CHAPTER 14

Hilbert System for Predicate Logic

1 Completeness Theorem for First Order Logic

There are many proofs of the Completeness Theorem for First Order Logic. We follow here a version of Henkin's proof, as presented in the *Handbook of Mathematical Logic*. It contains a method for reducing certain problems of first-order logic back to problems about propositional logic. We give independent proof of Compactness Theorem for propositional logic. The Compactness Theorem for first-order logic and Löwenheim-Skolem Theorems and the Gödel Completeness Theorem fall out of the Henkin method.

1.1 Compactness Theorem for Propositional Logic

Let $\mathcal{L} = \mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C})$ be a first order language with equality. We assume that the sets **P, F, C** are infinitely enumerable. We define a **propositional logic** within it as follows.

Prime formulas We consider a subset P of the set $\mathcal{F}$ of all formulas of $\mathcal{L}$. Intuitively these are formulas of $\mathcal{L}$ which are not direct propositional combination of simpler formulas, that is, *atomic formulas* ($A\mathcal{F}$) and formulas beginning with quantifiers.

Formally, we have that

$P = \{A \in \mathcal{F} : \quad A \in A\mathcal{F} \quad or \quad A = \forall x B, \quad A = \exists x B \quad for \quad B \in \mathcal{F}\}.$

Example 1.1 *The following are primitive formulas.*

$R(t_1, t_2), \quad \forall x(A(x) \Rightarrow \neg A(x)), \quad (c = c), \quad \exists x(Q(x, y) \cap \forall y A(y)).$

The following are not primitive formulas.

$(R(t_1, t_2) \Rightarrow (c = c)), \quad (R(t_1, t_2) \cup \forall x(A(x) \Rightarrow \neg A(x)).$

Given a set P of primitive formulas we define in a standard way the set $P\mathcal{F}$ of *propositional formulas* as follows.

Propositional formulas The smallest set $P\mathcal{F} \subset \mathcal{F}$ such that

1. $P \subset P\mathcal{F}$

2. If $A, B \in P\mathcal{F}$, then $(A \Rightarrow B), (A \cup B), (A \cap B)$, and $\neg A \in P\mathcal{F}$

is called a set of propositional formulas of the first order language $\mathcal{L}$.

We define propositional semantics for propositional formulas in $P\mathcal{F}$ as follows.

Truth assignment Let P be a set of prime formulas and $\{T, F\}$ be a two element set, thought as the set of logical values "true" and "false". Any function

$$v : P \longrightarrow \{T, F\}$$

is called *truth assignment* (or variable assignment).

Let $\mathbf{B} = (\{T, F\}, \Rightarrow, \cup, \cap, \neg)$ be a two-element Boolean algebra and $\mathbf{PF} = (P\mathcal{F}, \Rightarrow, \cup, \cap, \neg)$ a similar algebra of propositional formulas.

We extend v to a homomorphism

$v^* : \mathbf{PF} \longrightarrow \mathbf{B}$

in a usual way, i.e. we put $v^*(A) = v(A)$ for $A \in P$, and for any $A, B \in P\mathcal{F}$,

$v^*(A \Rightarrow B) = v^*(A) \Rightarrow v^*(B)$,

$v^*(A \cup B) = v^*(A) \cup v^*(B)$,

$v^*(A \cap B) = v^*(A) \cap v^*(B)$,

$v^*(\neg A) = \neg v^*(A)$.

Propositional Model A truth assignment v is called a *propositional model* for a formula $A \in P\mathcal{F}$ iff $v^*(A) = T$.

Propositional Tautology A formula $A \in P\mathcal{F}$ is a *propositional tautology* if $v^*(A) = T$ for all $v : P \longrightarrow \{T, F\}$.

For the sake of simplicity we will often say *model, tautology* instead *propositional model, propositional tautology.*

Model for the Set Given a set S of propositional formulas. We say that v is a *model for the set* S if v is a model for all formulas $A \in S$.

Consistent Set A set S of propositional formulas is *consistent* (in a sense of propositional logic) if it has a (propositional) model.

Theorem 1.1 (Compactness Theorem for Propositional Logic) *A set S of propositional formulas is* consistent *if and only if every finite subset of S is consistent.*

proof If S is a consistent set, then its model is also a model for all its finite subsets and all its finite subsets are consistent.

We prove the nontrivial half of the Compactness Theorem in a slightly modified form. To do so, we introduce the following definition.

Finitely Consistent Set (FC) Any set S such that all its subsets are consistent is called finitely consistent.

We use this definition to re-write the Compactness Theorem as: *A set S of propositional formulas is* consistent *if and only it is* finitely consistent. The nontrivial half of it is:

Every finitely consistent set of propositional formulas is consistent.

The proof of the nontrivial half of the Compactness Theorem, as stated above, consistes of the following four steps.

Step 1 We introduce the notion of a *maximal finitely consistent set.*

Step 2 We show that every *maximal finitely consistent set* is consistent by constructing its model.

Step 3 We show that every *finitely consistent set S* can be extended to a *maximal finitely consistent set S^**. I.e we show that for every finitely consistent set S there is a set S^*, such that $S \subset S^*$ and S^* is maximal finitely consistent.

Step 4 We use steps 2 and 3 to justify the following reasoning. Given a *finitely consistent* set S. We extend it, via construction defined in the step 2 to a *maximal finitely consistent* set S^*. By the step 2, S^* is consistent and hence so is the set S, what ends the proof.

Step 1: Maximal Finitely Consistent Set We call S *maximal finitely consistent* if S is finitely consistenst and for every formula A, either $A \in S$.

We use notation MFC for maximal finitely consistent set, and FC for the finitely consistent set.

Step 2: Any MFC set is consistent Given a MFC set S^*, we prove its consistency by constructing a truth assignment $v : P \longrightarrow \{T, F\}$ such that for all $A \in S^*$, $v^*(A) = T$.

Observe that the MFC sets have the following property.

MCF Property *For any MFC set S^*, for every $A \in P\mathcal{F}$, exactly one of the formulas $A \neg A$ belongs to S^*.*

In particular, for any $P \in P\mathcal{F}$, we have that exactly one of $P, \neg P \in S^*$. This justify the correctness of the following definition.

Let $v : \ P \ \longrightarrow \ \{T, F\}$ be a mapping such that

$$v(P) = \begin{cases} T & \text{if } P \in S^* \\ F & \text{if } P \notin S^* \end{cases}$$

We extend v to $v^* : \mathbf{PF} \ \longrightarrow \ \mathbf{B}$ in a usual way. In order to prove that v is a *model* for S^* we have to show that for any $A \in P\mathcal{F}$,

$$v^*(A) = \begin{cases} T & \text{if } A \in S^* \\ F & \text{if } A \notin S^* \end{cases}$$

We prove it by induction on the degree of the formula A. The base case of $A \in P$ follows immediately from the definition of v.

Case $A = \neg C$ Assume that $A \in S^*$. This means $\neg C \in S^*$ and by **MCF Property** we have that $C \notin S^*$. So by the inductive assumption $v^*(C) = F$ and $v^*(A) = v^*(\neg C) = \neg v^*(C) = \neg F = T$.

Assume now that $A \notin S^*$. By **MCF Property** we have that $C \in S^*$. By the inductive assumption $v^*(C) = T$ and $v^*(A) = v^*(\neg C) = \neg v^*(T) = \neg T = F$.

This proves that for any formula A,

$$v^*(\neg A) = \begin{cases} T & \text{if } \neg A \in S^* \\ F & \text{if } \neg A \notin S^* \end{cases}$$

Case $A = (B \cup C)$ Let $(B \cup C) \in S^*$. It is enough to prove that in this case $B \in S^*$ and $C \in S^*$, because then from the inductive assumption $v^*(C) = v^*(D) = T$ and $v^*(B \cup C) = v^*(B) \cup v^*(C) = T \cup T = T$.

Assume that $(B \cup C) \in S^*$, $B \notin S^*$ and $C \notin S^*$. Then by **MCF Property** we have that $\neg B \in S^*$, $\neg C \in S^*$ and consequently the set

$$\{(B \cup C), \neg B, \neg C\}$$

is a finite inconsistent subset of S^*, what contradicts the fact that S^* is finitely consistent.

Assume now that $(B \cup C) \notin S^*$. By **MCF Property**, $\neg(B \cup C) \in S^*$ and by the $A = \neg C$ we have that $v^*(\neg(B \cup C)) = T$. But $v^*(\neg(B \cup C)) = \neg v^*((B \cup C)) = T$ means that $v^*((B \cup C)) = F$, what end the proof of this case.

The remaining cases of $A = (B \cap C), A = (B \Rightarrow C)$ are similar to the above and are left to the reader as an exercise.

Step 3: Maximal finitely consistent extention Given a finitely consistent set S, we construct its *maximal finitely consistent extension* S^* as follows.

The set of all formulas of $\mathcal{L}$ is countable, so is $P\mathcal{F}$. We assume that all propositional formulas form a one-to-one sequence

$$A_1, A_2, ..., A_n, ... \tag{1}$$

We define a chain

$$S_0 \subset S_1 \subset S_2 .... \subset S_n \subset .... \tag{2}$$

of *extentions* of the set S by

$$S_0 = S;$$

$$S_{n+1} = \begin{cases} S_n \cup \{A_n\} & \text{if } S_n \cup \{A_n\} \text{ is finitely consistent} \\ S_n \cup \{\neg A_n\} & \text{otherwise.} \end{cases}$$

We take

$$S^* = \bigcup_{n \in N} S_n. \tag{3}$$

Clearly, $S \subset S^*$ and for every A, either $A \in S^*$ or $\neg A \in S^*$. To finish the proof that S^* is MCF we have to show that it is finitely consistent.

First, let observe that if all sets S_n are finitely consistent, so is $S^* = \bigcup_{n \in N} S_n$. Namely, let $S_F = \{B_1, ..., B_k\}$ be a finite subset of S^*. This means that there are sets $S_{i_1}, ...S_{i_k}$ in the chain (2) such that $B_m \in S_{i_m}$, $m = 1, ..k$. Let $M = max(i_1, ...i_k)$. Obviously $S_F \subset S_M$ and S_M is finitely consistent as an element of the chain (2). This proves the if all sets S_n are finitely consistent, so is S^*.

Now we have to prove only that all S_n in the chain (2) are finitely consistent. We carry the proof by induction over the length of the chain. $S_0 = S$, so it is FC by assumption of the Compactness Theorem. Assume now that S_n is FC, we prove that so is S_{n+1}. We have two cases to consider.

Case 1 $S_{n+1} = S_n \cup \{A_n\}$, then S_{n+1} is FC by the definition of the chain (2).

Case 2 $S_{n+1} = S_n \cup \{\neg A_n\}$. Observe that this can happen only if $S_n \cup \{A_n\}$ is not FC, i.e. there is a finite subset $S_n^{'} \subset S_n$, such that $S_n^{'} \cup \{A_n\}$ is not consistent.

Suppose now that S_{n+1} is not FC. This means that there is a finite subset $S_n^{''} \subset S_n$, such that $S_n^{''} \cup \{\neg A_n\}$ is not consistent.

Take $S_n^{'} \cup S_n^{''}$. It is a finite subset of S_n so is consistent by the inductive assumption. Let v be a model of $S_n^{'} \cup S_n^{''}$. Then *one* of $v^*(A), v^*(\neg A)$ *must be* T. This contradicts the inconsistency of both $S_n^{'} \cup \{A_n\}$ and $S_n^{'} \cup \{\neg A_n\}$.

Thus, in ether case, S_{n+1}, is after all consistent. This ends the proof of the Step 3 and of the Compactness Theorem via the argument presented in the Step 4.

1.2 Reduction of first-order logic to propositional logic

Propositional tautologies as defined in the previous section barely scratch the surface of the collection of first -order tautologies, or first order *valid* formulas, as they are often called. For example the following first-order formulas are propositional tautologies,

$$(\exists x A(x) \cup \neg \exists x A(x)),$$
$$(\forall x A(x) \cup \neg \forall x A(x)),$$
$$(\neg(\exists x A(x) \cup \forall x A(x)) \Rightarrow (\neg \exists x A(x) \cap \neg \forall x A(x))),$$

but the following are first order tautologies (valid formulas) that are not propositional tautologies:

$$\forall x (A(x) \cup \neg A(x)),$$
$$(\neg \forall x A(x) \Rightarrow \exists x \neg A(x)).$$

The first formula above is just a prime formula, the second is of the form $(\neg B \Rightarrow C)$, for B and C prime.

To stress the difference between the propositional and first order tautologies some books reserve the word *tautology* for the propositional tautologies alone, using the notion of *valid formula* for the first order tautologies. We use here both notions, with the preference to *first-order tautology* or *tautology* for short when there is no room for misunderstanding.

To make sure that there is no misunderstandings we remind the following definitions.

Given a first order language $\mathcal{L}$ with the set of variables VAR and the set of formulas $\mathcal{F}$. Let $\mathcal{M} = [M, I]$ be a structure for the language $\mathcal{L}$, with the universe M and the interpretation I and let $s : VAR \longrightarrow M$ be a valuation of $\mathcal{L}$ in M.

A is true in $\mathcal{M}$ Given a structure $\mathcal{M} = [M, I]$, we say that a formula A is true in $\mathcal{M}$ if there is a valuation $s : VAR \longrightarrow M$ such that

$$(\mathcal{M}, s) \models A.$$

A is valid in $\mathcal{M}$ Given a structure $\mathcal{M} = [M, I]$, we say that a formula A is valid in $\mathcal{M}$ if

$$(\mathcal{M}, s) \models A$$

for all valuations $s : VAR \longrightarrow M$.

Model $\mathcal{M}$ If A is valid in a structure $\mathcal{M} = [M, I]$, then $\mathcal{M}$ is called a model of A.

A is valid A formula A called is valid if it is valid in all structures $\mathcal{M} = [M, I]$, i.e. if all structures are models of A.

A is a first-order tautology A valid formula A is also called a first-order tautology, or tautology, for short.

Case: A is a sentence If A is a sentence, then the truth or falsity of $(\mathcal{M}, s) \models A$ is completely independent of s. Thus we write

$$\mathcal{M} \models A$$

and read $\mathcal{M}$ *is a model of* A, if for some (hence every) valuation s, $(\mathcal{M}, s) \models A$.

Model of a set of sentences $\mathcal{M}$ is a model of a set S of sentences if $\mathcal{M} \models A$ for all $A \in S$. We write it

$$\mathcal{M} \models S.$$

2 Completeness Theorem for Classical Predicate Logic

The relationship between the first order models defined in terms of structures $\mathcal{M} = [M, I]$ and valuations $s : VAR \longrightarrow M$ and propositional models defined in terms of truth assignments $v : P \longrightarrow \{T, F\}$ is established by the following lemma.

Lemma 2.1 (Predicate and Propositional Models)

Let $\mathcal{M} = [M, I]$ be a structure for the language $\mathcal{L}$ and let $s : VAR \longrightarrow M$ a valuation in $\mathcal{M}$. There is a truth assignments $v : P \longrightarrow \{T, F\}$ such that for all formulas A of $\mathcal{L}$,

$$(\mathcal{M}, s) \models A \;\; if \; and \; only \; if \;\; v^*(A) = T.$$

In particular, for any set S of sentences of $\mathcal{L}$,

if $\mathcal{M} \models S$ then S is consistent in sense of propositional logic.

Proof For any prime formula $A \in P$ we define

$$v(A) = \begin{cases} T & \text{if } (\mathcal{M}, s) \models A \\ F & \text{otherwise.} \end{cases}$$

Since every formula in $\mathcal{L}$ is either prime or is built up from prime formulas by means of propositional connectives, the conclusion is obvious.

Observe, that the converse of the lemma is far from true. Consider a set

$$S = \{\forall x(A(x) \Rightarrow B(x)), \forall x A(x), \exists x \neg B(x)\}.$$

All formulas of S are different prime formulas, S is hence consistent in the sense of propositional logic and obviously has no (predicate) model.

The language $\mathcal{L}$ is a predicate language with equality. We adopt a following set of axioms.

Equality Axioms For any free variable or constant of $\mathcal{L}$, i.e for any $u, w, u_i, w_i \in (VAR \cup \mathbf{C})$,

E1 $u = u$,

E2 $(u = w \Rightarrow w = u)$,

E3 $((u_1 = u_2 \cap u_2 = u_3) \Rightarrow u_1 = u_3)$,

E4 $((u_1 = w_1 \cap ... \cap u_n = w_n) \Rightarrow (R(u_1, ..., u_n) \Rightarrow R(w_1, ..., w_n)))$,

E5 $((u_1 = w_1 \cap ... \cap u_n = w_n) \Rightarrow (t(u_1, ..., u_n) \Rightarrow t(w_1, ..., w_n)))$,

where $R \in bfP$ and $t \in T$, i.e. R is an arbitrary n-ary relation symbol of $\mathcal{L}$ and t is an arbitrary n-ary term of $\mathcal{L}$.

Obviously, all equality axioms are first-order *tautologies*, or are *valid* formulas of $\mathcal{L}$, i.e. for all $\mathcal{M} = [M, I]$ and all $s : VAR \longrightarrow M$, and for all $A \in \{E1, E2, E3, E4, E5.E6\}, (\mathcal{M}, s) \models A$.

This is why we still call logic with equality axioms added a logic.

Now we are going to define notions that is fundamental to the Henkin's technique for reducing first-order logic to propositional logic. The first one is that of *witnessing expansion* of the language $\mathcal{L}$.

Witnessing expansion $\mathcal{L}(C)$ of $\mathcal{L}$ We construct an expansion of our language $\mathcal{L}$ by adding a set C of new constants to it, i.e. we define a new language $\mathcal{L}(C)$

$$\mathcal{L}(C) = \mathcal{L}(\mathbf{P}, \mathbf{F}, (\mathbf{C} \cup C))$$

which is usually denoted shortly as

$$\mathcal{L}(C) = \mathcal{L} \cup C.$$

Definition of C We define the set C of new constants by constructing an infinite sequence

$$C_0, C_1, ..., C_n, ... \tag{4}$$

of sets of constants together with an infinite sequence

$$\mathcal{L}_0, \mathcal{L}_1, ..., \mathcal{L}_n, ... \tag{5}$$

of languages, such that

$$\mathcal{L}_n = \mathcal{L} \cup C_n, \quad C = \bigcup_{n \in N} C_n$$

and

$$\mathcal{L}(C) = \mathcal{L} \cup C.$$

We define sequences (4), (5) as follows. Let

$$C_0 = \emptyset, \quad \mathcal{L}_0 = \mathcal{L} \cup C_0 = \mathcal{L}.$$

We denote by

$$A[x]$$

the fact that the formula A has exactly one free variable and for each such a formula we introduce a distinct new constant denoted by

$$c_{A[x]}.$$

We define

$$C_1 = \{c_{A[x]} : \quad A[x] \ \in \mathcal{L}_0\}, \quad \mathcal{L}_1 = \mathcal{L} \cup C_1.$$

Assume that we have defined C_n and $\mathcal{L}_n$. We introduce a distinct new constant $c_{A[x]}$ for each formula $A[x]$ of $\mathcal{L}_n$ which is not already a formula of $\mathcal{L}_{n-1}$ (i.e., if some constant from C_n appears in $A[x]$). We write it informally as $A[x] \in (\mathcal{L}_n - \mathcal{L}_{n-1})$.

We define

$$C_{n+1} = C_n \cup \{c_{A[x]} : \quad A[x] \in (\mathcal{L}_n - \mathcal{L}_{n-1})\},$$
$$\mathcal{L}_{n+1} = \mathcal{L} \cup C_{n+1}.$$

Witnessing constant For any formula A, a constant $c_{A[x]}$ as defined above is called a *witnessing constant.*

Henkin Axioms The following sentences

H1 $(\exists x A(x) \Rightarrow A(c_{A[x]}))$,

H2 $(A(c_{\neg A[x]}) \Rightarrow \forall x A(x))$

are called Henkin axioms.

The informal idea behind the Henkin axioms is the following.

The axiom H1 says:
If $\exists x A(x)$ *is true in a structure, choose an element a satisfying* $A(x)$ *and give it a new name* $c_{A[x]}$.

The axiom H2 says:
If $\forall x A(x)$ *is false, choose a counterexample b and call it by a new name* $c_{\neg A[x]}$.

Quantifier axioms The following sentences

Q1 $(\forall x A(x) \Rightarrow A(t))$, t is a closed term of $\mathcal{L}(C)$;

Q2 $(A(t) \Rightarrow \exists x A(x))$, t is a closed term of $\mathcal{L}(C)$

are called *quantifier axioms.* They obviously are first-order tautologies.

Henkin set Any set of sentences of $\mathcal{L}(C)$ which are either *Henkin axioms* H1, H2 or quantifier axioms Q1, Q2 is called *Henkin set* and denoted by

$$S_{Henkin}.$$

The set S_{Henkin} is obviously not true in every $\mathcal{L}(C)$-structure, but we are going to show that every $\mathcal{L}$ -structute can be turned into an $\mathcal{L}(C)$-structure which is *model* of S_{Henkin}. Before we do so we need to introduce two new notions.

Reduct and Expension Given two languages $\mathcal{L}$ and $\updownarrow^{'}$ such that $\mathcal{L} \subset \mathcal{L}^{'}$. Let $\mathcal{M}^{'} = [M, I^{'}]$ be a structure for $\mathcal{L}^{'}$. The structure

$$\mathcal{M} = [M, I^{'} \mid \mathcal{L}]$$

is called the *reduct* of $\mathcal{M}^{'}$ to the language $\mathcal{L}$ and $\mathcal{M}^{'}$ is called the *expension* of $\mathcal{M}$ to the language $\mathcal{L}^{'}$.

Thus the reduct and the expension $\mathcal{M}^{'}$ and $\mathcal{M}$ are the same except that $\mathcal{M}^{'}$ assigns meanings to the symbols in $(\mathcal{L} - \mathcal{L}^{'})$.

Lemma 2.2 *Let $\mathcal{M} = [M, I]$ be any structure for the language $\mathcal{L}$ and let $\mathcal{L}(C)$ be the witnessing expansion of $\mathcal{L}$. There is an extention $\mathcal{M}^{'} = [M, I^{'}]$ of $\mathcal{M} = [M, I]$ such that $\mathcal{M}^{'}$ is a model of the set S_{Henkin}.*

Proof In order to define the expansion of $\mathcal{M}$ to $\mathcal{M}^{'}$ we have to define the interpretation $I^{'}$ for the symbols of the language $\mathcal{L}(C) = \mathcal{L} \cup C$, such that $I^{'} \mid \mathcal{L} = I$. This means that we have to define $c_{I'}$ for all $c \in C$. By the definition, $c_{I'} \in M$, so this also means that we have to assign the elements of M to all constants $c \in C$ in such a way that the resulting expansion is a model for all sentences from S_{Henkin}.

The quantifier axioms $Q1, Q2$ are first order tautologies so they are going to be true regardless, so we have to worry only about the Henkin axioms $H1, H2$. Observe now that if the lemma holds for the Henkin axioms H1, then it must hold for the axioms $H2$. Namely, let's consider the axiom $H2$:

$$(A(c_{\neg A[x]}) \Rightarrow \forall x A(x)).$$

Assume that $A(c_{\neg A[x]})$ is true in the expansion $\mathcal{M}^{'}$, ie. that $\mathcal{M}^{'} \models A(c_{\neg A[x]})$ and that $\mathcal{M}^{'} \not\models \forall x A(x)$. This means that $\mathcal{M}^{'} \models \neg\forall x A(x)$ and by the de Morgan Laws, $\mathcal{M}^{'} \models \exists x \neg A(x)$. But we have assumed that $\mathcal{M}^{'}$ is a model for $H1$. In particular $\mathcal{M}^{'} \models (\exists x \neg A(x) \Rightarrow \neg A(c_{\neg A[x]}))$, and hence $\mathcal{M}^{'} \models \neg A(c_{\neg A[x]})$ and this contradicts the assumption that $\mathcal{M}^{'} \models A(c_{\neg A[x]})$. Thus if $\mathcal{M}^{'}$ is a model for all axioms of the type $H1$, it is also a model for all axioms of the type $H2$.

We define $c_{I'}$ for all $c \in C = \bigcup C_n$ by induction on n. Let $n = 1$ and $c_{A[x]} \in C_1$. By definition, $C_1 = \{c_{A[x]} : A[x] \in \mathcal{L}\}$. In this case we have that $\exists x A(x) \in \mathcal{L}$ and hence the notion $\mathcal{M} \models \exists x A(x)$ is well defined, as $\mathcal{M} = [M, I]$ is the structure for the language $\mathcal{L}$. As we consider arbitrary structure $\mathcal{M}$, there are two posibilities: $\mathcal{M} \models \exists x A(x)$ or $\mathcal{M} \not\models \exists x A(x)$. We define $c_{I'}$, for all $c \in C_1$ as follows.

If $\mathcal{M} \models \exists x A(x)$, then $(\mathcal{M}, v') \models A(x)$ for certain $v'(x) = a \in M$. We set $(c_{A[x]})_{I'} = a$. If $\mathcal{M} \not\models \exists x A(x)$, we set $(c_{A[x]})_{I'}$ arbitrarily. Obviously, $\mathcal{M}^{'} =$

$(M, I^{'}) \models (\exists x A(x) \Rightarrow A(c_{A[x]}))$. But once $c \in C_1$ are all interpreted in $\mathcal{M}^{'} = (M, I^{'})$, then the notion $\mathcal{M}^{'} \models A$ is defined for all formulas $A \in \mathcal{L} \cup C_1$. We carry the inductive step in the exactly the same way as the one above.

Canonical structure Given a structure $\mathcal{M} = [M, I]$ for the language $\mathcal{L}$. The extention $\mathcal{M}^{'} = [M, I^{'}]$ of $\mathcal{M} = [M, I]$ is called a *canonical structure for* $\mathcal{L}(C)$ if all $a \in M$ are denoted by some $c \in C$, i.e if

$$M = \{c_{I'} : \ c \in C\}.$$

Now we are ready to state and proof a lemma that provides the essential step in the proof of the Completeness Theorem.

Lemma 2.3 (The reduction to propositional logic) *Let $\mathcal{L}$ be a first order language and let $\mathcal{L}(C)$ be a witnessing expansion of $\mathcal{L}$. For any set S of* sentences *of $\mathcal{L}$ the following conditions are equivalent.*

(i) *S has a model, i.e. there is a structure $\mathcal{M} = [M, I]$ for the language $\mathcal{L}$ such that $\mathcal{M} \models A$ for all $A \in S$.*

(ii) *There is a* canonical $\mathcal{L}(C)$ structure $\mathcal{M}^{'} = [M, I^{'}]$ *which is a model for S, i.e. such that $\mathcal{M}^{'} \models A$ for all $A \in S$.*

(iii) *The set $S \cup S_{Henkin} \cup EQ$ is consistent in sense of propositional logic, where EQ denotes the equality axioms $E1 - E5$.*

Proof The implication $(ii) \rightarrow (i)$ is immediate. The implication $(i) \rightarrow (iii)$ follows from lemma 2.2. We have to prove only the implication $(iii) \rightarrow (ii)$.

Assume that the set $S \cup S_{Henkin} \cup EQ$ is consistent in sense of propositional logic and let v be a truth assignment to the prime sentences of $\mathcal{L}(C)$, such that $v^*(A) = T$ for all $A \in S \cup S_{Henkin} \cup EQ$. To prove the lemma, we construct a canonical model $\mathcal{M}^{'} = [M, I^{'}]$ such that, for all sentences A of $\mathcal{L}(C)$,

$$\mathcal{M}^{'} \models \ A \ \ if \ and \ only \ if \ v^*(A) = T.$$

v is a propositional model for the set S_{Henkin}, so v^* satisfies the following conditions:

$$v^*(\exists x A(x)) = T \ \ if \ and \ only \ if \ \ v^*(A(c_{A[x]})) = T, \tag{6}$$

$$v^*(\forall x A(x)) = T \ \ if \ and \ only \ if \ \ v^*(A(t)) = T, \tag{7}$$

for all closed terms t of $\mathcal{L}(C)$.

The conditions (6) and (7) allow us to construct the model $\mathcal{M}^{'} = [M, I^{'}]$ out of the constants in C in the following way.

TO BE DONE!

The Main Lemma provides not only a method of constructing models of theories out of symbols, but also gives us immediate proofs of the Compactness Theorem for the first order logic and Lowenheim-Skolem Theorem.

Theorem 2.1 (Compactness theorem for the first order logic)
Let S be any set of first order formulas. The set S has a model if and only if any finite subset S_0 of S has a model.

Proof Let S be a set of first order formulas such that every finite subset S_0 of S has a model. We need to show that S has a model. By the implication $(iii) \to (i)$ of the Main Lemma 2.3 this is equivalent to proving that $S \cup S_{Henkin} \cup EQ$ is consistent in the sense of propositional logic. By the Compactness Theorem 1.1 for propositional logic, it suffices to prove that for every finite subset $S_0 \subset S$, $S_0 \cup S_{Henkin} \cup EQ$ is consistent, which follows from the hypothesis and the implication $(i) \to (iii)$ of the Main Lemma 2.3.

Theorem 2.2 (Löwenheim-Skolem Theorem)
Let κ be an infinite cardinal and let S be a set of at most κ formulas of the first order language. If the set S has a model, then there is a model $\mathcal{M} = [M, I]$ of S such that $cardM \leq \kappa$.

Proof Let $\mathcal{L}$ be a first order language with the alphabet $\mathcal{A}$ such that $card(\mathcal{A}) \leq \kappa$. Obviously, $card(\mathcal{F}) \leq \kappa$. By the definition of the witnessing expansion $\mathcal{L}(C)$ of $\mathcal{L}$, $C = \bigcup_n C_n$ and for each n, $card(C_n) \leq \kappa$. So also $cardC \leq \kappa$. Thus any canonical structure for $\mathcal{L}(C)$ has $\leq \kappa$ elements. By the implication $(i) \to (ii)$ of the Main Lemma 2.3 there is a model of S (canonical structure) with $\leq \kappa$ elements.