CHAPTER 6

CLASSICAL TAUTOLOGIES

AND LOGICAL EQUIVALENCES

We present and discuss here a set of most widely used classical tautologies and logical equivalences. We also discuss the definability of classical connectives and as a consequence, the equivalence between classical propositional languages.

1 Implication

One of the most frequently used classical tautologies are the laws of detachment for implication and equivalence. The implication law was already known to the Stoics (3rd century B.C) and a rule of inference, based on it is called *Modus Ponens*, so we use the same name here.

Modus Ponens

$$\models ((A \cap (A \Rightarrow B)) \Rightarrow B) \tag{1}$$

Detachment

$$\models ((A \cap (A \Leftrightarrow B)) \Rightarrow B) \tag{2}$$

$$\models ((B \cap (A \Leftrightarrow B)) \Rightarrow A)$$

Mathematical and not only mathematical theorems are usually of the form of an implication, so we will discuss some terminology and more properties of implication.

Sufficient Given an implication

$$(A \Rightarrow B),$$

A is called a *sufficient condition* for B to hold.

Necessary Given an implication

$$(A \Rightarrow B),$$

B is called a *necessary condition* for A to hold.

Simple The implication $(A \Rightarrow B)$ is called *a simple implication*.

Converse Given a simple implication $(A \Rightarrow B)$, the implication $(B \Rightarrow A)$ is called *a converse implication*.

Opposite Given a simple implication $(A \Rightarrow B)$, the implication $(\neg B \Rightarrow \neg A)$ is called *an opposite implication*.

Contrary Given a simple implication $(A \Rightarrow B)$, the implication $(\neg A \Rightarrow \neg B)$ is called *a contrary implication*.

Each of the following pairs of implications: *a simple* and *an opposite*, and *a converse* and *a contrary* are equivalent, i.e. the following formulas are tautologies:

Laws of contraposition (1)

$$\models ((A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)), \tag{3}$$

$$\models ((B \Rightarrow A) \Leftrightarrow (\neg A \Rightarrow \neg B)).$$

The laws of contraposition make it possible to replace, in any deductive argument, a sentence of the form $(A \Rightarrow B)$ by $\neg B \Rightarrow \neg A)$, and conversely. The relationships between all implications involved in the contraposition laws are usually shown graphically in a following form, which is called the *square of opposition*.

$$\begin{array}{cc} (A \Rightarrow B) & (B \Rightarrow A) \\ \\ \\ (\neg A \Rightarrow \neg B) & (\neg B \Rightarrow \neg A) \end{array}$$

Equivalent implications are situated at the vertices of one and the same diagonal. It follows from the contraposition laws that to prove all of the following implications: $(A \Rightarrow B)$, $(B \Rightarrow A)$, $(\neg A \Rightarrow \neg B)$, $(\neg B \Rightarrow \neg A)$, it suffices to prove any pairs of those implications which are situated at one and the same side of the square, since the remaining two implications are equivalent to those already proved to be true.

Consider now the following tautology:

$$\models ((A \Leftrightarrow B)) \Leftrightarrow ((A \Rightarrow B) \cap (B \Rightarrow A))). \tag{4}$$

Necessary and sufficient The above tautology 4 says that in order to prove a theorem $(A \Leftrightarrow B)$ it suffices to prove two implications: the simple one $(A \Rightarrow B)$ and the converse one $(B \Rightarrow A)$. Conversely, if $(A \Leftrightarrow B)$ is a theorem, then the implications $(A \Rightarrow B)$ and $(B \Rightarrow A)$ are also theorems.

In other words, B is then a necessary condition for A, and at the same time B is a sufficient condition for A. Accordingly, we say that a theorem of the form
$$(A \Leftrightarrow B)$$
is often formulated as:

B is necessary and sufficient condition for A.

It follows from the square of opposition that to prove $(A \Leftrightarrow B)$ it suffices to prove one of the pairs of implications situated in the square of opposition along one and the same side. Conversely, if $(A \Leftrightarrow B)$ is a theorem, then all the implications: simple $(A \Rightarrow B)$, converse $(B \Rightarrow A)$, contrary $(\neg A \Rightarrow \neg B)$, and opposite $(\neg B \Rightarrow \neg A)$, are also theorems.

Other laws developed by the Stoics are the hypothetical syllogism and modus tollendo ponens. We present them here in a form of logical tautology, not as the rule of reasoning, as it was developed. The relationship between those two approaches is quite obvious and will be discussed in detail in the proof theory chapter.

Hypothetical syllogism

$$\begin{gathered} \models (((A \Rightarrow B) \cap (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)) \\ \models ((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))) \\ \models ((B \Rightarrow C) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))). \end{gathered} \tag{5}$$

Modus tollendo ponens

$$\begin{gathered} \models (((A \cup B) \cap \neg A) \Rightarrow B) \\ \models (((A \cup B) \cap \neg B) \Rightarrow A) \end{gathered} \tag{6}$$

Here are some other tautologies with a history centuries old. First is called *Duns Scotus Law* after an eminent medieval philosopher who lived at the turn of the 13th century. Second is called *Clavius Law* , after Clavius, a Euclid commentator who lived in the late 16th century. The reasonings based on this law were already known to Euclid, but this type of inference became popular in scholarly circles owing to Clavius, hence the name. The third is called *Frege Laws* after G. Frege who was first to give a formulation of the classical propositional logic as a formalized axiomatic system in 1879, adopting the second of them as one of his axioms.

Duns Scotus

$$\models (\neg A \Rightarrow (A \Rightarrow B)) \tag{7}$$

Clavius

$$\models ((\neg A \Rightarrow A) \Rightarrow A) \tag{8}$$

Frege

$$\models (((A \Rightarrow (B \Rightarrow C)) \cap (A \Rightarrow B)) \Rightarrow (A \Rightarrow C)) \tag{9}$$

$$\models ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$$

Double Negation

$$\models (\neg\neg A \Leftrightarrow A) \tag{10}$$

2 Apagogic Proofs

Next set of tautologies deal with *apagogic proofs* which are the proofs by *reductio ad absurdum*. The method of apagogic proof consists in negating the theorem which is to be proved. If the assumption that the theorem is false yields a contradiction, then we conclude that the theorem is true. The correctness of this reasoning is based on the following tautology.

Reductio ad absurdum

$$\models ((\neg A \Rightarrow (B \cap \neg B)) \Rightarrow A) \tag{11}$$

If the theorem to be proved by reductio ad absurdum is of the form of an implication $(A \Rightarrow B)$, then the prove often follows a following pattern: it is assumed that $\neg(A \Rightarrow B)$ is true, and we try to deduce a contradiction from this assumption. If we succeed in doing so, then we infer that the implication $(A \Rightarrow B)$ is true. The correctness of this reasoning is based on the following tautology.

$$\models (((\neg(A \Rightarrow B) \Rightarrow (C \cap \neg C)) \Rightarrow (A \Rightarrow B)).$$

Sometimes to prove $(A \Rightarrow B)$ it is assumed that $(A \cap \neg B)$ is true and if the assumption leads to contradiction, then we deduce that the implication $(A \Rightarrow B)$ is true. In this case a tautology, which guarantee the correctness this kind of argument is:

$$\models (((A \cap \neg B) \Rightarrow (C \cap \neg C)) \Rightarrow (A \Rightarrow B)).$$

Often, when assuming $(A \cap \neg B)$, we arrive, by deductive reasoning, at the conclusion $\neg A$. Then we need the following tautology:

$$\models (((A \cap \neg B) \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)).$$

Sometimes, on assuming $(A \cap \neg B)$ we arrive by deductive reasoning at the conclusion B. The following tautology is then applied:

$$\models (((A \cap \neg B) \Rightarrow B) \Rightarrow (A \Rightarrow B)).$$

The proofs based on the application of the laws of contraposition 3 are also classed as apagogic. Instead of proving a simple theorem $(A \Rightarrow B)$ we prove the opposite theorem $(\neg B \Rightarrow \neg A)$, which is equivalent to the simple one. The following two tautologies, also called laws of contraposition, are used, respectively, when the hypothesis or the thesis of the theorem to be proved is in the form of a negation.

Laws of contraposition (2)

$$\begin{aligned} &\models ((\neg A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow A)), \\ &\models ((A \Rightarrow \neg B) \Leftrightarrow (B \Rightarrow \neg A)). \end{aligned} \tag{12}$$

3 Conjunction, Disjunction

We present here some tautologies characterizing basic properties of conjunction and disjunction.

Conjunction

$$\models ((A \cap B) \Rightarrow A),$$
$$\models ((A \cap B) \Rightarrow B),$$
$$\models (((A \Rightarrow B) \cap (A \Rightarrow C)) \Rightarrow (A \Rightarrow (B \cap C))),$$
$$\models (((A \Rightarrow B) \cap (C \Rightarrow D)) \Rightarrow ((A \cap C) \Rightarrow (B \cap D))),$$
$$\models (A \Rightarrow (B \Rightarrow (A \cap B))).$$

Disjunction

$$\models ((A \Rightarrow (A \cup B)),$$
$$\models ((B \Rightarrow (A \cup B)),$$
$$\models (((A \Rightarrow B) \cap (B \Rightarrow C)) \Rightarrow ((A \cup B) \Rightarrow C)),$$
$$\models (((A \Rightarrow B) \cap (C \Rightarrow D)) \Rightarrow ((A \cup C) \Rightarrow (B \cup D))),$$
$$\models (A \cup \neg A).$$

4 Logical equivalence

We discuss here propositional tautologies which have a form of an equivalence. i.e in a form

$$\models (A \Leftrightarrow B).$$

We present them in a form of a logical equivalence

$$A \equiv B$$

rather then in a form of a formula $(A \Leftrightarrow B)$. The logical equivalence $\equiv$ is defined below.

Definition 4.1 (Logical Equivalence) *For any A, $B \in \mathcal{F}$, we say that*

A and B are logically equivalent iff $v^*(A) = v^*(B)$, for any v.
We denote it as

$$A \equiv B.$$

Observe that the following property follows directly from the definition 4.1.

$$A \equiv B \quad iff \quad \models (A \Leftrightarrow B) \tag{13}$$

For example we write the laws of contraposition 3, 12, and the law of double negation 10 as logical equivalences as follows.

Laws of contraposition (1)

$$(A \Rightarrow B) \equiv (\neg B \Rightarrow \neg A),$$

$$(B \Rightarrow A) \equiv (\neg A \Rightarrow \neg B),$$

$$(\neg A \Rightarrow B) \equiv (\neg B \Rightarrow A),$$

$$(A \Rightarrow \neg B) \equiv (B \Rightarrow \neg A).$$

Laws of contraposition (2)

$$((\neg A \Rightarrow B) \equiv (\neg B \Rightarrow A)),$$

$$((A \Rightarrow \neg B) \equiv (B \Rightarrow \neg A)).$$

Double Negation

$$(\neg\neg A \equiv A).$$

Logical equivalence is a very useful notion when we want to obtain new formulas, or tautologies, if needed, on a base of some already known in a way that guarantee preservation of the logical value of the initial formula.

Foe example, we easily obtain equivalences for Laws of contraposition (2) from equivalences for Laws of contraposition (1) and the Double Negation equivalence as follows.

$$(\neg A \Rightarrow B) \equiv (\neg B \Rightarrow \neg\neg A) \equiv (\neg B \Rightarrow A), i.e.$$
$$((\neg A \Rightarrow B) \equiv (\neg B \Rightarrow A)).$$
$$(A \Rightarrow \neg B) \equiv (\neg\neg B \Rightarrow \neg A) \equiv (B \Rightarrow \neg A), i.e.$$
$$(A \Rightarrow \neg B) \equiv (B \Rightarrow \neg A).$$

The correctness of the above procedure of proving new equivalences from the known ones is established by the following theorem

Theorem 4.1 *Let B_1 be obtained from A_1 by substitution of a formula B for one or more occurrences of a sub-formula A of A_1, what we denote as*

$$B_1 = A_1(A/B).$$

Then the following holds.

$$If \quad A \equiv B, \quad then \quad A_1 \equiv B_1,$$

i.e. by the equation 4.1

$$\models ((A \Leftrightarrow B) \Rightarrow (A_1 \Leftrightarrow B_1)).$$

Proof Consider a truth assignment v. If $v^*(A) \neq v^*(B)$, then obviously $v^*(A \Leftrightarrow B) = F$, and so $v^*((A \Leftrightarrow B) \Rightarrow (A_1 \Leftrightarrow B_1)) = T$.

If $v^*(A) = v^*(B)$, then so $v^*(A_1) = v^*(B_1)$, since B_1 differs from A_1 only in containing B in some places where A_1 contains A. Hence in this case if $v^*(A \Leftrightarrow B) = T$, $v^*(A_1 \Leftrightarrow B_1) = T$, and therefore $v^*((A \Leftrightarrow B) \Rightarrow (A_1 \Leftrightarrow B_1)) = T$, what proves that $((A \Leftrightarrow B) \Rightarrow (A_1 \Leftrightarrow B_1))$.

Example Let $A_1 = (C \cup D)$ and $B = \neg\neg C$. Obviously, $\neg\neg C \equiv C$. Let $B_1 = A_1(C/B) = A_1(C/\neg\neg C) = (\neg\neg C \cup D)$ and

$$(C \cup D) \equiv (\neg\neg C \cup D).$$

5 Definability of Connectives and Equivalence of Languages

The next set of equivalences, or corresponding tautologies, deals with what is called *a definability of connectives* in classical semantics. For example, a tautology

$$\models ((A \Rightarrow B) \Leftrightarrow (\neg A \cup B))$$

makes it possible to define implication in terms of disjunction and negation. We state it in a form of logical equivalence as follows.

Definability of Implication in terms of negation and disjunction:

$$(A \Rightarrow B) \equiv (\neg A \cup B) \tag{14}$$

We are using the logical equivalence notion, instead of the tautology notion, as it makes the manipulation of formulas much easier.

The equivalence 14 allows us, by the force of Theorem 4.1 to replace any formula of the form $(A \Rightarrow B)$ placed anywhere in another formula by a formula $(\neg A \cup B)$. Hence we can transform a given formula containing implication into an logically equivalent formula that does contain implication (but contains negation and disjunction).

We usually use the equation 14 to transform any formula A of language containing implication into a formula B of language containing disjunction and negation and not containing implication at all, such that $A \equiv B$.

Example 1
Consider a formula A

$$((C \Rightarrow \neg B) \Rightarrow (B \cup C)).$$

We use equality 14 to transform A into its logically equivalent form not containing $\Rightarrow$ as follows.

$$((C \Rightarrow \neg B) \Rightarrow (B \cup C)) \equiv (\neg(C \Rightarrow \neg B) \cup (B \cup C)))$$

$$\equiv (\neg(\neg C \cup \neg B) \cup (B \cup C))).$$

It means that for example that we can, by the Theorem 4.1 transform a language $\mathcal{L}_1 = \mathcal{L}_{\{\neg,\cap,\Rightarrow\}}$ into a language $\mathcal{L}_2 = \mathcal{L}_{\{\neg,\cap,\cup\}}$ with all its formulas being logically equivalent. I.e. that the following condition holds.

C1: for any formula A of $\mathcal{L}_1$, there is a formula B of $\mathcal{L}_2$, such that $A \equiv B$.

Example 2
Let A be a formula $(\neg A \cup (\neg A \cup \neg B))$, we use equivalence 14 to **eliminate**

disjunction from A by replacing it by logically equivalent formula containing implication only as follows.

$$(\neg A \cup (\neg A \cup \neg B)) \equiv (\neg A \cup (A \Rightarrow \neg B))$$
$$\equiv (A \Rightarrow (A \Rightarrow \neg B)).$$

Observe, that we can't always use the equivalence 14 to eliminate any disjunction. For example, we can't use it for a formula $A = ((a \cup b) \cap \neg a)$.

In order to be able to transform any formula of a language containing **disjunction** (and some other connectives) into a language with negation and implication (and some other connectives), but **without disjunction** we need the following logical equivalence.

Definability of Disjunction in terms of negation and implication:

$$(A \cup B) \equiv (\neg A \Rightarrow B) \tag{15}$$

Example 3
Consider a formula A

$$(a \cup b) \cap \neg a).$$

We use equality 15 to transform A into its logically equivalent form not containing $\cup$ as follows.

$$((a \cup b) \cap \neg a) \equiv ((\neg a \Rightarrow b) \cap \neg a).$$

In general, we use the equality 15 and Theorem 4.1 to transform the language $\mathcal{L}_2 = \mathcal{L}_{\{\neg,\cap,\cup\}}$ to the language $\mathcal{L}_1 = \mathcal{L}_{\{\neg,\cap,\Rightarrow\}}$ with logically equivalent formulas. I.e. that the following condition holds.

C2: for any formula C of $\mathcal{L}_2$, there is a formula D of $\mathcal{L}_1$, such that $C \equiv D$.

The languages $\mathcal{L}_1$ and $\mathcal{L}_2$ for which we the conditions **C1, C2** hold are logically equivalent and denote it by

$$\mathcal{L}_1 \equiv \mathcal{L}_2.$$

We put it in a general, formal definition as follows.

Definition 5.1 (Equivalence of Languages) *Given two languages:*
$\mathcal{L}_1 = \mathcal{L}_{CON_1}$ *and* $\mathcal{L}_2 = \mathcal{L}_{CON_2}$, *for* $CON_1 \neq CON_2$.
We say that they are **logically equivalent**, *i.e.*

$$\mathcal{L}_1 \equiv \mathcal{L}_2$$

if and only if the following conditions **C1, C2** *hold.*

C1: *For every formula A of $\mathcal{L}_1$, there is a formula B of $\mathcal{L}_2$, such that*

$$A \equiv B,$$

C2: *For every formula C of $\mathcal{L}_2$, there is a formula D of $\mathcal{L}_1$, such that*

$$C \equiv D.$$

Example 4

To prove the logical equivalence of the languages

$$\mathcal{L}_{\{\neg,\cup\}} \equiv \mathcal{L}_{\{\neg,\Rightarrow\}}$$

we need two definability equivalences 14 and 15, and the substitution theorem 4.1.

Example 5

To prove the logical equivalence of the languages

$$\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}} \equiv \mathcal{L}_{\{\neg,\cap,\cup\}}$$

we needed only the definability equivalence 14. It proves, by Theorem 4.1 that for any formula A of $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$ there is B of $\mathcal{L}_{\{\neg,\cap,\cup\}}$ that equivalent to A, i.e. condition **C1** holds.

Any formula A of language $\mathcal{L}_{\{\neg,\cap,\cup\}}$ is also a formula of $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$ and of course $A \equiv A$, so both conditions **C1** and **C2** of definition 5.1 are satisfied.

The logical equalities below

Definability of Conjunction in terms of implication and negation

$$(A \cap B) \equiv \neg(A \Rightarrow \neg B), \tag{16}$$

Definability of Implication in terms of conjunction and negation

$$(A \Rightarrow B) \equiv \neg(A \cap \neg B), \tag{17}$$

prove, by Theorem 4.1 that

$$\mathcal{L}_{\{\neg,\cap\}} \equiv \mathcal{L}_{\{\neg,\Rightarrow\}}.$$

Definability of Disjunction in terms of negation and conjunction

$$(A \cup B) \equiv \neg(\neg A \cap \neg B), \tag{18}$$

Definability of Conjunction in terms of negation and disjunction

$$(A \cap B) \equiv \neg(\neg A \cup \neg B). \tag{19}$$

The above equalities and Theorem 4.1 prove that

$$\mathcal{L}_{\{\neg,\cap\}} \equiv \mathcal{L}_{\{\neg,\cup\}}.$$

Definability of Equivalence in terms of implication and conjunction

$$(A \Leftrightarrow B) \equiv ((A \Rightarrow B) \cap (B \Rightarrow A)). \tag{20}$$

This proves, with Theorem 4.1 that for example

$$\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}} \equiv \mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow,\Leftrightarrow\}}.$$

We leave other definability equivalences and language equivalences as homework problems at the end of the chapter.

Here are some more important equivalence laws which are also frequently used.

Idempotents

$$(A \cap A) \equiv A, \qquad (A \cup A) \equiv A,$$

Associativity

$$((A \cap B) \cap C) \equiv (A \cap (B \cap C)),$$
$$((A \cup B) \cup C) \equiv (A \cup (B \cup C)),$$

Commutativity

$$(A \cap B) \equiv (B \cap A), \qquad (A \cup B) \equiv (B \cup A),$$

Distributivity

$$(A \cap (B \cup C)) \equiv ((A \cap B) \cup (A \cap C)),$$
$$(A \cup (B \cap C)) \equiv ((A \cup B) \cap (A \cup C)),$$

De Morgan

$$\neg(A \cup B) \equiv (\neg A \cap \neg B),$$
$$\neg(A \cap B) \equiv (\neg A \cup \neg B).$$

Negation of implication

$$\neg(A \Rightarrow B) \equiv (A \cap \neg B).$$

Negation of equivalence

$$\neg(A \Leftrightarrow B) \equiv (A \cap \neg B) \cup (B \cap \neg A.)$$

Double negation

$$\neg\neg A \equiv A,$$

Excluded Middle

$$(A \cup \neg A),$$

Exportation and Importation

$$((A \cap B) \Rightarrow C) \equiv (A \Rightarrow (B \Rightarrow C)).$$

De Morgan laws are named after A. De Morgan (1806 - 1871), an English logician, who discovered analogous laws for the algebra of sets. They stated that for any sets A,B the complement of their union is the same as the intersection of their complements, and vice versa, the complement of the intersection of two sets is equal to the union of their complements. The laws of the propositional calculus were formulated later, but they are usually also called De Morgan Laws.

Observe that De Morgan Laws tell us how to negate disjunction and conjunction, so the laws stating how to negate other connectives follows them.

Example 3
Consider the following A,

$$\models ((\neg(A \Rightarrow B) \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)).$$

We know that $(A \Rightarrow B) \equiv (\neg A \cup B)$, by Theorem 4.1 if we replace $(A \Rightarrow B)$ by $(\neg A \cup B)$ in A, the logical value of A will remain the same and

$$((\neg(A \Rightarrow B) \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)) \equiv ((\neg(\neg A \cup B) \Rightarrow \neg A) \Rightarrow (\neg A \cup B)).$$

Now we can use de Morgan Laws and Double Negation Laws and by Theorem 4.1 we get

$$((\neg(A \Rightarrow B) \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)) \equiv ((\neg(\neg A \cup B) \Rightarrow \neg A) \Rightarrow (\neg A \cup B))$$

$$\equiv (((\neg\neg A \cap \neg B) \Rightarrow \neg A) \Rightarrow (\neg A \cup B)) \equiv (((A \cap \neg B) \Rightarrow \neg A) \Rightarrow (\neg A \cup B)).$$

This proves that

$$\models (((A \cap \neg B) \Rightarrow \neg A) \Rightarrow (\neg A \cup B)).$$

5.1 Exercises and Homework Problems

Exercise 1

Prove by transformation, using proper logical equivalences that

1. $\neg(A \Leftrightarrow B) \equiv ((A \cap \neg B) \cup (\neg A \cap B))$,

2. $((B \cap \neg C) \Rightarrow (\neg A \cup B)) \equiv ((B \Rightarrow C) \cup (A \Rightarrow B))$.

Solution 1.

$$\neg(A \Leftrightarrow B) \equiv^{def} \neg((A \Rightarrow B) \cap (B \Rightarrow A))$$
$$\equiv^{de\ Morgan} (\neg(A \Rightarrow B) \cup \neg(B \Rightarrow A))$$
$$\equiv^{neg\ impl} ((A \cap \neg B) \cup (B \cap \neg A)) \equiv^{commut} ((A \cap \neg B) \cup (\neg A \cap B)).$$

Solution 2.

$$((B \cap \neg C) \Rightarrow (\neg A \cup B)) \equiv^{impl} (\neg(B \cap \neg C) \cup (\neg A \cup B))$$
$$\equiv^{de\ Morgan} ((\neg B \cup \neg\neg C) \cup (\neg A \cup B)) \equiv^{neg} ((\neg B \cup C) \cup (\neg A \cup B))$$
$$\equiv^{impl} ((B \Rightarrow C) \cup (A \Rightarrow B)).$$

Exercise 2

(a) Prove that $\mathcal{L}_{\{\cap,\neg\}} \equiv \mathcal{L}_{\{\cup,\neg\}}$.

(b) Transform a formula $A = \neg(\neg(\neg a \cap \neg b) \cap a)$ of $\mathcal{L}_{\{\cap,\neg\}}$ into a logically equivalent formula B of $\mathcal{L}_{\{\cup,\neg\}}$.

(c) Transform a formula $A = (((\neg a \cup \neg b) \cup a) \cup (a \cup \neg c))$ of $\mathcal{L}_{\{\cup,\neg\}}$ into a formula B of $\mathcal{L}_{\{\cap,\neg\}}$, such that $A \equiv B$.

(d) Prove/disaprove: $\models \neg(\neg(\neg a \cap \neg b) \cap a)$.

(e) Prove/disaprove: $\models (((\neg a \cup \neg b) \cup a) \cup (a \cup \neg c))$.

Solution (a)

True due to the Theorem 4.1 and two definability of connectives equivalences: (19, 18, respectively.)

$$(A \cap B) \equiv \neg(\neg A \cup \neg B), \quad (A \cup B) \equiv \neg(\neg A \cap \neg B).$$

Solution (b)

$$\neg(\neg(\neg a \cap \neg b) \cap a) \equiv^{19} \neg(\neg\neg(\neg\neg a \cup \neg\neg b) \cap a)$$

$$\equiv^{dneg} \neg((a \cup b) \cap a) \equiv^{19} \neg(\neg(a \cup b) \cup \neg a).$$

The formula B of $\mathcal{L}_{\{\cup,\neg\}}$ equivalent to A is $B = \neg(\neg(a \cup b) \cup \neg a)$.

Solution (c)

$$(((\neg a \cup \neg b) \cup a) \cup (a \cup \neg c)) \equiv^{18} ((\neg(\neg\neg a \cap \neg\neg b) \cup a) \cup \neg(\neg a \cap \neg\neg c))$$

$$\equiv^{dneg} ((\neg(a \cap b) \cup a) \cup \neg(\neg a \cap c)) \equiv^{18} (\neg(\neg\neg(a \cap b) \cap \neg a) \cup \neg(\neg a \cap c))$$

$$\equiv^{dneg} (\neg((a \cap b) \cap \neg a) \cup \neg(\neg a \cap c)) \equiv^{18} \neg(\neg\neg((a \cap b) \cap \neg a) \cap \neg\neg(\neg a \cap c))$$

$$\equiv^{dneg} \neg(((a \cap b) \cap \neg a) \cap (\neg a \cap c))$$

There are two formulas B of $\mathcal{L}_{\{\cap,\neg\}}$, such that $A \equiv B$.

$B = B_1 = \neg(\neg\neg((a \cap b) \cap \neg a) \cap \neg\neg(\neg a \cap c))$, $\quad B = B_2 = \neg(((a \cap b) \cap \neg a) \cap (\neg a \cap c))$.

Solution (d) $\not\models \neg(\neg(\neg a \cap \neg b) \cap a)$. Our formula is logically equivalent, as proved in **(c)** with the formula $B = \neg(\neg(a \cup b) \cup \neg a)$. Consider any truth assignment v, such that $v(a) = F$, then $(\neg(a \cup b) \cup T) = T$, and hence $v^*(B) = F$.

Solution (e) $\models (((\neg a \cup \neg b) \cup a) \cup (a \cup \neg c))$ because it was proved in **(c)** that

$$(((\neg a \cup \neg b) \cup a) \cup (a \cup \neg c)) \equiv \neg(((a \cap b) \cap \neg a) \cap (\neg a \cap c))$$

and obviously the formula $(((a \cap b) \cap \neg a) \cap (\neg a \cap c))$ is a contradiction. Hence its negation is a tautology.

Exercise 3

Prove that $\mathcal{L}_{\{\neg,\cap\}} \equiv \mathcal{L}_{\{\neg,\Rightarrow\}}$.

Solution
The equivalence of languages holds due to two definability of connectives equivalences:

$$(A \cap B) \equiv \neg(A \Rightarrow \neg B), \quad (A \Rightarrow B) \equiv \neg(A \cap \neg B).$$

Exercise 4 Prove using proper logical equivalences (list them at each step) that

1. $\neg(A \Leftrightarrow B) \equiv ((A \cap \neg B) \cup (\neg A \cap B))$.

Solution: $\neg(A \Leftrightarrow B) \equiv^{def} \neg((A \Rightarrow B) \cap (B \Rightarrow A)) \equiv^{deMorgan} (\neg(A \Rightarrow B) \cup \neg(B \Rightarrow A))$
$\equiv^{negimpl} ((A \cap \neg B) \cup (B \cap \neg A)) \equiv^{commut} ((A \cap \neg B) \cup (\neg A \cap B))$.

2. $((B \cap \neg C) \Rightarrow (\neg A \cup B)) \equiv ((B \Rightarrow C) \cup (A \Rightarrow B))$.

Solution: $((B\cap\neg C) \Rightarrow (\neg A\cup B))\equiv^{impl}(\neg(B\cap\neg C)\cup(\neg A\cup B))\equiv^{deMorgan}((\neg B\cup \neg\neg C) \cup (\neg A \cup B))$
$\equiv^{dneg}((\neg B \cup C) \cup (\neg A \cup B))\equiv^{impl}((B \Rightarrow C) \cup (A \Rightarrow B))$.

Exercise 5 Prove that $\mathcal{L}_{\{\neg,\cap\}} \equiv \mathcal{L}_{\{\neg,\Rightarrow\}}$.

Solution: The equivalence of languages holds due to two definability of connectives equivalences:

$$(A \cap B) \equiv \neg(A \Rightarrow \neg B), \quad (A \Rightarrow B) \equiv \neg(A \cap \neg B).$$

Homework Problems

1. Prove that 3 equivalences of your choice are tautologies.

2. Prove by transformation, using proper logical equivalences that

(a) $\neg(\neg A \cup \neg(B \Rightarrow \neg C)) \equiv (A \cap \neg(B \cap C))$,

(b) $(\neg A \cap (\neg A \cup B)) \equiv (\neg A \cup (\neg A \cap B))$.

3. Prove the following equivalences of languages (for classical semantics).

(a) $\mathcal{L}_{\{\cap,\neg\}} \equiv \mathcal{L}_{\{\Rightarrow,\neg\}}$,

(b) $\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}} \equiv \mathcal{L}_{\{\cup,\neg\}}$,

(c) $\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}} \equiv \mathcal{L}_{\{\ \uparrow\}}$.

4. Determine which (if any) the language equivalences listed in **3.** hold for

(a) **Ł** semantics,

(b) **K** semantics,

(c) **H** semantics,

(d) **B** semantics.

5a. Transform a formula $A = (((a \cup \neg b) \Rightarrow a) \cap (\neg a \Rightarrow \neg b))$ of $\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}}$ into a logically equivalent formula B of $\mathcal{L}_{\{\cup,\neg\}}$.

5b. Find all B of $\mathcal{L}_{\{\cup,\neg\}}$, such that $B \equiv A$, for A from **5a.**

6a. Transform a formula $A = (((\neg a \cup \neg b) \cup a) \cup (a \cup \neg c))$ of $\mathcal{L}_{\{\cup,\neg\}}$ into a formula B of $\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}}$, such that $A \equiv B$.

6b. Find all B of $\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}}$, such that $B \equiv A$, for A from **6a.**