Chapter 4: Classical Propositional Semantics

Language :

$$\mathcal{L}_{\{\neg, \cup, \cap, \Rightarrow\}}.$$

Classical Semantics assumptions:

TWO VALUES: there are only two logical values: truth (T) and false (F), and

EXTENSIONALITY: the logical value of a formula depends only on a main connective and logical values of its sub-formulas.

We define formally a **classical semantics** for $\mathcal{L}$ in terms of two factors: classical truth tables and a truth assignment.

We summarize now here the chapter 2 tables for $\mathcal{L}_{\{\neg,\cup,\cap,\Rightarrow\}}$ in one simplified table as follows.

A	B	$\neg A$	$(A \cap B)$	$(A \cup B)$	$(A \Rightarrow B)$
T	T	F	T	T	T
T	F	F	F	T	F
F	T	T	F	T	T
F	F	T	F	F	T

Observe that The first row of the above table reads:

For any formulas A, B, if the logical value of $A = T$ and $B = T$, then logical values of $\neg A = T$, $(A \cap B) = T$, $(A \cup B) = T$ and $(A \Rightarrow B) = T$.

We read and write the other rows in a similar manner.

Our table indicates that the logical value of of propositional connectives depends **only** on the logical values of its factors; i.e. it is **independent of the formulas** A, B.

EXTENSIONAL CONNECTIVES : The logical value of a given connective depend only of the logical values of its factors.

We write now the last table as the following equations.

$$\neg T = F, \quad \neg F = T;$$

$$(T \cap T) = T, \quad (T \cap F) = F, \quad (F \cap T) = F, \quad (F \cap F) = F;$$

$$(T \cup T) = T, \quad (T \cup F) = T, \quad (F \cup T) = T, \quad (F \cup F) = F;$$

$$(T \Rightarrow T) = T, \quad (T \Rightarrow F) = F, \quad (F \Rightarrow T) = T, \quad (F \Rightarrow F) = T.$$

Observe now that the above equations describe a set of unary and binary operations (functions) defined on a set $\{T, F\}$ and a set $\{T, F\} \times \{T, F\}$, respectively.

Negation $\neg$ is a function:

$$\neg : \quad \{T, F\} \longrightarrow \{T, F\},$$

such that $\neg T = F$, $\neg F = T$.

Conjunction $\cap$ is a function:

$$\cap : \quad \{T, F\} \times \{T, F\} \longrightarrow \{T, F\},$$

such that
$(T \cap T) = T, \quad (T \cap F) = F,$
$(F \cap T) = F, \quad (F \cap F) = F.$

Dissjunction $\cup$ is a function:

$$\cup: \quad \{T, F\} \times \{T, F\} \longrightarrow \{T, F\},$$

such that
$(T \cup T) = T, \quad (T \cup F) = T,$
$(F \cup T) = T, \quad (F \cup F) = F.$

Implication $\Rightarrow$ is a function:

$$\Rightarrow: \quad \{T, F\} \times \{T, F\} \longrightarrow \{T, F\},$$

such that
$(T \Rightarrow T) = T, \quad (T \Rightarrow F) = F,$
$(F \Rightarrow T) = T, \quad (F \Rightarrow F) = T.$

Observe that if we have have a language $\mathcal{L}_{\{\neg,\cup,\cap,\Rightarrow,\Leftrightarrow\}}$ containing also the equivalence connective $\Leftrightarrow$ we define

$$\Leftrightarrow: \quad \{T, F\} \times \{T, F\} \longrightarrow \{T, F\},$$

as a function such that
$(T \Leftrightarrow T) = T, \quad (T \Leftrightarrow F) = F,$
$(F \Leftrightarrow T) = F, \quad (T \Leftrightarrow T) = T.$

We write these definitions of connectives as the following tables, usually called the **classical truth tables.**

Negation :

$\neg$	T	F
	F	T

Disjunction :

$\cup$	T	F
T	T	T
F	T	F

Conjunction :

$\cap$	T	F
T	T	F
F	F	F

Implication :

$\Rightarrow$	T	F
T	T	F
F	T	T

Equivalence :

$\Leftrightarrow$	T	F
T	T	F
F	F	T

A truth assignment is any function

$$v : VAR \longrightarrow \{T, F\}.$$

Observe that the truth assignment is defined only on variables (atomic formulas).

We define its **extension** v^* to the set $\mathcal{F}$ of all formulas of $\mathcal{L}$ as follows.

$$v^* : \mathcal{F} \longrightarrow \{T, F\}$$

is such that

(i) for any $a \in VAR$,

$$v^*(a) = v(a);$$

(ii) and for any $A, B \in \mathcal{F}$,

$$v^*(\neg A) = \neg v^*(A);$$

$$v^*(A \cap B) = (v^*(A) \cap v^*(B));$$

$$v^*(A \cup B) = (v^*(A) \cup v^*(B));$$

$$v^*(A \Rightarrow B) = (v^*(A) \Rightarrow v^*(B)),$$

$$v^*(A \Leftrightarrow B) = (v^*(A) \Leftrightarrow v^*(B)),$$

where

the symbols on the **left-hand side** of the equations represent connectives in their **natural language meaning** and

the symbols on the **right-hand side** represent connectives in their **logical meaning** given by the classical truth tables.

Example

Consider a formula

$$((a \Rightarrow b) \cup \neg a))$$

a truth assignment v such that

$$v(a) = T, v(b) = F.$$

We calculate the logical value of the formula A as follows: $v^*(A) = v^*((a \Rightarrow b) \cup \neg a)) = (v^*(a \Rightarrow b) \cup v^*(\neg a)) = ((v(a) \Rightarrow v(b)) \cup \neg v(a)) = ((T \Rightarrow F) \cup \neg T) = (F \cup F) = \cup(F, F) = F$.

Observe that we did not need (and usually we don't) to specify the $v(x)$ of any $x \in VAR - \{a, b\}$, as these values do not influence the computation of the logical value $v^*(A)$.

SATISFACTION relation

Definition: Let $v : VAR \longrightarrow \{T, F\}$. We say that
v **satisfies a formula** $A \in \mathcal{F}$ iff $v^*(A) = T$

Notation: $v \models A.$

Definition: We sat that
v **does not satisfy a formula** $A \in \mathcal{F}$ iff $v^*(A) \neq T$.

Notation: $v \not\models A.$

REMARK In our classical semantics we have that
$v \not\models A$ iff $v^*(A) = F$ and we say that v **falsifies the formula** A.

OBSERVE $v^*(A) \neq T$ is is equivalent to the fact that $v^*(A) = F$ ONLY in 2-valued logic!

This is why we adopt the following

Definition: For any v,
v **does not satisfy a formula** $A \in \mathcal{F}$ iff $v^*(A) \neq T$

Example

$$A = ((a \Rightarrow b) \cup \neg a))$$

$$v : VAR \longrightarrow \{T, F\}$$

such that $v(a) = T, v(b) = F$.

Calculation of $v^*(A)$ using the short hand notation:

$((T \Rightarrow F) \cup \neg T) = (F \cup F) = F.$

$$v \not\models ((a \Rightarrow b) \cup \neg a)).$$

Observe that we did not need (and usually we don't) to specify the $v(x)$ of any $x \in VAR - \{a, b\}$, as these values do not influence the computation of the logical value $v^*(A)$.

Example

$$A = ((a \cap \neg b) \cup \neg c)$$

$$v : VAR \longrightarrow \{T, F\}$$

such that $v(a) = T, v(b) = F, v(c) = T$.

Calculation in a short hand notation:

$$(T \cap \neg F) \cup \neg T = (T \cap T) \cup F = T \cup F = T.$$

$$v \models ((a \cap \neg b) \cup \neg c).$$

Formula: $A = ((a \cap \neg b) \cup \neg c)$.

Consider now $v_1 : VAR \longrightarrow \{T, F\}$ such that $v_1(a) = T, v_1(b) = F, v_1(c) = T$, and $v_1(x) = F$, for all $x \in VAR - \{a, b, c\}$,

Observe: $v(a) = v_1(a), v(b) = v_1(b), v(c) = v_1(c)$, so we get

$$v_1 \models ((a \cap \neg b) \cup \neg c).$$

Consider $v_2 : VAR \longrightarrow \{T, F\}$ such that
$v_2(a) = T, v_2(b) = F, v_2(c) = T, v_2(d) = T,$
and
$v_2(x) = F, \quad$ for all $x \in VAR - \{a, b, c, d\}$,

Observe: $v(a) = v_2(a), v(b) = v_2(b), v(c) = v_2(c)$, so we get

$$v_2 \models ((a \cap \neg b) \cup \neg c).$$

We are going to prove that there are as many of such truth assignments as real numbers! but they are all *the same* as the first v with respect to the formula A.

When we ask a question: *"How many truth assignments satisfy/fasify a formula A?"* we mean to find all assignment that are *different on the formula A*, not just different on a set VAR of all variables, as all of our v_1, v_2's were.

To address and to answer this question formally we first introduce some notations and definitions.

Notation: for any formula A, we denote by

$$VAR_A$$

a set of **all variables that appear in** A.

Definition: Given a formula $A \in \mathcal{F}$, any function

$$w : VAR_A \longrightarrow \{T, F\}$$

is called a **truth assignment restricted** to A.

Example

$$A = ((a \cap \neg b) \cup \neg c)$$

$$VAR_A = \{a, b, c\}$$

Truth assignment restricted to A is any function:

$$w : \{a, b, c\} \longrightarrow \{T, F\}.$$

We use the following theorem to count all possible truth assignment restricted to A.

Counting Functions Theorem (1) For any **finite sets** A and A, if A has **n** elements and B has **m** elements, then there are $\mathbf{m}^{\mathbf{n}}$ possible functions that map A into B.

There are $2^3 = 8$ truth assignment restricted to $A = ((a \Rightarrow \neg b) \cup \neg c)$.

General case For any A there are

$$2^{|VAR_A|}$$

possible truth assignments w restricted to A.

All w **restricted to** A are listed in the table below.

$$A = ((a \cap \neg b) \cup \neg c)$$

w	a	b	c	$w^*(A)$ *computation*	$w^*(A)$
w_1	T	T	T	$(T \Rightarrow T) \cup \neg T = T \cup F = T$	T
w_2	T	T	F	$(T \Rightarrow T) \cup \neg F = T \cup T = T$	T
w_3	T	F	F	$(T \Rightarrow F) \cup \neg F = F \cup T = T$	T
w_4	F	F	T	$(F \Rightarrow F) \cup \neg T = T \cup F = T$	T
w_5	F	T	T	$(F \Rightarrow T) \cup \neg T = T \cup F = T$	T
w_6	F	T	F	$(F \Rightarrow T) \cup \neg F = T \cup T = T$	T
w_7	T	F	T	$(T \Rightarrow F) \cup \neg T = F \cup F = F$	F
w_8	F	F	F	$(F \Rightarrow F) \cup \neg F = T \cup T = T$	T

Model for A is a v such that

$$v \models A.$$

$w_1, w_2, w_3, w_4 w_5, w_6, w_8$ are **models** for A.

Counter- Model for A is a v such that

$$v \not\models A.$$

w_7 is **a counter- model** for A.

Tautology :

A is **a tautology** iff any v is **a model** for A, i.e.

$$\forall v \ (v \models A).$$

Not a tautology :

A is **not a tautology** iff there is $v : VAR \longrightarrow \{T, F\}$, such that v is **a counter-model** for A, i.e.

$$\exists v \ (v \not\models A).$$

Tautology Notation $\models A$

Example

$$\not\models ((a \cap \neg b) \cup \neg c)$$

because the truth assignment w_7 is a counter-model for A.

Tautology Verification

Truth Table Method: list and evaluate all possible truth assignments restricted to A.

Example: $(a \Rightarrow (a \cup b))$.

v	a	b	$v^*(A)$ *computation*	$v^*(A)$
v_1	T	T	$(T \Rightarrow (T \cup T)) = (T \Rightarrow T) = T$	T
v_2	T	F	$(T \Rightarrow (T \cup F)) = (T \Rightarrow T) = T$	T
v_3	F	T	$(F \Rightarrow (F \cup T)) = (F \Rightarrow T) = T$	T
v_4	F	F	$(F \Rightarrow (F \cup F)) = (F \Rightarrow F) = T$	T

for all $v : VAR \longrightarrow \{T, F\}$, $v \models A$, i.e.

$$\models (a \Rightarrow (a \cup b)).$$

Proof by Contradiction Method

One works backwards, trying to find a truth assignment v which makes a formula A false.

If we find one, it means that A is not a tautology,

if we prove that it is impossible ,

it means that the formula is a tautology.

Example $A = (a \Rightarrow (a \cup b)$

Step 1 Assume that $\not\models A$, i.e. $A = F$.

Step 2 Analyze Strep 1:

$(a \Rightarrow (a \cup b)) = F$ iff $a = T$ and $a \cup b = F$.

Step 3 Analyze Step 2:

$a = T$ and $a \cup b = F$, i.e. $T \cup b = F$.

This is impossible by the definition of $\cup$.

Conclusion:

$$\models \quad (a \Rightarrow (a \cup b)).$$

Observe that exactly the same reasoning proves that for any formulas $A, B \in \mathcal{F}$,

$$\models (A \Rightarrow (A \cup B)).$$

Observe that he following formulas are tautologies

$$((((a \Rightarrow b) \cap \neg c) \Rightarrow ((((a \Rightarrow b) \cap \neg c) \cup \neg d)),$$

$$(((a \Rightarrow b) \cap \neg C) \cup d) \cap \neg e) \Rightarrow$$

$$(((a \Rightarrow b) \cap \neg C) \cup d) \cap \neg e) \cup ((a \Rightarrow \neg e)))$$

because they are of the form

$$(A \Rightarrow (A \cup B)).$$

Tautologies, Contradictions

$$\mathbf{T} = \{A \in \mathcal{F} : \ \models A\},$$

$$\mathbf{C} = \{A \in \mathcal{F} : \ \forall v \ (v \not\models A)\}.$$

Theorem 1 For any formula $A \in \mathcal{F}$ the following conditions are equivalent.

(1) A is a tautology

(2) $A \in \mathbf{T}$

(3) $\neg A$ is a contradiction

(4) $\neg A \in \mathbf{C}$

(5) $\forall v \ (v^*(A) = T)$

(6) $\forall v \ (v \models A)$

(7) Every v is a model for A

Theorem 2 For any formula $A \in \mathcal{F}$ the following conditions are equivalent.

(1) A is a contradiction

(2) $A \in \mathbf{C}$

(3) $\neg A$ is a tautology

(4) $\neg A \in \mathbf{T}$

(5) $\forall v \ (v^*(A) = F)$

(6) $\forall v \ (v \not\models A)$

(7) A does not have a model.