Chapter 6

Propositional Tautologies, Logical Equivalences, Definability of Connectives and Equivalence of Languages

Propositional Tautologies for Implication

Modus Ponens known to the Stoics (3rd century B.C)

$$\models ((A \cap (A \Rightarrow B)) \Rightarrow B)$$

Detachment

$$\models ((A \cap (A \Leftrightarrow B)) \Rightarrow B)$$

$$\models ((B \cap (A \Leftrightarrow B)) \Rightarrow A)$$

Sufficient Given an implication

$$(A \Rightarrow B),$$

A is called a *sufficient condition* for B to hold.

Necessary Given an implication

$$(A \Rightarrow B),$$

B is called a *necessary condition* for A to hold.

Implication Names

Simple $(A \Rightarrow B)$ is called *a simple implication*.

Converse $(B \Rightarrow A)$ is called *a converse implication* to $(A \Rightarrow B)$.

Opposite $(\neg B \Rightarrow \neg A)$ is called *an opposite implication* to $(A \Rightarrow B)$.

Contrary $(\neg A \Rightarrow \neg B)$ is called *a contrary implication* to $(A \Rightarrow B)$.

Laws of contraposition

$$\models ((A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)),$$

$$\models ((B \Rightarrow A) \Leftrightarrow (\neg A \Rightarrow \neg B)).$$

The laws of contraposition make it possible to replace, in any deductive argument, a sentence of the form $(A \Rightarrow B)$ by $\neg B \Rightarrow \neg A)$, and conversely.

Necessary and sufficient :

We read $(A \Leftrightarrow B)$ as

B is necessary and sufficient for A

because of the following tautology.

$$\models ((A \Leftrightarrow B)) \Leftrightarrow ((A \Rightarrow B) \cap (B \Rightarrow A))).$$

Hypothetical syllogism (Stoics, 3rd century B.C.)

$$\models (((A \Rightarrow B) \cap (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)),$$

$$\models ((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))),$$

$$\models ((B \Rightarrow C) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))).$$

Modus Tollendo Ponens (Stoics, 3rd century B.C.)

$$\models (((A \cup B) \cap \neg A) \Rightarrow B),$$

$$\models (((A \cup B) \cap \neg B) \Rightarrow A)$$

Duns Scotus (12/13 century)

$$\models (\neg A \Rightarrow (A \Rightarrow B))$$

Clavius (16th century)

$$\models ((\neg A \Rightarrow A) \Rightarrow A)$$

Frege (1879, first formulation of the classical propositional logic as a formalized axiomatic system)

$$\models (((A \Rightarrow (B \Rightarrow C)) \cap (A \Rightarrow B)) \Rightarrow (A \Rightarrow C)),$$

$$\models ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$$

Apagogic Proofs : means proofs by *reductio ad absurdum*.

Reductio ad absurdum : to prove A to be true, we assume $\neg A$.

If we get a contradiction, means we have proved A to be true.

$$\models ((\neg A \Rightarrow (B \cap \neg B)) \Rightarrow A)$$

Implication form : we want to prove $(A \Rightarrow B)$ by *reductio ad absurdum*. Correctness of reasoning is based on the following tautologies.

$$\models (((\neg(A \Rightarrow B) \Rightarrow (C \cap \neg C)) \Rightarrow (A \Rightarrow B)),$$

We use the equivalence: $\neg(A \Rightarrow B) \equiv (A \cap \neg B)$ and get

$$\models (((A \cap \neg B) \Rightarrow (C \cap \neg C)) \Rightarrow (A \Rightarrow B)).$$

$$\models (((A \cap \neg B) \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)).$$

$$\models (((A \cap \neg B) \Rightarrow B) \Rightarrow (A \Rightarrow B)).$$

Logical equivalence : For any formulas A, B,

$$A \equiv B \quad iff \quad \models (A \Leftrightarrow B).$$

Property:

$$A \equiv B \quad iff \quad \models (A \Rightarrow B) \;\; and \;\; \models (B \Rightarrow A).$$

Laws of contraposition

$$(A \Rightarrow B) \equiv (\neg B \Rightarrow \neg A),$$

$$(B \Rightarrow A) \equiv (\neg A \Rightarrow \neg B),$$

$$(\neg A \Rightarrow B) \equiv (\neg B \Rightarrow A),$$

$$(A \Rightarrow \neg B) \equiv (B \Rightarrow \neg A).$$

Theorem Let B_1 be obtained from A_1 by substitution of a formula B for one or more occurrences of a sub-formula A of A_1, what we denote as

$$B_1 = A_1(A/B).$$

Then the following holds.

$$If \quad A \equiv B, \quad then \quad A_1 \equiv B_1,$$

Definability of Connectives

$$(A \Rightarrow B) \equiv (\neg A \cup B)$$

Transform a formula with implication into a logically equivalent formula without implication.

We transform (via our Theorem) a formula

$$((C \Rightarrow \neg B) \Rightarrow (B \cup C))$$

into its logically equivalent form not containing $\Rightarrow$ as follows.

$$((C \Rightarrow \neg B) \Rightarrow (B \cup C)) \equiv (\neg(C \Rightarrow \neg B) \cup (B \cup C)))$$

$$\equiv (\neg(\neg C \cup B) \cup (B \cup C))).$$

We get

$$((C \Rightarrow \neg B) \Rightarrow (B \cup C)) \equiv (\neg(\neg C \cup B) \cup (B \cup C))).$$

Substitution Theorem Let B_1 be obtained from A_1 by substitution of a formula B for one or more occurrences of a sub-formula A of A_1.

We denote it as

$$B_1 = A_1(A/B).$$

Then the following holds.

$$If \quad A \equiv B, \quad then \quad A_1 \equiv B_1,$$

The next set of equivalences, or corresponding tautologies, deals with what is called *a definability of connectives* in classical semantics.

For example, a tautology

$$\models ((A \Rightarrow B) \Leftrightarrow (\neg A \cup B))$$

makes it possible to define implication in terms of disjunction and negation.

We state it in a form of logical equivalence as follows.

Definability of Implication in terms of negation and disjunction:

$$(A \Rightarrow B) \equiv (\neg A \cup B)$$

We use logical equivalence notion, instead of the tautology notion, as it makes the manipulation of formulas much easier.

Definability of Implication equivalence allows us, by the force of **Substitution Theorem to replace** any formula of the form $(A \Rightarrow B)$ placed anywhere in another formula by a formula $(\neg A \cup B)$.

Hence we transform a given formula containing implication into an logically equivalent formula that does contain implication (but contains negation and disjunction).

Example 1 We transform (via Substitution Theorem) a formula

$$((C \Rightarrow \neg B) \Rightarrow (B \cup C))$$

into its logically equivalent form not containing $\Rightarrow$ as follows.

$$((C \Rightarrow \neg B) \Rightarrow (B \cup C))$$

$$\equiv (\neg(C \Rightarrow \neg B) \cup (B \cup C)))$$

$$\equiv (\neg(\neg C \cup B) \cup (B \cup C))).$$

We get

$$((C \Rightarrow \neg B) \Rightarrow (B \cup C))$$

$$\equiv (\neg(\neg C \cup B) \cup (B \cup C))).$$

It means that that we can, by the Substitution Theorem transform a language

$$\mathcal{L}_1 = \mathcal{L}_{\{\neg, \cap, \Rightarrow\}}$$

into a language

$$\mathcal{L}_2 = \mathcal{L}_{\{\neg, \cap, \cup\}}$$

with all its formulas being logically equivalent.

We write it as the following condition.

C1: for any formula A of $\mathcal{L}_1$, there is a formula B of $\mathcal{L}_2$, such that $A \equiv B$.

Example 2 : Let A be a formula

$$(\neg A \cup (\neg A \cup \neg B))$$

.

We use the definability of implication equivalence to **eliminate disjunction** as follows

$$(\neg A \cup (\neg A \cup \neg B)) \equiv (\neg A \cup (A \Rightarrow \neg B))$$

$$\equiv (A \Rightarrow (A \Rightarrow \neg B)).$$

Observe, that we can't always use the equivalence $(A \Rightarrow B) \equiv (\neg A \cup B)$ to eliminate any disjunction.

For example, we can't use it for a formula

$$A = ((a \cup b) \cap \neg a).$$

In order to be able to transform *any formula* of a language containing **disjunction** (and some other connectives) into a language with negation and implication (and some other connectives), but **without disjunction** we need the following logical equivalence.

Definability of Disjunction in terms of negation and implication:

$$(A \cup B) \equiv (\neg A \Rightarrow B)$$

Example 3 Consider a formula A

$$(a \cup b) \cap \neg a).$$

We transform A into its logically equivalent form not containing $\cup$ as follows.

$$((a \cup b) \cap \neg a) \equiv ((\neg a \Rightarrow b) \cap \neg a).$$

In general, we transform the language $\mathcal{L}_2 = \mathcal{L}_{\{\neg, \cap, \cup\}}$ to the language $\mathcal{L}_1 = \mathcal{L}_{\{\neg, \cap, \Rightarrow\}}$ with all its formulas being logically equivalent.

We write it as the following condition.

C1: for any formula C of $\mathcal{L}_2$, there is a formula D of $\mathcal{L}_1$, such that $C \equiv D$.

The languages $\mathcal{L}_1$ and $\mathcal{L}_2$ for which we the conditions **C1, C2** hold are called **logically equivalent**.

We denote it by

$$\mathcal{L}_1 \equiv \mathcal{L}_2.$$

A general, formal definition goes as follows.

Definition of Equivalence of Languages

Given two languages: $\mathcal{L}_1 = \mathcal{L}_{CON_1}$ and $\mathcal{L}_2 = \mathcal{L}_{CON_2}$, for $CON_1 \neq CON_2$.

We say that they are logically equivalent, i.e.

$$\mathcal{L}_1 \equiv \mathcal{L}_2$$

if and only if the following conditions **C1, C2** hold.

C1: For every formula A of $\mathcal{L}_1$, there is a formula B of $\mathcal{L}_2$, such that

$$A \equiv B,$$

C2: For every formula C of $\mathcal{L}_2$, there is a formula D of $\mathcal{L}_1$, such that

$$C \equiv D.$$

Example 4 To prove the logical equivalence of the languages

$$\mathcal{L}_{\{\neg,\cup\}} \equiv \mathcal{L}_{\{\neg,\Rightarrow\}}$$

we need two definability equivalences:

implication in terms of disjunction and negation,

disjunction in terms of implication and negation, and the **Substitution Theorem.**

Example 5 To prove the logical equivalence of the languages

$$\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}} \equiv \mathcal{L}_{\{\neg,\cap,\cup\}}$$

we need only the definability of implication equivalence.

It proves, by Substitution Theorem that *for any* formula A of

$$\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$$

there is B of $\mathcal{L}_{\{\neg,\cap,\cup\}}$ that equivalent to A, i.e.

$$A \equiv B$$

and condition **C1** holds.

Observe, that any formula A of language

$$\mathcal{L}_{\{\neg,\cap,\cup\}}$$

is also a formula of

$$\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$$

and of course

$$A \equiv A,$$

so **C2** also holds.

The logical equalities below

Definability of Conjunction in terms of implication and negation

$$(A \cap B) \equiv \neg(A \Rightarrow \neg B),$$

Definability of Implication in terms of conjunction and negation

$$(A \Rightarrow B) \equiv \neg(A \cap \neg B),$$

and the **Substitution Theorem** prove that

$$\mathcal{L}_{\{\neg,\cap\}} \equiv \mathcal{L}_{\{\neg,\Rightarrow\}}.$$

Exercise 1

(a) Prove that

$$\mathcal{L}_{\{\cap,\neg\}} \equiv \mathcal{L}_{\{\cup,\neg\}}.$$

(b) Transform a formula $A = \neg(\neg(\neg a \cap \neg b) \cap a)$ of $\mathcal{L}_{\{\cap,\neg\}}$ into a logically equivalent formula B of $\mathcal{L}_{\{\cup,\neg\}}$.

(c) Transform a formula $A = (((\neg a \cup \neg b) \cup a) \cup (a \cup \neg c))$ of $\mathcal{L}_{\{\cup,\neg\}}$ into a formula B of $\mathcal{L}_{\{\cap,\neg\}}$, such that $A \equiv B$.

(d) Prove/disaprove: $\models \neg(\neg(\neg a \cap \neg b) \cap a)$.

(e) Prove/disaprove: $\models (((\neg a \cup \neg b) \cup a) \cup (a \cup \neg c))$.

Solution (a) True due to the Substitution Theorem and two definability of connectives equivalences:

$$(A \cap B) \equiv \neg(\neg A \cup \neg B), \quad (A \cup B) \equiv \neg(\neg A \cap \neg B).$$

Solution (b)

$$\neg(\neg(\neg a \cap \neg b) \cap a)$$

$$\equiv \neg(\neg\neg(\neg\neg a \cup \neg\neg b) \cap a)$$

$$\equiv \neg((a \cup b) \cap a)$$

$$\equiv \neg(\neg(a \cup b) \cup \neg a).$$

The formula B of $\mathcal{L}_{\{\cup,\neg\}}$ equivalent to A is

$$B = \neg(\neg(a \cup b) \cup \neg a).$$

Solution (c)

$$(((\neg a \cup \neg b) \cup a) \cup (a \cup \neg c))$$

$$\equiv((\neg(\neg\neg a \cap \neg\neg b) \cup a) \cup \neg(\neg a \cap \neg\neg c))$$

$$\equiv ((\neg(a \cap b) \cup a) \cup \neg(\neg a \cap c))$$

$$\equiv (\neg(\neg\neg(a \cap b) \cap \neg a) \cup \neg(\neg a \cap c))$$

$$\equiv (\neg((a \cap b) \cap \neg a) \cup \neg(\neg a \cap c))$$

$$\equiv \neg(\neg\neg((a \cap b) \cap \neg a) \cap \neg\neg(\neg a \cap c))$$

$$\equiv\neg(((a \cap b) \cap \neg a) \cap (\neg a \cap c))$$

There are two formulas B of $\mathcal{L}_{\{\cap,\neg\}}$, such that $A \equiv B$.

$$B = B_1 = \neg(\neg\neg((a \cap b) \cap \neg a) \cap \neg\neg(\neg a \cap c)),$$

$$B = B_2 = \neg(((a \cap b) \cap \neg a) \cap (\neg a \cap c)).$$

Solution (d)

$$\not\models \neg(\neg(\neg a \cap \neg b) \cap a)$$

Our formula A is logically equivalent, as proved in **(c)** with the formula
$B = \neg(\neg(a \cup b) \cup \neg a)$.

Consider any truth assignment v, such that
$v(a) = F$, then
$(\neg(a \cup b) \cup T) = T$,
and hence $v^*(B) = F$.

Solution (e)

$$\models (((\neg a \cup \neg b) \cup a) \cup (a \cup \neg c))$$

because it was proved in **(c)** that

$$(((\neg a \cup \neg b) \cup a) \cup (a \cup \neg c))$$

$$\equiv \neg(((a \cap b) \cap \neg a) \cap (\neg a \cap c))$$

and obviously the formula

$$(((a \cap b) \cap \neg a) \cap (\neg a \cap c))$$

is a contradiction.

Hence its negation is a tautology.

Exercise 2 Prove by transformation, using proper logical equivalences that

1.

$$\neg(A \Leftrightarrow B) \equiv ((A \cap \neg B) \cup (\neg A \cap B)),$$

2.

$$((B \cap \neg C) \Rightarrow (\neg A \cup B))$$

$$\equiv ((B \Rightarrow C) \cup (A \Rightarrow B)).$$

Solution 1.

$$\neg(A \Leftrightarrow B)$$

$$\equiv^{def} \neg((A \Rightarrow B) \cap (B \Rightarrow A))$$

$$\equiv^{de\ Morgan} (\neg(A \Rightarrow B) \cup \neg(B \Rightarrow A))$$

$$\equiv^{neg\ impl} ((A \cap \neg B) \cup (B \cap \neg A))$$

$$\equiv^{commut} ((A \cap \neg B) \cup (\neg A \cap B)).$$

Solution 2.

$$((B \cap \neg C) \Rightarrow (\neg A \cup B))$$

$$\equiv^{impl} (\neg(B \cap \neg C) \cup (\neg A \cup B))$$

$$\equiv^{de\ Morgan} ((\neg B \cup \neg\neg C) \cup (\neg A \cup B))$$

$$\equiv^{neg} ((\neg B \cup C) \cup (\neg A \cup B))$$

$$\equiv^{impl} ((B \Rightarrow C) \cup (A \Rightarrow B)).$$

SOME PROBLEMS: chapters 5,6

Reminder: We define **H** semantics operations $\cup$ and $\cap$ as follows

$$a \cup b = max\{a, b\}, \qquad a \cap b = min\{a, b\}.$$

The Truth Tables for Implication and Negation are:

H-Implication

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	F	T	T
T	F	$\perp$	T

H Negation

$\neg$	F	$\perp$	T
	T	F	F

QUESTION 1 We know that

$$v : VAR \longrightarrow \{F, \perp, T\}$$

is such that

$$v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \perp$$

under **H** semantics.

evaluate:

$v^*((((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b))$.

Solution : $v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \perp$ under H semantics if and only if (we use shorthand notation) $(a \cap b) = T$ and $(a \Rightarrow c) = \perp$ if and only if $a = T, b = T$ and $(T \Rightarrow c) = \perp$ if and only i f $c = \perp$. I.e. we have that

$$v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \perp \;\; iff \;\; a = T, \; b = T, c = \perp$$

Now we can we **evaluate** $v^*(((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b))$ as follows (in shorthand notation).

$v^*(((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b)) =$
$(((T \Rightarrow T) \Rightarrow (T \Rightarrow \neg \perp)) \cup (T \Rightarrow T)) =$
$((T \Rightarrow (T \Rightarrow F)) \cup T) = T$.

We define a 4 valued $\L_4$ logic semantics as follows. The language is $\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$.

We define the logical connectives $\neg, \Rightarrow, \cup, \cap$ of $\L_4$ as the following operations in the set $\{F, \bot_1, \bot_2, T\}$, where $\{F < \bot_1 < \bot_2 < T\}$.

Negation $\neg : \{F, \bot_1, \bot_2, T\} \longrightarrow \{F, \bot_1, \bot_2, T\}$,

such that

$$\neg\bot_1 = \bot_1, \ \neg\bot_2 = \bot_2, \quad \neg F = T, \quad \neg T = F.$$

Conjunction $\cap : \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \longrightarrow \{F, \ \bot_1, \bot_2, , T\}$

such that for any $a, b \in \{F, \bot_1, \bot_2, T\}$,

$$a \cap b = min\{a, b\}.$$

Disjunction $\cup : \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \longrightarrow \{F, \bot_1, \bot_2, T\}$

such that for any $a, b \in \{F, \bot_1, \bot_2, T\}$,

$$a \cup b = max\{a, b\}.$$

Implication $\Rightarrow: \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \longrightarrow \{F, \bot_1, \bot_2, T\}$,

such that for any $a, b \in \{F, \bot_1, \bot_2, T\}$,

$$a \Rightarrow b = \begin{cases} \neg a \cup b & \text{if } a > b \\ T & \text{otherwise} \end{cases}$$

QUESTION 2

Part 1 Write all Tables for **Ł**$_4$

Solution :

Ł$_4$ Negation

$\neg$	F	$\bot_1$	$\bot_2$	T
	T	$\bot_1$	$\bot_2$	F

Ł$_4$ Conjunction

$\cap$	F	$\bot_1$	$\bot_2$	T
F	F	F	F	F
$\bot_1$	F	$\bot_1$	$\bot_1$	$\bot_1$
$\bot_2$	F	$\bot_1$	$\bot_2$	$\bot_2$
T	F	$\bot_1$	$\bot_2$	T

Ł$_4$ Disjunction

$\cup$	F	$\bot_1$	$\bot_2$	T
F	F	$\bot_1$	$\bot_2$	T
$\bot_1$	$\bot_1$	$\bot_1$	$\bot_2$	T
$\bot_2$	$\bot_2$	$\bot_2$	$\bot_2$	T
T	T	T	T	T

Ł$_4$-Implication

$\Rightarrow$	F	$\bot_1$	$\bot_2$	T
F	T	T	T	T
$\bot_1$	$\bot_1$	T	T	T
$\bot_2$	$\bot_2$	$\bot_2$	T	T
T	F	$\bot_1$	$\bot_2$	T

Part 2 Verify whether

$$\models_{Ł_4}((a \Rightarrow b) \Rightarrow (\neg a \cup b))$$

Solution : Let v be a truth assignment such that $v(a) = v(b) = \bot_1$.

We evaluate $v^*((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = ((\bot_1 \Rightarrow \bot_1) \Rightarrow (\neg\bot_1 \cup \bot_1)) = (T \Rightarrow (\bot_1 \cup \bot_1)) = (T \Rightarrow \bot_1) = \bot_1$.

This proves that v is a counter-model for our formula and

$$\not\models_{\text{Ł}_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b)).$$

Observe that a v such that $v(a) = v(b) = \bot_2$ is also a counter model, as $v^*((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = ((\bot_2 \Rightarrow \bot_2) \Rightarrow (\neg \bot_2 \cup \bot_2)) = (T \Rightarrow (\bot_2 \cup \bot_2)) = (T \Rightarrow \bot_2) = \bot_2$.

QUESTION 3 Prove using proper logical equivalences (list them at each step) that

1. $\neg(A \Leftrightarrow B) \equiv ((A \cap \neg B) \cup (\neg A \cap B))$,

Solution: $\neg(A \Leftrightarrow B) \equiv^{def} \neg((A \Rightarrow B) \cap (B \Rightarrow A)) \equiv^{deMorgan} (\neg(A \Rightarrow B) \cup \neg(B \Rightarrow A))$ $\equiv^{negimpl} ((A \cap \neg B) \cup (B \cap \neg A)) \equiv^{commut} ((A \cap \neg B) \cup (\neg A \cap B))$.

2. $((B \cap \neg C) \Rightarrow (\neg A \cup B)) \equiv ((B \Rightarrow C) \cup (A \Rightarrow B))$.

Solution: $((B \cap \neg C) \Rightarrow (\neg A \cup B)) \equiv^{impl} (\neg(B \cap \neg C) \cup (\neg A \cup B)) \equiv^{deMorgan} ((\neg B \cup \neg\neg C) \cup (\neg A \cup B))$
$\equiv^{dneg} ((\neg B \cup C) \cup (\neg A \cup B)) \equiv^{impl} ((B \Rightarrow C) \cup (A \Rightarrow B))$.

QUESTION 4 We define an EQUIVALENCE of LANGUAGES as follows:

Given two languages:
$\mathcal{L}_1 = \mathcal{L}_{CON_1}$ and $\mathcal{L}_2 = \mathcal{L}_{CON_2}$, for $CON_1 \neq CON_2$.
We say that they are **logically equivalent**, i.e.

$$\mathcal{L}_1 \equiv \mathcal{L}_2$$

if and only if the following conditions **C1, C2** hold.

C1: For every formula A of $\mathcal{L}_1$, there is a formula B of $\mathcal{L}_2$, such that

$$A \equiv B,$$

C2: For every formula C of $\mathcal{L}_2$, there is a formula D of $\mathcal{L}_1$, such that

$$C \equiv D.$$

Prove that $\mathcal{L}_{\{\neg,\cap\}} \equiv \mathcal{L}_{\{\neg,\Rightarrow\}}$.

Solution: The equivalence of languages holds due to two definability of connectives equivalences:

$$(A \cap B) \equiv \neg(A \Rightarrow \neg B), \quad (A \Rightarrow B) \equiv \neg(A \cap \neg B).$$