The Logic of Compound Statements

CSE 215, Foundations of Computer Science

Stony Brook University

http://www.cs.stonybrook.edu/~cse215

Mathematical Formalization

- Why formalize?
  - to remove ambiguity
  - to represent facts on a computer and use it for proving, proof-checking, etc.
  - to detect unsound reasoning in arguments

All people are mortal.
Socrates is a person.
$\rightarrow$ Socrates is mortal.

$$\left.\begin{array}{l}\forall x\ P(x) \rightarrow M(x) \\ P(S)\end{array}\right\} \rightarrow \left.\begin{array}{l}P(S) \rightarrow M(S) \\ P(S)\end{array}\right\} \rightarrow M(S)$$

Logic

- Mathematical logic is a tool for dealing with formal reasoning
  - formalization of natural language and reasoning methods
- Logic does:
  - Assess if an argument is Valid/inValid
- Logic does not directly:
  - Assess the truth of atomic statements

Propositional Logic

- Propositional logic is the study of:
  - the structure (syntax) and
  - the meaning (semantics) of (simple and complex) propositions.
- The key questions are:
  - How is the truth value of a complex proposition obtained from the truth value of its simpler components?
  - Which propositions represent correct reasoning arguments?

Propositional Logic

- A **proposition** is a sentence that is either true or false, but not both.
- Examples of simple propositions:
  - John is a student
  - 5+1 = 6
  - 426 > 1721
  - It is 52 degrees outside right now.
- Example of a complex proposition:
  - Tom is five and Mary is six
- Sentences which are not propositions:
  - Did Steve get an A on the 215 exam?
  - Go away!

Propositional Logic

- In studying properties of propositions we represent them by expressions called **proposition forms** or **formulas** built from propositional variables (atoms), which represent simple propositions and symbols representing logical connectives
  - **Proposition** or **propositional variables**: $p, q, \ldots$

    each can be true or false

    Examples: p="*Socrates is mortal*"

    q="*Plato is mortal*"
  - **Connectives**: $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\sim$
    - connect propositions: $p \vee q$
- Example: "I passed the exam or I did not pass it." $p \vee \sim p$
  - The formula expresses the logical structure of the proposition, where p is an abbreviation for the simple proposition "I passed the exam."

Connectives

- $\sim$ not
- $\wedge$ and
- $\vee$ or (non-exclusive!)
- $\rightarrow$ implies (if … then …)
- $\leftrightarrow$ if and only if
- $\forall$ for all
- $\exists$ exists

Formulas

- Atomic: $p, q, x, y, \ldots$
- Unit Formula: $p, \sim p, (\text{formula}), \ldots$
- Conjunctive: $p \wedge q, p \wedge \sim q, \ldots$
- Disjunctive: $p \vee q, p \vee (q \wedge x), \ldots$
- Conditional: $p \rightarrow q$
- Biconditional: $p \leftrightarrow q$

Negation (~ or ¬ or !)

- We use the symbol ~ to denote negation (same with the textbook)
- Formalization (syntax): If p is a formula, then $\sim p$ is also a formula. We say that the second formula is the *negation* of the first
  - Examples: p, $\sim p$, and $\sim\sim p$ are all formulas.
- Meaning (semantics): If a proposition is true, then its negation is false. If it is false, then its negation is true.
  - The similarity in the structure of a formula and its negation reflects a relationship between the meaning of propositions of this form

Negation (~ or ¬ or !)

- Examples:
  - John went to the store yesterday (p).
  - John did not go to the store yesterday ($\sim p$).
- At the formula level we express the connection via a so-called **truth table**:
  - If p is true, then $\sim p$ is false
  - If p is false, then $\sim p$ is true

Truth Table for $\sim p$

p	$\sim p$
T	F
F	T

Negation (~ or ¬ or !)

- Note: $\sim\sim p \equiv p$

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

p and $\sim(\sim p)$ always have the same truth values, so they are logically equivalent

Conjunction ($\wedge$ or & or •)

- We use the symbol $\wedge$ to denote conjunction (same with the textbook)
- Syntax: If p and q are formulas, then $p \wedge q$ is also a formula.
- Semantics: If p is true and q is true, then $p \wedge q$ is true. In all other cases, $p \wedge q$ is false.

Truth Table for $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Conjunction ($\wedge$ or & or $\bullet$)

- Example:
  1. Bill went to the store.
  2. Mary ate cantaloupe.
  3. Bill went to the store and Mary ate cantaloupe.
- If p and q abbreviate the first and second sentence, then the third is represented by the conjunction $p \wedge q$.

Inclusive Disjunction (∨ or | or +)

- We use the symbol ∨ to denote (inclusive) disjunction.
- Syntax: If p and q are formulas, then $p \lor q$ is also a formula.
- Semantics: If p is true or q is true or both are true, then $p \lor q$ is true. If p and q are both false, then $p \lor q$ is false.

Truth Table for $p \lor q$

p	q	$p \lor q$
T	T	T
T	F	T
F	T	T
F	F	F

Inclusive Disjunction (∨ or | or +)

- Example:
  - John works hard (p).
  - Mary is happy (q).
  - John works hard or Mary is happy ($p \vee q$).

Exclusive Disjunction ($\oplus$, XOR)

- We use the symbol $\oplus$ to denote exclusive disjunction.
- Syntax: If p and q are formulas, then $p \oplus q$ is also a formula.
- Semantics: An exclusive disjunction $p \oplus q$ is true if, and only if, one of p or q is true, but not both.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

- Example:
  - Either John works hard or Mary is happy ($p \oplus q$)

Implication

- Example of proposition:

  If I do not pass the exam I will fail the course.

- Corresponding formula: $\sim p \rightarrow q$

Determining Truth of A Formula

- Atomic formulae: given
- Compound formulae: via meaning of the connectives
  - The semantics of logical connectives determines how propositional formulas are evaluated depending on the truth values assigned to propositional variables
  - Each possible truth assignment or valuation for the propositional variables of a formula yields a truth value. The different possibilities can be summarized in a truth table.

Determining Truth of A Formula

- Example 1: $p \wedge \sim q$ (read "p and not q")

p	q	$\sim q$	$p \wedge \sim q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Determining Truth of A Formula

- Example 2: $p \land (q \lor r)$ (read "p and, in addition, q or r")

p	q	r	$q \lor r$	$p \land (q \lor r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

- Note : it is usually necessary to evaluate all subformulas

Evaluation of formulas - Truth Tables

- A truth table for a formula lists all possible "situations" of truth or falsity, depending on the values assigned to the propositional variables of the formula

Truth Tables

- Example: If p, q and r are the propositions "Peter [Quincy, Richard] will lend Sam money," then Sam can deduce logically correct, that he will be able to borrow money whenever one of his three friends is willing to lend him some ($p \lor q \lor r$)

p	q	r	$p \lor q \lor r$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

- Each row in the truth table corresponds to one possible situation of assigning truth values to p, q and r

Truth Tables

- How many rows are there in a truth table with n propositional variables?
  - For n = 1, there are two rows,
  - for n = 2, there are four rows,
  - for n = 3, there are eight rows, and so on.

Truth Table for $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- Do you see a pattern?

Constructing Truth Tables

- There are two choices (true or false) for each of n variables, so in general there are 2x2x2x…x2 = 2^n rows for n variables.
- A systematic procedure (an algorithm) is necessary to make sure you construct all rows without duplicates
  - construct the rows systematically:
    - count in binary: 000, 001, 010, 011,100, . . .
    - the rightmost column must be computed as a function of all the truth values in the row.

Truth Table for $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Constructing Truth Tables

- Because it is clumsy and time-consuming to build large explicit truth tables, we will be interested in more efficient logical evaluation procedures.

Syntax of Formulas

- The **formal language** of propositional logic can be specified by **grammar rules**
- The **syntactic structure** of a complex logical expression (i.e., its parse tree) must be **unambiguous**

$$
\begin{aligned}
\langle\text{proposition}\rangle ::= \quad & \langle\text{variable}\rangle \\
& \mid (\sim\langle\text{proposition}\rangle) \\
& \mid (\langle\text{proposition}\rangle \wedge \langle\text{proposition}\rangle) \\
& \mid (\langle\text{proposition}\rangle \vee \langle\text{proposition}\rangle) \\
& \ldots
\end{aligned}
$$

$$\langle\text{variable}\rangle ::= p \mid q \mid r \mid \ldots$$

Ambiguities in syntax of Formulas

- For example, the expression $p \land q \lor r$ can be interpreted in two different ways:

p	q	r	$p \land q$	$(p \land q) \lor r$	$q \lor r$	$p \land (q \lor r)$
F	F	T	F	**T**	T	**F**

- Parentheses are needed to avoid ambiguities.
  - Without parentheses the meaning of the formula is not clear!
- The same problem arises in arithmetic: does 5+2 x 4 mean (5+2) x 4 or 5+(2 x 4)?
  - priorities

Simplified Syntax

- In arithmetic one often species a precedence among operators (say, times ahead of plus) to eliminate the need for some parentheses in certain programming languages.
- The same can be done for the logical connectives, though deleting parentheses may cause confusion.
- Example: If $\wedge$ is ahead of $\vee$ in the precedence, there is no ambiguity in $p \wedge q \vee r$

Precedence

- $\sim$
- $\wedge$
- $\vee$
- $\rightarrow$, $\leftrightarrow$

highest

lowest

- Avoid confusion - use '(' and ')':
  - $(p \wedge q) \vee x$

Simplified Syntax

- The properties of the logical connectives can also be exploited to simplify the notation.
  - Example: Disjunction is commutative

p	q	$p \lor q$	$q \lor p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

Simplified Syntax

- AND Disjunction is associative

p	q	r	$(p \lor q) \lor r$	$p \lor (q \lor r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	F	F

- We will therefore ambiguously write $p \lor q \lor r$ to denote either $(p \lor q) \lor r$ or $p \lor (q \lor r)$. The ambiguity is usually of no consequence, as both formulas have the same meaning.

Logical Equivalence

- If two formulas evaluate to the same truth value in all situations, so that their truth tables are the same, they are said to be logically equivalent
- We write $p \equiv q$ to indicate that two formulas p and q are logically equivalent
- If two formulas are logically equivalent, their syntax may be different, but their semantics is the same. The logical equivalence of two formulas can be established by inspecting the associated truth tables.
- Substituting logically inequivalent formulas is the source of most real-world reasoning errors

Logical Equivalence

- Example 1:
  - Is $\sim(p \wedge q)$ logically equivalent to $\sim p \wedge \sim q$?

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	T	F	F	F	F
T	**F**	**F**	**T**	**F**	**T**	**F**
F	**T**	**F**	**T**	**T**	**F**	**F**
F	F	F	T	T	T	T

  - Lines 2 and 3 prove that this is not the case.

Logical Equivalence

- Example 2:
  - Is $\sim(p \land q)$ logically equivalent to $\sim p \lor \sim q$?

p	q	$p \land q$	$\sim(p \land q)$	$\sim p$	$\sim q$	$\sim p \lor \sim q$
T	T	T	**F**	F	F	**F**
T	F	F	**T**	F	T	**T**
F	T	F	**T**	T	F	**T**
F	F	F	**T**	T	T	**T**

  - Yes.

De Morgan's Laws

- There are a number of important equivalences, including the following De Morgan's Laws:
  - $\sim(p \wedge q) \equiv \sim p \vee \sim q$
  - $\sim(p \vee q) \equiv \sim p \wedge \sim q$
  - These equivalences can be used to transform a formula into a logically equivalent one of a certain syntactic form, called a "normal form“
- Another useful logical equivalence is double negation:
  - $\sim\sim p \equiv p$

De Morgan's Laws

- Example:
  - $\sim(\sim p \wedge \sim q) \equiv \sim\sim(p \vee q) \equiv p \vee q$
  - The first equivalence is by De Morgan's Law, the second by double negation
  - We have just derived a new equivalence: $p \vee q \equiv \sim(\sim p \wedge \sim q)$ (as equivalence can be used in both directions) which shows that disjunction can be expressed in terms of conjunction and negation!

Some Logical Equivalences

- You should be able to convince yourself of (i.e., prove) each of these:
  - Commutativity of $\wedge$: $p \wedge q \equiv q \wedge p$
  - Commutativity of $\vee$: $p \vee q \equiv q \vee p$
  - Associativity of $\wedge$: $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
  - Associativity of $\vee$: $p \vee (q \vee r) \equiv (p \vee q) \vee r$
  - Idempotence: $p \equiv p \wedge p \equiv p \vee p$
  - Absorption: $p \equiv p \wedge (p \vee q) \equiv p \vee (p \wedge q)$

Some Logical Equivalences

- You should be able to convince yourself of (i.e., prove) each of these:
  - Distributivity of $\wedge$: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - Distributivity of $\vee$: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
  - Contradictions: $p \wedge \mathrm{F} \equiv \mathrm{F} \equiv p \wedge \sim p$
  - Identities: $p \wedge \mathrm{T} \equiv p \equiv p \vee \mathrm{F}$
  - Tautologies: $p \vee \mathrm{T} \equiv \mathrm{T} \equiv p \vee \sim p$

Tautologies

- A tautology is a formula that is always true, no matter which truth values we assign to its variables.
- Consider the proposition "I passed the exam or I did not pass the exam," the logical form of which is represented by the formula $p \vee \sim p$

p	$\sim p$	$p \vee \sim p$
T	F	**T**
F	T	**T**

  - This is a tautology, as we get T in every row of its truth table.

Contradictions

- A contradiction is a formula that is always false.
- The logical form of the proposition "I passed the exam and I did not pass the exam" is represented by $p \land \sim p$

p	$\sim p$	$p \land \sim p$
T	F	**F**
F	T	**F**

  - This is a contradiction, as we get F in every row of its truth table

Tautologies and contradictions

- Tautologies and contradictions are related

Theorem: If p is a tautology (contradiction) then $\sim p$ is a contradiction (tautology).

$$\sim(p \vee \sim p) \equiv \sim p \wedge \sim\sim p \equiv \sim p \wedge p \equiv p \wedge \sim p$$

Implication ($\rightarrow$)

- Syntax: If p and q are formulas, then $p \rightarrow q$ (read "p implies q") is also a formula.
- We call p the premise and q the conclusion of the implication.
- Semantics: If p is true and q is false, then $p \rightarrow q$ is false. In all other cases, $p \rightarrow q$ is true.
- Truth table:

Truth Table for $p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Implication ($\longrightarrow$)

- Example:
  - p: You get A's on all exams.
  - q: You get an A in this course.
  - $p \longrightarrow q$: If you get A's on all exams, then you will get an A in this course.

Implication ($\rightarrow$)

- The semantics of implication is trickier than for the other connectives
  - if p and q are both true, clearly the implication $p \rightarrow q$ is true
  - if p is true but q is false, clearly the implication $p \rightarrow q$ is false
  - If the premise p is false no conclusion can be drawn, but both q being true and being false are consistent, so that the implication $p \rightarrow q$ is true in both cases
- Implication can also be expressed by other connectives, for example, $p \rightarrow q$ is logically equivalent to $\sim(p \land \sim q)$.

Example: The Case of the Bad Defense Attorney

- Prosecutor:
  - *"If the defendant is guilty, then he had an accomplice."*
- Defense Attorney:
  - *"That's not true!!"*
- What did the defense attorney just claim??
  - $\sim(p \rightarrow q) \equiv \sim\sim(p \wedge \sim q) \equiv p \wedge \sim q$

Biconditional

- Syntax: If p and q are formulas, then $p \leftrightarrow q$ (read "p if and only if (iff) q") is also a formula.
- Semantics: If p and q are either both true or both false, then $p \leftrightarrow q$ is true. Otherwise, $p \leftrightarrow q$ is false.
- Truth table:

Truth Table for $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Biconditional

- Example:
  - p: Bill will get an A.
  - q: Bill studies hard.
  - $p \leftrightarrow q$: Bill will get an A if and only if Bill studies hard.
- The biconditional may be viewed as a shorthand for a conjunction of two implications, as $p \leftrightarrow q$ is logically equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$

Necessary and Sufficient Conditions

- The phrase "necessary and sufficient conditions" appears often in mathematics
- A proposition p is ***necessary*** for q if q cannot be true without it: $\sim p \rightarrow \sim q$ (equivalent to $q \rightarrow p$ is a tautology).
  - Example: It is necessary for a student to have a 3.0 GPA in the core courses to be admitted to become a CSE major.
- A proposition p is ***sufficient*** for q if $p \rightarrow q$ is a tautology.
  - Example: It is sufficient for a student to get A's in CSE114, CSE215, CSE214, and CSE220 in order to be admitted to become a CSE major

Necessary and Sufficient Conditions

Theorem: If a proposition p is both necessary and sufficient for q, then p and q are logically equivalent (and vice versa).

Tautologies and Logical Equivalence

Theorem: A propositional formula p is logically equivalent to q if and only if $p \leftrightarrow q$ is a tautology

- Proof:
  - (a) If $p \leftrightarrow q$ is a tautology, then p is logically equivalent to q

  Why? If $p \leftrightarrow q$ is a tautology, then it is true for all truth assignments. By the semantics of the biconditional, this means that p and q agree on every row of the truth table. Hence the two formulas are logically equivalent.

  - (b) If p is logically equivalent to q, then $p \leftrightarrow q$ is a tautology

  Why? If p and q logically equivalent, then they evaluate to the same truth value for each truth assignment. By the semantics of the biconditional, the formula $p \leftrightarrow q$ is true in all situations. ■

Related Implications

- **Implication:** $p \rightarrow q$
  - If you get A's on all exams, you get an A in the course.
- **Contrapositive:** $\sim q \rightarrow \sim p$
  - If you didn't get an A in the course, then you didn't get A's on all exams
- Note that implication is logically equivalent to the contrapositive

p	q	$p \rightarrow q$	$\sim q$	$\sim p$	$\sim q \rightarrow \sim p$
T	T	**T**	F	F	**T**
T	F	**F**	T	F	**F**
F	T	**T**	F	T	**T**
F	F	**T**	T	T	**T**

Related Implications

- **Converse:** $q \rightarrow p$
  - If you get an A in the course, then you got A's on all exams.
- **Inverse:** $\sim p \rightarrow \sim q$
  - If you didn't get A's on all exams, then you didn't get an A in the course.
- Note that converse is logically equivalent to the inverse

p	q	$q \rightarrow p$	$\sim p$	$\sim q$	$\sim p \rightarrow \sim q$
T	T	**T**	F	F	**T**
T	F	**T**	F	T	**T**
F	T	**F**	T	F	**F**
F	F	**T**	T	T	**T**

Deriving Logical Equivalences

- We can establish logical equivalence either via truth tables OR symbolically
- Example: $p \leftrightarrow q$ is logically equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$

p	q	$q \leftrightarrow p$		$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	**T**		T	T	**T**
T	F	**F**		F	T	**F**
F	T	**F**		T	F	**F**
F	F	**T**		T	T	**T**

- Symbolic proofs are much like the simplifications you did in high school algebra - trial-and-error leads to experience and finally cunning

Symbolic proofs

- Example: $p \wedge q \equiv (p \vee \sim q) \wedge q$
  - Proof:

$$
\begin{aligned}
(p \vee \sim q) \wedge q &\equiv q \wedge (p \vee \sim q) && (1)\\
&\equiv (q \wedge p) \vee (q \wedge \sim q) && (2)\\
&\equiv (q \wedge p) \vee \mathrm{F} && (3)\\
&\equiv (q \wedge p) && (4)\\
&\equiv p \wedge q && (5)
\end{aligned}
$$

Symbolic proofs

- Example: $p \land q \equiv (p \lor \sim q) \land q$
  - Proof: which laws are used at each step?

$$
\begin{aligned}
(p \lor \sim q) \land q &\equiv q \land (p \lor \sim q) && (1)\\
&\equiv (q \land p) \lor (q \land \sim q) && (2)\\
&\equiv (q \land p) \lor \mathrm{F} && (3)\\
&\equiv (q \land p) && (4)\\
&\equiv p \land q && (5)
\end{aligned}
$$

Symbolic proofs

- Example: $p \wedge q \equiv (p \vee \sim q) \wedge q$
  - Proof: which laws are used at each step?

$$
\begin{aligned}
(p \vee \sim q) \wedge q &\equiv q \wedge (p \vee \sim q) && \text{(1) Commutativity of } \wedge \\
&\equiv (q \wedge p) \vee (q \wedge \sim q) && \text{(2)} \\
&\equiv (q \wedge p) \vee \mathrm{F} && \text{(3)} \\
&\equiv (q \wedge p) && \text{(4)} \\
&\equiv p \wedge q && \text{(5)}
\end{aligned}
$$

Symbolic proofs

- Example: $p \wedge q \equiv (p \vee \sim q) \wedge q$
  - Proof: which laws are used at each step?

$(p \vee \sim q) \wedge q$	$\equiv q \wedge (p \vee \sim q)$	(1) Commutativity of $\wedge$
	$\equiv (q \wedge p) \vee (q \wedge \sim q)$	(2) Distributivity of $\wedge$
	$\equiv (q \wedge p) \vee \mathrm{F}$	(3)
	$\equiv (q \wedge p)$	(4)
	$\equiv p \wedge q$	(5)

Symbolic proofs

- Example: $p \wedge q \equiv (p \vee \sim q) \wedge q$
  - Proof: which laws are used at each step?

$(p \vee \sim q) \wedge q$	$\equiv q \wedge (p \vee \sim q)$	(1) Commutativity of $\wedge$
	$\equiv (q \wedge p) \vee (q \wedge \sim q)$	(2) Distributivity of $\wedge$
	$\equiv (q \wedge p) \vee \mathrm{F}$	(3) Contradiction
	$\equiv (q \wedge p)$	(4)
	$\equiv p \wedge q$	(5)

Symbolic proofs

- Example: $p \land q \land r \equiv (p \lor \sim q) \land q$
  - Proof: which laws are used at each step?

$(p \lor \sim q) \land q$	$\equiv q \land (p \lor \sim q)$	(1) Commutativity of $\land$
	$\equiv (q \land p) \lor (q \land \sim q)$	(2) Distributivity of $\land$
	$\equiv (q \land p) \lor \mathrm{F}$	(3) Contradiction
	$\equiv (q \land p)$	(4) Identity
	$\equiv p \land q$	(5)

Symbolic proofs

- Example: $p \wedge q \equiv (p \vee \sim q) \wedge q$
  - Proof: which laws are used at each step?

$(p \vee \sim q) \wedge q$	$\equiv q \wedge (p \vee \sim q)$	(1) Commutativity of $\wedge$
	$\equiv (q \wedge p) \vee (q \wedge \sim q)$	(2) Distributivity of $\wedge$
	$\equiv (q \wedge p) \vee \mathrm{F}$	(3) Contradiction
	$\equiv (q \wedge p)$	(4) Identity
	$\equiv p \wedge q$	(5) Commutativity of $\wedge$

Logical Consequence

- We say that p logically implies q, or that q is a logical consequence of p, if q is true whenever p is true.
- Example: p logically implies $p \vee q$

p	q	$p \vee q$
T	T	**T**
T	F	**T**
F	T	T
F	F	F

- Note that logical consequence is a weaker condition than logical equivalence

Logical Consequence

Theorem: A formula p logically implies q if and only if $p \longrightarrow q$ is a tautology.

- This gives us a tool to infer truths!
- A rule of inference is a rule of the form:

  "From premises p_1, p_2, …, p_n infer conclusion q"

  - A rule of inference is **sound** or **valid** if the conclusion q is a logical consequence of the conjunction $p_1 \wedge p_2 \wedge \ldots \wedge p_n$ of all premises
  - A rule of inference is **unsound** or **bogus** if it isn't!