg \neg A_1 \) we get

\[ B_1, B_2, ..., B_n \vdash \neg A, \]

\[ B_1, B_2, ..., B_n \vdash A'. \]

**Case:** \( v^*(A_1) = F \)

If \( v^*(A_1) = F \) then \( A'_1 = \neg A_1 \) and \( v^*(A) = T \) so \( A' = A. \)

**Therefore** by the inductive assumption we have that

\[ B_1, B_2, ..., B_n \vdash \neg A_1 \]

that is (as \( A = \neg A_1 \) and \( A' = A \))

\[ B_1, B_2, ..., B_n \vdash A'. \]
**Case:** \( A \) is \((A_1 \Rightarrow A_2)\)

If \( A \) is \((A_1 \Rightarrow A_2)\) then \( A_1 \) and \( A_2 \) have less than \( n \) connectives.

By the inductive assumption and monotonicity we have
\[
B_1, B_2, ..., B_n \vdash A_1'
\]
and
\[
B_1, B_2, ..., B_n \vdash A_2',
\]
where \( B_1, B_2, ..., B_n \) are formulas corresponding to the propositional variables in \( A \).

Now we have the following subcases to consider.
Case: \( v^*(A_1) = v^*(A_2) = T \)

If \( v^*(A_1) = T \) then \( A_1' \) is \( A_1 \) and if \( v^*(A_2) = T \) then \( A_2' \) is \( A_2 \).

We also have \( v^*(A_1 \Rightarrow A_2) = T \) and so \( A' \)
is \( (A_1 \Rightarrow A_2) \).

By the above and the inductive assumption,
\( B_1, B_2, ..., B_n \models A_2 \) and since we have assumed about \( S \) that \( \vdash (A_2 \Rightarrow (A_1 \Rightarrow A_2)) \),
we have by monotonicity and Modus Ponens,
that \( B_1, B_2, ..., B_n \models (A_1 \Rightarrow A_2) \), that is
\( B_1, B_2, ..., B_n \models A' \).
Case: \(v^*(A_1) = T, v^*(A_2) = F\)

If \(v^*(A_1) = T\) then \(A_1' = A_1\) and

if \(v^*(A_2) = F\) then \(A_2' = \neg A_2\).

Also we have in this case \(v^*(A_1 \Rightarrow A_2) = F\) and so \(A' = \neg(A_1 \Rightarrow A_2)\).

By the above and the inductive assumption, therefore, \(B_1, B_2, ..., B_n \vdash \neg A_2\). Since we have assumed \(\vdash (A_1 \Rightarrow (\neg A_2 \Rightarrow \neg(A_1 \Rightarrow A_2)))\), we have by monotonicity and Modus Ponens twice, that \(B_1, B_2, ..., B_n \vdash \neg(A_1 \Rightarrow A_2)\), that is

\[B_1, B_2, ..., B_n \vdash A'.\]
Case: $v^*(A_1) = F$

If $v^*(A_1) = F$ then $A_1' = \neg A_1$ and, whatever value $v$ gives $A_2$, we have that $v^*(A_1 \Rightarrow A_2) = T$ and so $A' = (A_1 \Rightarrow A_2)$.

Therefore,

$$B_1, B_2, ..., B_n \vdash \neg A_1$$

and since $\vdash (\neg A_1 \Rightarrow (A_1 \Rightarrow A_2))$, by monotonicity and Modus Ponens we get that

$$B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow A_2),$$

that is

$$B_1, B_2, ..., B_n \vdash A'.$$
With that we have covered all cases and, by induction on $n$, the proof of the lemma is complete.

Proof of the Completeness Theorem

Assume that $\models A$.

Let $b_1, b_2, ..., b_n$ be all propositional variables that occur in $A$, i.e. $A = A(b_1, b_2, ..., b_n)$.

By the lemma we know that, for any variable assignment $\nu$, the corresponding formulas $\hat{A}', B_1, B_2, ..., B_n$ can be found such that

$$B_1, B_2, ..., B_n \vdash \hat{A}'$$.
Note here that $A'$ of the definition is $A$ for any $v$ since $\models A$.

Hence, if $v$ is such that $v(b_n) = T$, then $B_n$ is $b_n$ and

$$B_1, B_2, ..., b_n \vdash A.$$ 

If $w$ is such that $w(b_n) = F$, then $B_n$ is $\neg b_n$ and by the lemma

$$B_1, B_2, ..., \neg b_n \vdash A.$$ 

So, by the Deduction Theorem, we have

$$B_1, B_2, ..., B_{n-1} \vdash (b_n \Rightarrow A)$$ 

and

$$B_1, B_2, ..., B_{n-1} \vdash (\neg b_n \Rightarrow A).$$
By monotonicity and assumed formula 9

\[ \vdash_S ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B)) \]

we have that

\[ B_1, B_2, ..., B_{n-1} \vdash ((b_n \Rightarrow A) \Rightarrow ((\neg b_n \Rightarrow A) \Rightarrow A)). \]

Applying Modus Ponens twice we get that

\[ B_1, B_2, ..., B_{n-1} \vdash A. \]

Similarly, \( v^*(B_{n-1}) \) may be T or F, and, again applying Deduction Theorem, monotonicity, and \( \vdash_S ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B)) \), and Modus Ponens twice we can eliminate \( B_{n-1} \) just as we eliminated \( B_n \).

After n steps, we finally obtain proof of \( A \) in \( S \), i.e. we have that

\[ \vdash A. \]
Observe that our proof of the fact that $\vdash A$ is a constructive one. Moreover, we have used in it only Main Lemma and Deduction Theorem which both have a constructive proofs.

We can hence reconstruct proofs in each case when we apply these theorems back to the original axioms $A1 – A3$ of $H_2$. The same applies to the proofs in $H_2$ of all formulas 1 -9 of the system $S$.

It means that for any $A$, such that $\models A$, each $v$ restricted to $A$ provides us the method of a construction of the formal proof of $A$ in $H_2$, or in any system $S$ in which formulas 1 -9 are provable.
EXAMPLE As an example of how the Completeness Theorem proof works, we consider the case in which $A$ is a tautology

$$(a \Rightarrow (\neg a \Rightarrow b))$$

and show how the construction described in the Proof 1 works; i.e how we construct the proof of $A$.

Step 1. We apply Main Lemma to all different variable assignments for $A$. We have 4 cases to consider. As $\models A$ in all cases we have that $A' = A$.

Case 1: $v(a) = T, v(b) = T$.
In this case $B_1 = a, B_2 = b$ and, as in all cases $A' = A$.

By the Main Lemma,

$$a, b \vdash (a \Rightarrow (\neg a \Rightarrow b)).$$
Case 2: \( v(a) = T, v(b) = F \).
In this case \( B_1 = a, B_2 = \neg b, A' = A \) and by the Main Lemma,

\[
  a, \neg b \vdash (a \Rightarrow (\neg a \Rightarrow b)).
\]

Case 3: \( v(a) = F, v(b) = T \).
In this case \( B_1 = \neg a, B_2 = b, A' = A \) and by the Main Lemma,

\[
  \neg a, b \vdash (a \Rightarrow (\neg a \Rightarrow b)).
\]

Case 4: \( v(a) = F, v(b) = F \).
In this case \( B_1 = \neg a, B_2 = \neg b, A' = A \) and by the Main Lemma,

\[
  \neg a, \neg b \vdash (a \Rightarrow (\neg a \Rightarrow b)).
\]
We apply Deduction Theorem on formulas $b, \neg b$ to all the cases 1-4. This is the case of $B_n$ elimination in the Proof 1.

**D1** (Cases 1 and 2)

$$a \vdash (b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))),$$

$$a \vdash (\neg b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))).$$

**D2** (Cases 3 and 4)

$$\neg a \vdash (b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))),$$

$$\neg a \vdash (\neg b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))).$$
By the monotonicity and proper substitution of the formula 8 we have that

\[ a \vdash ((b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow ((\neg b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))), \]

\[ \neg a \vdash ((b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow ((\neg b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))). \]

Applying Modus Ponens twice to \( \textbf{D1}, \textbf{D2} \) and these above, respectively, gives us

\[ a \vdash (a \Rightarrow (\neg a \Rightarrow b)) \text{ and } \]

\[ \neg a \vdash (a \Rightarrow (\neg a \Rightarrow b)). \]
Applying the Deduction Theorem to the above we obtain

\[ \mathbf{D3} \vdash (a \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \] and

\[ \mathbf{D4} \vdash (\neg a \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))). \]

Applying Modus Ponens twice to \( \mathbf{D3} \) and \( \mathbf{D4} \) and the following form of formula 8,

\[
\vdash ((a \Rightarrow (a \Rightarrow (\neg a \Rightarrow b)))
\Rightarrow ((\neg a \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))))
\]

we get finally that

\[ \vdash (a \Rightarrow (\neg a \Rightarrow b)). \]
Completeness Theorem: Proof 2
A Counter-Model Existence Method

We prove now the Completeness Theorem by proving the opposite implication:

If $\not\vdash A$, then $\not\models A$

We will show now how one can define a counter-model for $A$ from the fact that $A$ is not provable.

This means that we deduce that a formula $A$ is not a tautology from the fact that it does not have a proof.

We hence call it a counter-model existence method.
The construction of a counter-model for any non-provable $A$ is much more general (and less constructive) then in the case of our first proof.

It can be generalized to the case of predicate logic, and many of non-classical logics; propositional and predicate.

It is hence a much more general method then the first one and this is the reason we present it here.
We remind that $\not\models A$ means that there is a variable assignment $v : VAR \rightarrow \{T, F\}$, such that $v^*(A) \neq T$, i.e. in classical semantics that $v^*(A) = F$. Such $v$ is called a counter-model for $A$, hence the proof provides a counter-model construction method.

Since we assume that $A$ does not have a proof in $S$ ($\not\vdash A$) the method uses this information in order to show that $A$ is not a tautology, i.e. to define $v$ such that $v^*(A) = F$.

We also have to prove that all steps in that method are correct. This is done in the following steps.
Step 1: Definition of $\Delta^*$

We use the information $\not\models A$ to define a special set $\Delta^*$, such that $\neg A \in \Delta^*$.

Step 2: Counter-model definition

We define the variable assignment $v : VAR \rightarrow \{T, F\}$ as follows:

$$v(a) = \begin{cases} T & \text{if } \Delta^* \models a \\ F & \text{if } \Delta^* \models \neg a. \end{cases}$$
Step 3: Prove that $v$ is a counter-model

We first prove a more general property, namely we prove that the set $\Delta^*$ and $v$ defined in the steps 1 and 2, respectively, are such that for every formula $B \in \mathcal{F}$,

$$v^*(B) = \begin{cases} T & \text{if } \Delta^* \vdash B \\ F & \text{if } \Delta^* \vdash \neg B. \end{cases}$$

Then we use the Step 1 to prove that $v^*(A) = F$.

The definition and the properties of the set $\Delta^*$, and hence the Step 1, are the most essential for the proof.

The other steps have only technical character.
The main notions involved in this step are: **consistent set, complete set** and a **consistent complete extension of a set**.

We are going now to introduce them and to prove some essential facts about them.

**Consistent and Inconsistent Sets**

There exist two definitions of consistency; **semantical and syntactical**.
**Semantical** definition uses the notion of a model and says:

*a set is consistent if it has a model.*

**Syntactical** definition uses the notion of provability and says:

*a set is consistent if one can’t prove a contradiction from it.*

In our proof of the Completeness Theorem we use assumption that a given formula $A$ does not have a proof to deduce that $A$ is not a tautology.

We hence use the following syntactical definition of consistency.
Consistent set

We say that a set $\Delta \subseteq \mathcal{F}$ of formulas is **consistent** if and only if **there is no** a formula $A \in \mathcal{F}$ such that

$$\Delta \vdash A \quad \text{and} \quad \Delta \vdash \neg A.$$ 

Inconsistent set

A set $\Delta \subseteq \mathcal{F}$ is **inconsistent** if and only if **there is** a formula $A \in \mathcal{F}$ such that $\Delta \vdash A$ and $\Delta \vdash \neg A$.

The notion of consistency, as defined above, is characterized by the following lemma.
LEMMA: Consistency Condition

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) $\Delta$ is consistent,

(ii) there is a formula $A \in \mathcal{F}$ such that $\Delta \not\vdash A$. 
**Proof:** The implications:

(i) $\Delta$ is **consistent**, implies

(ii) there is a formula $A \in \mathcal{F}$ such that $\Delta \nvdash A$ and vice-versa are proved by showing the corresponding opposite implications.

I.e. to establish the equivalence of (i) and (ii), we first show that

**Case 1:** not (ii) implies not (i), and then that

**Case 2:** not (i) implies not (ii).
Case 1

Assume that not (ii).

It means that for all formulas $A \in \mathcal{F}$ we have that $\Delta \vdash A$.

In particular it is true for a certain $A = B$ and $A = \neg B$ and hence it proves that $\Delta$ is inconsistent,

i.e. not (i) holds.
Case 2

Assume that not (i), i.e that $\Delta$ is inconsistent.

Then there is a formula $A$ such that $\Delta \vdash A$ and $\Delta \vdash \neg A$.

Let $B$ be any formula. Since $(\neg A \Rightarrow (A \Rightarrow B))$ is provable in $S$ (formula 6),

hence by monotonicity and applying Modus Ponens twice and by detaching from it $\neg A$ first, and $A$ next, we obtain a formal proof of $B$ from the set $\Delta$.

This proves that $\Delta \vdash B$ for any formula $B$. Thus not (ii).
The inconsistent sets are hence characterized by the following fact.

**LEMMA: Inconsistency Condition**

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) $\Delta$ is inconsistent,

(ii) for all formulas $A \in \mathcal{F}$, $\Delta \vdash A$. 
We remind here the property of the finiteness of the consequence operation.

**LEMMA: Finite Consequence**

For every set $\Delta$ of formulas and for every formula $A \in \mathcal{F}$,

$\Delta \vdash A$ if and only if there is a finite subset $\Delta_0 \subseteq \Delta$ such that $\Delta_0 \vdash A$.

**Proof:**

If $\Delta_0 \vdash A$ for a certain $\Delta_0 \subseteq \Delta$,
then by the monotonicity of the consequence, also $\Delta \vdash A$.

Assume now that $\Delta \vdash A$ and let $A_1, A_2, ..., A_n$ be a formal proof of $A$ from $\Delta$.

Let $\Delta_0 = \{A_1, A_2, ..., A_n\} \cap \Delta$.

Obviously, $\Delta_0$ is finite and $A_1, A_2, ..., A_n$ is a formal proof of $A$ from $\Delta_0$. 
The following theorem is a simply corollary of the above Finite Consequence Lemma.

**Finite Inconsistency THEOREM**

1. If a set $\Delta$ is **inconsistent**, then there is a finite subset $\Delta_0 \subseteq \Delta$ which is inconsistent.

   It follows therefore from that

2. if every finite subset of a set $\Delta$ is consistent, then the set $\Delta$ is also consistent.

**Proof:**

If $\Delta$ is inconsistent, then for some formula $A$,

$$\Delta \vdash A \quad \text{and} \quad \Delta \vdash \neg A.$$
By the Finite Consequence Lemma, there are finite subsets $\Delta_1$ and $\Delta_2$ of $\Delta$ such that

$$\Delta_1 \vdash A \quad \text{and} \quad \Delta_2 \vdash \neg A.$$ 

By monotonicity, the union $\Delta_1 \cup \Delta_2$ is a finite subset of $\Delta$, such that

$$\Delta_1 \cup \Delta_2 \vdash A \quad \text{and} \quad \Delta_1 \cup \Delta_2 \vdash \neg A.$$ 

Hence $\Delta_1 \cup \Delta_2$ is a finite inconsistent subset of $\Delta$.

The second implication is the opposite to the one just proved and hence also holds.
The following lemma links the notion of non-provability and consistency.

It will be used as an important step in our proof of the Completeness Theorem.

**LEMMA**

For any formula $A \in \mathcal{F}$,

if $\not\vdash A$, then the set $\{\neg A\}$ is consistent.

**Proof:** If $\{\neg A\}$ is inconsistent, then by the Inconsistency Condition Lemma we have $\{\neg A\} \vdash A$. 
\{\neg A \} \vdash A \text{ and the Deduction Theorem imply } \\
\vdash (\neg A \Rightarrow A).

Applying the Modus Ponens rule to \((\neg A \Rightarrow A)\) 
and assumed provable formula 9 
\[((\neg A \Rightarrow A) \Rightarrow A)\),

we get that \(\vdash A\), contrary to the assumption of the lemma.
Complete and Incomplete Sets

Another important notion, is that of a **complete set** of formulas. Complete sets, as defined here are sometimes called **maximal**, but we use the first name for them.

They are defined as follows.

**Complete set**

A set $\Delta$ of formulas is called complete if

for every formula $A \in \mathcal{F}$,

$$\Delta \vdash A \text{ or } \Delta \vdash \neg A.$$ 

The complete sets are characterized by the following fact.
Complete Set Condition Lemma

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) $\Delta$ is complete,

(ii) for every formula $A \in \mathcal{F}$, if $\Delta \not\vdash A$,

then the set $\Delta \cup \{A\}$ is inconsistent.

Proof: We consider two cases.

Case 1 We show that (i) implies (ii) and

Case 2 we show that (ii) implies (i).
Proof of Case 1:

Assume that (i) and that for every formula \( A \in \mathcal{F}, \Delta \nvdash A \).

We have to show that in this case \( \Delta \cup \{A\} \) is inconsistent.

But if \( \Delta \nvdash A \), then from the definition of complete set and assumption that \( \Delta \) is complete set, we get that

\[ \Delta \vdash \neg A. \]
By the monotonicity of the consequence we have that

\[ \Delta \cup \{A\} \vdash \neg A. \]

By formula 4 \( \vdash (A \Rightarrow A) \) and monotonicity we get \( \Delta \vdash (A \Rightarrow A) \) and by Deduction Theorem

\[ \Delta \cup \{A\} \vdash A. \]

This proves that \( \Delta \cup \{A\} \) is inconsistent. Hence \( (ii) \) holds.
Case 2

Assume that (ii), i.e. for every formula \( A \in \mathcal{F} \), if \( \Delta \nvdash A \), then the set \( \Delta \cup \{ A \} \) is inconsistent.

Let \( A \) be any formula. We want to show (i), i.e. to show that the condition:

\[
\Delta \vdash A \text{ or } \Delta \vdash \neg A
\]

is satisfied.

If

\[
\Delta \vdash \neg A,
\]

then the condition is obviously satisfied.
If, on other hand,

$$\Delta \not\vdash \neg A,$$

then we are going to show now that it must be, under the assumption of \((ii)\), that $\Delta \vdash A$, i.e. that \((i)\) holds.

**Assume** that

$$\Delta \not\vdash \neg A,$$

then by \((ii)\), the set $\Delta \cup \{\neg A\}$ is inconsistent.

It means, by the Consistency Condition Lemma, that

$$\Delta \cup \{\neg A\} \vdash A.$$
By the Deduction Theorem, this implies that

$$\Delta \vdash (\neg A \Rightarrow A).$$

**Observe** that

$$((\neg A \Rightarrow A) \Rightarrow A)$$

is a provable formula 4 in $S$.

By monotonicity,

$$\Delta \vdash ((\neg A \Rightarrow A) \Rightarrow A).$$

Detaching $(\neg A \Rightarrow A)$, we obtain that

$$\Delta \vdash A.$$

This ends the proof that (i) holds.
Incomplete set

A set $\Delta$ of formulas is called incomplete if it is not complete, i.e. if there exists a formula $A \in \mathcal{F}$ such that

$$\Delta \not\vdash A \text{ and } \Delta \not\vdash \neg A$$

We get as a direct consequence of the Complete Set Condition Lemma the following characterization of incomplete sets.

Incomplete Set Condition Lemma

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) $\Delta$ is incomplete,

(ii) there is formula $A \in \mathcal{F}$ such that $\Delta \not\vdash A$, and the set $\Delta \cup \{A\}$ is consistent.
Main Lemma: Complete Consistent Extension

Now we are going to prove a lemma that is essential to the construction of the special set $\Delta^*$ mentioned in the Step 1 of the proof of the Completeness Theorem, and hence to the proof of the theorem itself.

Let’s first introduce one more notion.

Extension $\Delta^*$ of the set $\Delta$.

A set $\Delta^*$ of formulas is called an extension of a set $\Delta$ of formulas if the following condition holds:

$$\{A \in F : \Delta \vdash A\} \subseteq \{A \in F : \Delta^* \vdash A\}.$$

In this case we say also that $\Delta$ extends to the set of formulas $\Delta^*$. 
The Main Lemma states as follows.

Complete Consistent Extension Lemma

Every consistent set \( \Delta \) of formulas can be extended to a complete consistent set \( \Delta^* \) of formulas.

Proof: Assume that the lemma does not hold, i.e. that there is a consistent set \( \Delta \), such that all its consistent extensions are not complete.

In particular, as \( \Delta \) is an consistent extension of itself, we have that \( \Delta \) is not complete.
The proof consists of a construction of a particular set $\Delta^*$ and proving that it forms a complete consistent extension of $\Delta$, contrary to the assumption that all its consistent extensions are not complete.

**CONSTRUCTION of $\Delta^*$.

As we know, the set $\mathcal{F}$ of all formulas is enumerable. They can hence be put in an infinite sequence

$$F \quad A_1, A_2, \ldots, A_n, \ldots$$

such that every formula of $\mathcal{F}$ occurs in that sequence exactly once.
We define, by mathematical induction, an infinite sequence $\{\Delta_n\}_{n \in \mathbb{N}}$ of consistent subsets of formulas together with a sequence $\{B_n\}_{n \in \mathbb{N}}$ of formulas as follows.

**Initial Step**

In this step we define the sets $\Delta_1, \Delta_2$ and the formula $B_1$ and prove that $\Delta_1$ and $\Delta_2$ are consistent, incomplete extensions of $\Delta$.

We take as the first set, the set $\Delta$, i.e. we define

$$\Delta_1 = \Delta.$$
By assumption the set $\Delta$, and hence also $\Delta_1$ is not complete.

From the Incomplete Set Condition we get that there is a formula $B \in \mathcal{F}$ such that

$$\Delta_1 \nvdash B \quad \text{and} \quad \Delta_1 \cup \{B\} \quad \text{is consistent.}$$

Let

$$B_1$$

be the first formula with this property in the sequence $\mathcal{F}$ of all formulas;
We define

$$\Delta_2 = \Delta_1 \cup \{B_1\}.$$ 

Observe that the set $\Delta_2$ is consistent and

$$\Delta_1 = \Delta \subseteq \Delta_2,$$

so by the monotonicity, $\Delta_2$ is a consistent extension of $\Delta$.

Hence, as we assumed that all consistent extensions of $\Delta$ are not complete, we get that $\Delta_2$ cannot be complete, i.e.

$\Delta_2$ is incomplete.
Inductive Step

Suppose that we have defined a sequence

$$\Delta_1, \Delta_2, ..., \Delta_n$$

of incomplete, consistent extensions of \(\Delta\), and a sequence

$$B_1, B_2, ...B_{n-1}$$

of formulas, for \(n \geq 2\).

Since \(\Delta_n\) is incomplete, it follows from the Incomplete Set Condition that

there is a formula \(B \in \mathcal{F}\) such that \(\Delta_n \nvdash B\),

then and the set \(\Delta_n \cup \{B\}\) is consistent.
Let $B_n$ be the first formula with this property in the sequence $F$ of all formulas.

We define:

$$\Delta_{n+1} = \Delta_n \cup \{B_n\}.$$ 

By the definition,

$$\Delta \subseteq \Delta_n \subseteq \Delta_{n+1}$$

and the set $\Delta_{n+1}$ is a consistent extension of $\Delta$.

Hence by our assumption that all all consistent extensions of $\Delta$ are incomplete we get that $\Delta_{n+1}$ is an incomplete consistent extension of $\Delta$. 

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By the principle of mathematical induction we have defined an infinite sequence

\[ \Delta = \Delta_1 \subseteq \Delta_2 \subseteq \ldots, \subseteq \Delta_n \subseteq \Delta_{n+1} \subseteq \ldots \]

such that for all \( n \in N \), \( \Delta_n \) is consistent, and each \( \Delta_n \) an incomplete consistent extension of \( \Delta \).
Moreover, we have also defined a sequence $B_1, B_2, ..., B_n, ...$ of formulas, such that for all $n \in \mathbb{N}$,

\[ \Delta_n \nvdash B_n, \text{ and the set } \Delta_n \cup \{B_n\} \text{ is consistent.} \]

Observe that $B_n \in \Delta_{n+1}$ for all $n \geq 1$. 
Now we are ready to define $\Delta^*$. 

**Definition of $\Delta^*$**

$$\Delta^* = \bigcup_{n \in N} \Delta_n.$$ 

To complete the proof our theorem we have now to prove that

$\Delta^*$ is a **complete consistent extension** of $\Delta$.

**Obviously,** by the definition,

$\Delta^*$ is an extension of $\Delta$. 

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Fact 1 $\Delta^*$ is consistent.

proof: assume that $\Delta^*$ is inconsistent. By the Finite Inconsistency theorem there is a finite subset $\Delta_0$ of $\Delta^*$ that is inconsistent, i.e.

$$\Delta_0 = \{C_1, \ldots, C_n\} \subseteq \bigcup_{n \in \mathbb{N}} \Delta_n$$

and $\Delta_0$ is inconsistent.
By the definition, \( C_i \in \Delta_{k_i} \) for certain \( \Delta_{k_i} \) in the sequence \( \mathbf{D} \) and \( 1 \leq i \leq n \).

Hence \( \Delta_0 \subseteq \Delta_m \) for \( m = \max\{k_1, k_2, \ldots, k_n\} \).

But all sets of the sequence \( \mathbf{D} \) are consistent.

This contradicts the fact that \( \Delta_m \) is inconsistent, as it contains an inconsistent sub-set \( \Delta_0 \).

Hence \( \Delta^* \) must be consistent.
Fact 2 \( \Delta^* \) is complete.

**proof:** assume that \( \Delta^* \) is not complete. By the Incomplete Set Condition, there is a formula \( B \in \mathcal{F} \) such that

\[
\Delta^* \not\vdash B, \text{ and the set } \Delta^* \cup \{B\} \text{ is consistent.}
\]

By definition \( \mathcal{D} \) of the sequence \( \Delta_n \), for every \( n \in N, \Delta_n \not\vdash B \) and the set \( \Delta_n \cup \{B\} \) is consistent.
Since the formula $B$ is one of the formulas of the sequence $\mathcal{B}$ and it would have to be one of the formulas of the sequence i.e. $B = B_j$ for certain $j$.

By definition, $B_j \in \Delta_{j+1}$, it proves that $B \in \Delta^* = \bigcup_{n \in N} \Delta_n$.

But this means that

$$\Delta^* \vdash B,$$

contrary to the assumption.

This proves that $\Delta^*$ is a complete consistent extension of $\Delta$ and completes the proof out our lemma.

Now we are ready to prove the completeness theorem for the system $S$. 
Proof of the Completeness Theorem

As by assumption our system $S$ is sound, we have to prove only the Completeness part of the Completeness Theorem, i.e. for any formula $A$,

If $\models A$, then $\vdash A$

We prove it by proving the opposite implication

If $\not\models A$, then $\not\vdash A$. 
Reminder: \( \not\models A \) means that there is a variable assignment \( v : VAR \rightarrow \{T, F\} \), such that \( v^*(A) \neq T \).

In classical case it means that \( v^*(A) = F \), i.e. that there is a variable assignment that falsifies \( A \). Such \( v \) is also called a counter-model for \( A \).
Assume that $A$ doesn’t have a proof in $S$, we want to define a counter-model for $A$.

But if $\not\vdash A$, then by the Inconsistency Lemma the set $\{\neg A\}$ is consistent.

By the Main Lemma there is a complete, consistent extension of the set $\{\neg A\}$, i.e. there is a set set $\Delta^*$ such that $\{\neg A\} \subseteq \Delta^*$, i.e.

$$E \quad \neg A \in \Delta^*.$$

Since $\Delta^*$ is a consistent, complete set, it satisfies the following form consistency condition, which says that for any $A$,

$$\Delta^* \not\vdash A \quad \text{or} \quad \Delta^* \not\vdash \neg A.$$
It also satisfies the completeness condition, which says that for any $A$,

$$\Delta^* \vdash A \text{ or } \Delta^* \vdash \neg A.$$ 

This means that for any $A$, **exactly one** of the following conditions is satisfied:

(1) $\Delta^* \vdash A$, or

(2) $\Delta^* \vdash \neg A$.

In particular, for every propositional variable $a \in VAR$ **exactly one** of the following conditions is satisfied:

(1) $\Delta^* \vdash a$, or

(2) $\Delta^* \vdash \neg a$. 

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This justifies the correctness of the following definition.

**Definition of** $v$

We define the variable assignment

$$v : VAR \rightarrow \{T, F\}$$

as follows:

$$v(a) = \begin{cases} T & \text{if } \Delta^* \vdash a \\ F & \text{if } \Delta^* \vdash \neg a. \end{cases}$$

We show, as a separate lemma below, that such defined variable assignment $v$ has the following property.
Property of $v$ Lemma

Let $v$ be the variable assignment defined above and $v^*$ its extension to the set $\mathcal{F}$ of all formulas.

For every formula $B \in \mathcal{F}$, the following is true

$$v^*(B) = \begin{cases} 
T & \text{if } \Delta^* \vdash B \\
F & \text{if } \Delta^* \vdash \neg B.
\end{cases}$$

Given Property of $v$ Lemma (still to be proved) we now prove that the $v$ is in fact, a **counter model for** any formula $A$, such that $\not\vdash A$. 
Let $A$ be such that $\not\models A$. By $E \neg A \in \Delta^*$ and obviously,

$\Delta^* \vdash \neg A$.

**Hence,** by the property of $v$,

$v^*(A) = F$,

what proves that $v$ is a **counter-model** for $A$ and hence **ends** the proof of the completeness theorem.

In order to really complete the proof we still have to show the Property of $v$ Lemma.
Proof of the Lemma (Property of $v$ lemma)

The proof is conducted by the induction on the degree of the formula $A$.

Initial step If $A$ is a propositional variable, then the Lemma is true holds by definition of $v$.

Inductive Step If $A$ is not a propositional variable, then $A$ is of the form $\neg C$ or $(C \Rightarrow D)$, for certain formulas $C, D$.

By the inductive assumption the Lemma holds for the formulas $C$ and $D$. 
Case $A = \neg C$

We have to consider two possibilities:

1. $\Delta^* \vdash A$,

2. $\Delta^* \vdash \neg A$.

Consider case 1. i.e. assume

$$\Delta^* \vdash A.$$  

It means that $\Delta^* \vdash \neg C$.

Then from the fact that $\Delta^*$ is consistent it must be that

$$\Delta^* \not\vdash C.$$
By the inductive assumption we have that
$v^*(C) = F$, and accordingly

\[ v^*(A) = v^*(\neg C) = \neg v^*(C) = \neg F = T. \]
Consider case 2. i.e. assume that
\[ \Delta^* \vdash \neg A. \]

Then from the fact that \( \Delta^* \) is consistent it must be that \( \Delta^* \not\vdash A \) and
\[ \Delta^* \not\vdash \neg C. \]

If so, then \( \Delta^* \vdash C \), as the set \( \Delta^* \) is complete.

By the inductive assumption, \( v^*(C) = T \), and accordingly
\[ v^*(A) = v^*(\neg C) = \neg v^*(C) = \neg T = F. \]
Thus \( A \) satisfies the \( v \) property Lemma.
Case \( A = (C \Rightarrow D) \).

As in the previous case, we assume that the Lemma holds for the formulas \( C, D \) and we consider two possibilities:

1. \( \Delta^* \vdash A \) and

2. \( \Delta^* \vdash \neg A \).

Case 1. Assume \( \Delta^* \vdash A \). It means that \( \Delta^* \vdash (C \Rightarrow D) \).

If at the same time \( \Delta^* \not\vdash C \), then \( v^*(C) = F \), and accordingly

\[
v^*(A) = v^*(C \Rightarrow D) = v^*(C) \Rightarrow v^*(D) = F \Rightarrow v^*(D) = T.
\]
If at the same time $\Delta^* \vdash C$, then since $\Delta^* \vdash (C \Rightarrow D)$, we infer, by Modus Ponens, that $\Delta^* \vdash D$.

If so, then

$$v^*(C) = v^*(D) = T,$$

and accordingly

$$v^*(A) = v^*(C \Rightarrow D) =$$

$$v^*(C) \Rightarrow v^*(D) = T \Rightarrow T = T.$$

Thus, if $\Delta^* \vdash A$, then $v^*(A) = T$. 
Case 2. Assume now, as before, that
\[ \Delta^* \vdash \neg A. \]

Then from the fact that \( \Delta^* \) is consistent it must be that \( \Delta^* \nvdash A \), i.e.,
\[ \Delta^* \nvdash (C \Rightarrow D). \]

It follows from this that
\[ \Delta^* \nvdash D, \]
for if \( \Delta^* \vdash D \), then, as \( (D \Rightarrow (C \Rightarrow D)) \) is provable formula 1 in \( S \), by monotonicity also
\[ \Delta^* \vdash (D \Rightarrow (C \Rightarrow D)). \]

Applying Modus Ponens we obtain
\[ \Delta^* \vdash (C \Rightarrow D), \]
which is contrary to the assumption.
Also we must have

$$\Delta^* \vdash C,$$

for otherwise, as $\Delta^*$ is complete we would have $\Delta^* \vdash \neg C$.

But this is impossible, since the formula $\neg C \Rightarrow (C \Rightarrow D)$ is assumed to be provable formula 9 in $S$ and by monotonicity

$$\Delta^* \vdash (\neg C \Rightarrow (C \Rightarrow D)).$$

Applying Modus Ponens we would get

$$\Delta^* \vdash (C \Rightarrow D),$$

which is contrary to the assumption.

This ends the proof of the lemma and completes the counter-model existence proof of the Completeness Theorem.