Introduction to Medical Imaging

Lecture 11: Cone-Beam CT Theory

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Introduction

Available cone-beam reconstruction methods:

- exact
- approximate
- algebraic

Our discussion:

- exact (now)
- approximate (next)

The Radon transform and its inverse are important mechanisms to understand cone-beam CT

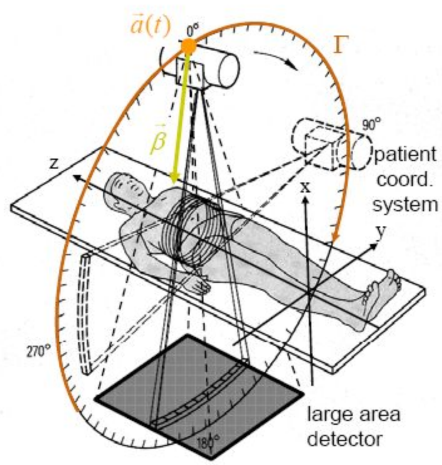
Cone-Beam Transform

$$D\mu\left(\vec{a}(t),\vec{\beta}\right)=\int_0^\infty \mu\left(\vec{a}(t)+s\vec{\beta}\right)ds,\quad \left(\vec{a},\vec{\beta}\right)\in\Gamma\times S^2$$

$\vec{a}(t)$ is the source position along trajectory Γ

$\vec{\beta}$ the unit vector pointing along a particular x-ray beam

The cone-beam transform reflects the data acquisition process of measuring line integrals of the attenuation coefficient μ.



from: Dr. Günter Lauritsch, Siemens

2D Radon Transform

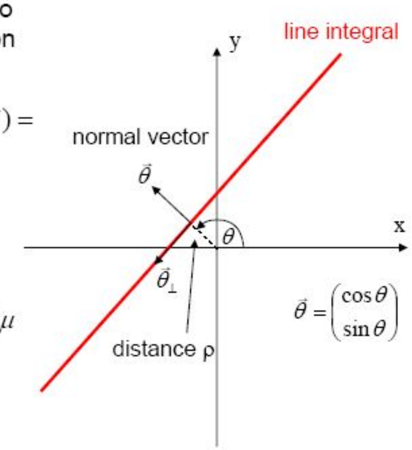
The analytical approach of reconstruction by projections has to be done in the context of the Radon transform $\Re$

$$\Re\mu(\rho,\vec{\theta})=\int d^2r\,\delta(\vec{r}\cdot\vec{\theta}-\rho)\cdot\mu(\vec{r})=\int_{-\infty}^{+\infty} dl\;\mu(\rho\cdot\vec{\theta}+l\cdot\vec{\theta}_\perp)$$

Thus in the 2D case the Radon transform $\Re\mu$ is identical to the measured cone beam transform $D\mu$

$$D\mu(\vec{a},\vec{\theta}_\perp)\Big|_{\vec{a}\cdot\vec{\theta}=\rho}=\Re\mu(\rho,\vec{\theta})$$

with projection angle θ.



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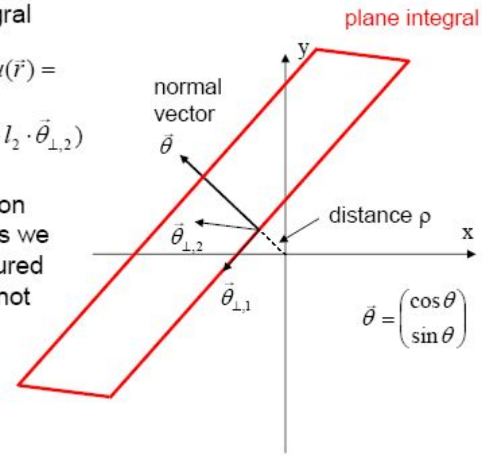
3D Radon Transform

In three dimensions the Radon transform $\Re$ is a plane integral

$$\Re\mu(\rho,\vec{\theta}) = \int d^3r\ \delta(\vec{r}\cdot\vec{\theta}-\rho)\cdot\mu(\vec{r}) = \int_{-\infty}^{+\infty} dl_1 \int_{-\infty}^{+\infty} dl_2\ \mu(\rho\cdot\vec{\theta}+l_1\cdot\vec{\theta}_{\perp,1}+l_2\cdot\vec{\theta}_{\perp,2})$$

which is a severe complication compared to the 2D case. As we will see the link to the measured cone beam transform $D\mu$ is not trivial.



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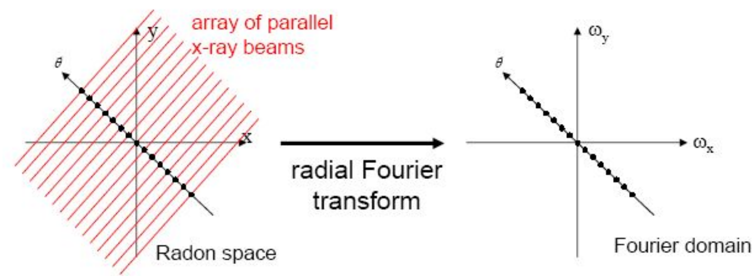
Fourier-Slice Theorem in 2D

$$F_\rho\Re\mu(\rho,\vec{\theta}) = (F_2\mu)(\omega_\rho\cdot\vec{\theta})$$

The radial 1D Fourier transform F_ρ of the Radon transform $\Re\mu$ along $\vec{\theta}$ is equal to the 2D Fourier transform F_2 of the object μ along $\vec{\theta}$ perpendicular to the direction of the projection.



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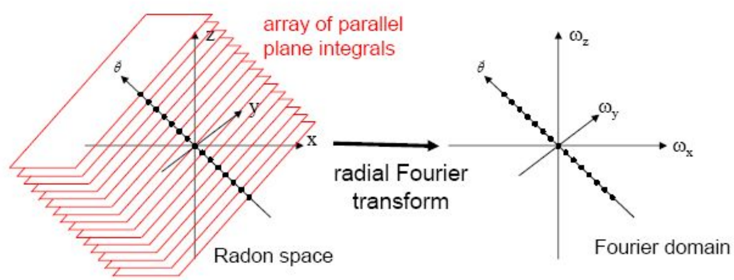
Fourier-Slice Theorem in 3D

$$F_\rho\Re\mu(\rho,\vec{\theta}) = (F_3\mu)(\omega_\rho\cdot\vec{\theta})$$

The radial 1D Fourier transform F_ρ of the Radon transform $\Re\mu$ along $\vec{\theta}$ is equal to the 3D Fourier transform F_3 of the object μ along $\vec{\theta}$ perpendicular to the direction of the projection.



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Exact Reconstruction in 2D and 3D

In 2D:

- use 2D inversion formula: the filtered backprojection procedure
- we have seen a spatial technique, only performing filtering in the frequency domain (in a polar grid)
- but may also interpolate the polar grid in the frequency domain and invert the resulting cartesian lattice
- employ linogram techniques for the latter (see later)

In 3D:

- use 3D inversion formula: not nearly as straightforward than 2D inversion
- full frequency-space methods also exist
- more details next (on all)

Exact Inversion Formula

The basic 3D inversion filtered backprojection formula, due to Natterer (1986):

$$f(\boldsymbol{x}) = \frac{-1}{8\pi^2} \int_{S^2} \frac{\partial^2}{\partial \rho^2} \Re f(|\rho|\boldsymbol{\theta})\, \mathrm{d}\boldsymbol{\theta}.$$

- θ is the angle, a unit vector on a unit sphere
- x, ρ are object and Radon space coordinates, resp.: $|\rho| = x \cdot \theta$
- involves a 2^{nd} derivative of the 3D Radon transform
- the second derivative operator can be treated as a convolution kernel

Some manipulations can reduce the second derivative to a first derivative, along with convolution operators

$$f(x) = \frac{1}{2} \int_{S^2} \frac{-1}{4\pi^2} \frac{\partial^2}{\partial \rho^2} \Re f(|\rho|\boldsymbol{\theta})\, \mathrm{d}\boldsymbol{\theta} = \frac{1}{2} \int_{S^2} \frac{-1}{2\pi^2 \rho^2} * \frac{\partial}{\partial \rho} \left[\frac{1}{2\pi^2 \rho} * \Re f(|\rho|\boldsymbol{\theta}) \right] \mathrm{d}\boldsymbol{\theta}$$

- many different variants have been proposed
  - for example: Kudo/Saito (1990), Smith (1985)

Grangeat's Algorithm

Phase 1:

- from cone-beam data to derivatives of Radon data

Phase 2:

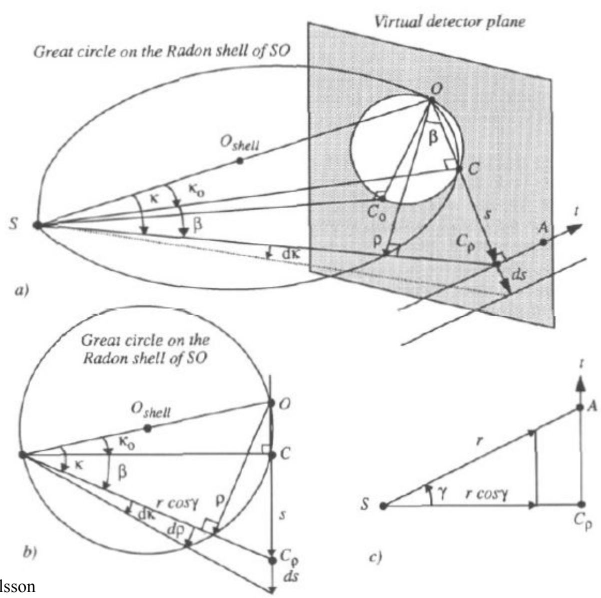
- from derivatives of Radon data to reconstructed 3D object

There are many ways to achieve Phase 2

- direct, $O(N^5)$
- a two-step procedure, $O(N^4)$ [Marr et al, 1981]
- a Fourier method, $O(N^3 \log N)$, [Axelsson/Danielsson, 1994]
- a divide-and-conquer strategy, $O(N^3 \log N)$ [Basu/Bresler, 2002]
- we shall discuss the first three here

But first let us see how Radon data are generated from cone-beam data

Transforming Cone-Beam to Radon Data



from Axelsson/Danielsson

Transforming Cone-Beam to Radon Data

$$\frac{\mathrm{d}}{\mathrm{d}\rho}[\Re f(\rho)] = \int_{-\pi/2}^{\pi/2} \int_0^{\infty} \frac{\mathrm{d}}{\mathrm{d}\rho} f(\rho, r, \gamma) r \,\mathrm{d}r\, \mathrm{d}\gamma = \frac{\mathrm{d}}{\mathrm{d}\kappa} \int_{-\pi/2}^{\pi/2} \boldsymbol{X} f(\rho, \gamma) \frac{1}{\cos\gamma} \mathrm{d}\gamma$$
$$= \frac{SC}{\cos^2\beta} \frac{\mathrm{d}}{\mathrm{d}s} \int_{-\infty}^{\infty} \frac{1}{SA} \boldsymbol{X} f(\rho, t)\, \mathrm{d}t.$$

Strategy:

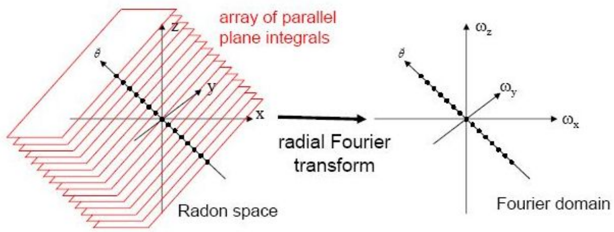
- weigh detector data with a factor 1/SA
- integrate along all intersections (lines) between the detector plane and the required Radon planes
  - there are N^2 such lines (N lines and N rotations)
- take the derivative in the s-direction (in the detector plane perpendicular to t)
- weight the 2D data set resulting from a single source position by the factor SC / $\cos^2 \beta$

The order of these operations can be switched since they are all linear (Grangeat swapped the order of operation 2 and 3)

Radon Data to Object: Direct Method

There are $O(N^3)$ data points in Radon (derivative) space

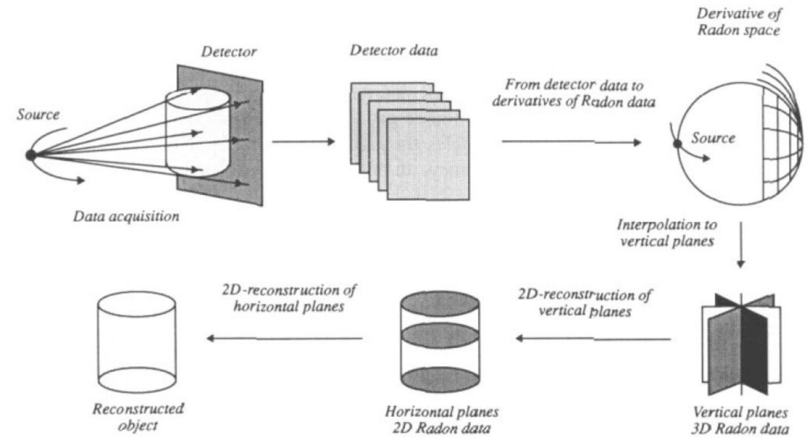
Each is due to a plane integral



The direct method simply inserts the plane data into the object space, one by one

- this is basically the expansion of a point into a plane, defined by (θ, ρ)
- this gives rise to an $O(N^5)$ algorithm

Radon Data to Object: Two-Step Method



from Axelsson/Danielsson

Radon Data to Object: Two-Step Method

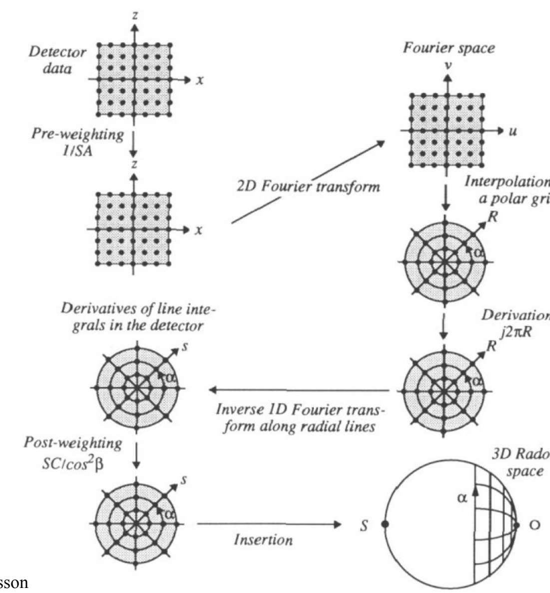
Each vertical plane holds all Radon points due to plane integrals of perpendicularly intersecting planes

- filtered backprojection reduces the plane integrals to line integrals, confined to horizontal planes

The horizontal planes are then reconstructed with another filtered backprojection

Each such operation is $O(N^3)$ and there are $O(N)$ of them, resulting in a complexity of $O(N^4)$

Radon Data to Object: Fourier Space Approach



from Axelsson/Danielsson

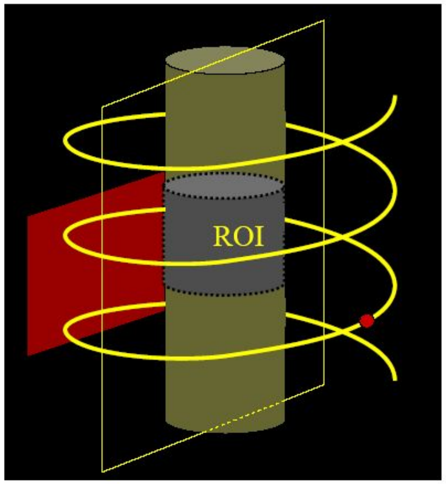
Radon Data to Object: Fourier Space Approach

Takes advantage of the O(N log N) complexity of the FFT at various steps

It also uses linograms [Edholm/Herman, 1987] to reduce 2D interpolation to 1D interpolation

The complexity is then O(N^3 log N)

Long Object Problem



- Reconstruction of an ROI should be feasible from projection data restricted to the ROI and some surrounding.
- The basic 3D Radon inversion formula does not fulfill this request.

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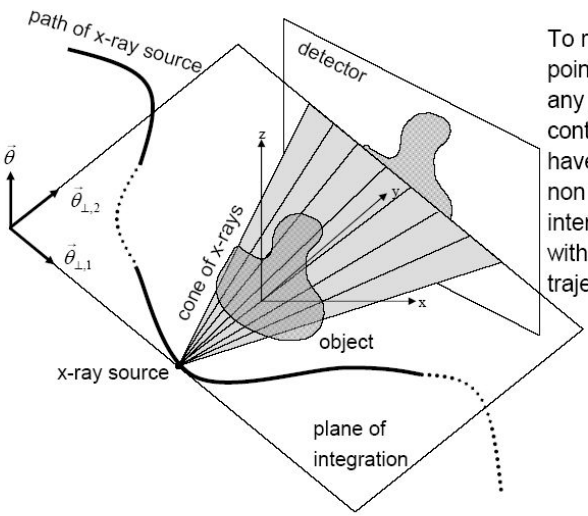
Tuy's Sufficiency Condition



To reconstruct a point x of the object any plane containing x must have at least one non tangential intersection point with the source trajectory.

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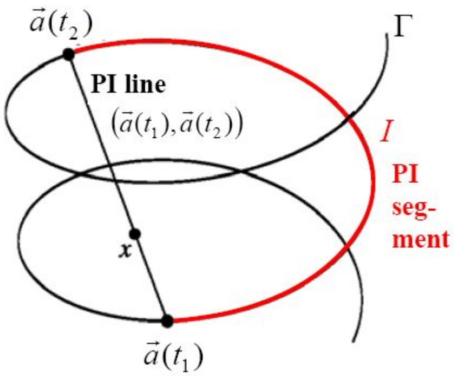
Concept of PI-Lines



For a point x on a PI line any plane containing x has at least one intersection point with the PI segment associated with the PI line.

The PI segment is that portion of the source trajectory needed for reconstructing the point x.

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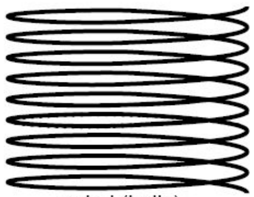
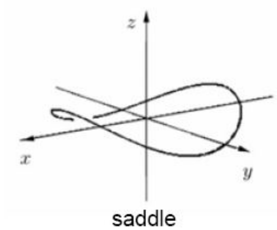
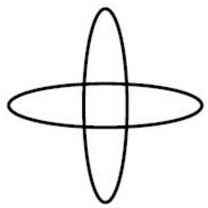
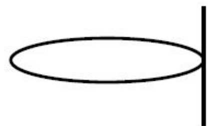
Examples of Complete Trajectories


spiral (helix)



saddle


two orthogonal (tilted) circles


circle and line

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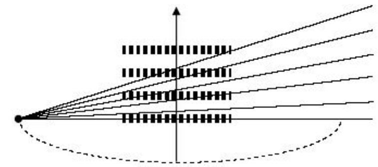
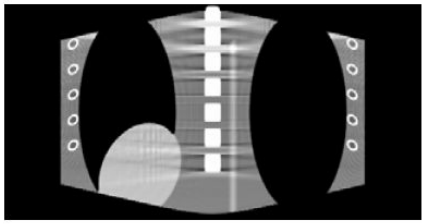
Circular Source Path

A prominent example of an incomplete trajectory




Thorax simulation study.
Coronal slice. C=0, W=200

- Due to incomplete data sampling cone artifacts show up at sharp z-edges of objects with high contrast.
- Almost horizontal rays (or integration planes) are missing to distinguish between the members of the object stack.

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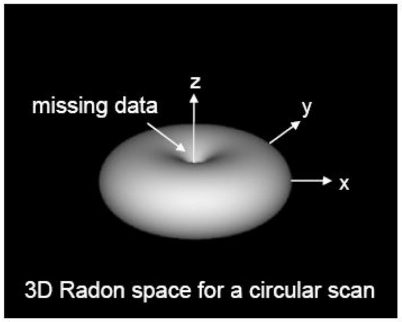
3D Radon Data Acquired by a Circular Trajectory



3D Radon space for a circular scan

By a circular source trajectory a donut shaped region is acquired in 3D Radon space. At the z-axis a cone-like region is missing.

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Challenges in Cone-Beam Reconstruction

The naive application of the 3D Radon inversion formula is prohibitive due to

- long object problem
- enormous computational expense

Simplifications have to be found to end up in an efficient and numerically stable reconstruction algorithm preferably in a shift-invariant 1D-filtered backprojection algorithm

Utilization of redundant data is obscure. Ideally redundancy in collected Radon planes has to be considered. However, this approach is suboptimal because:

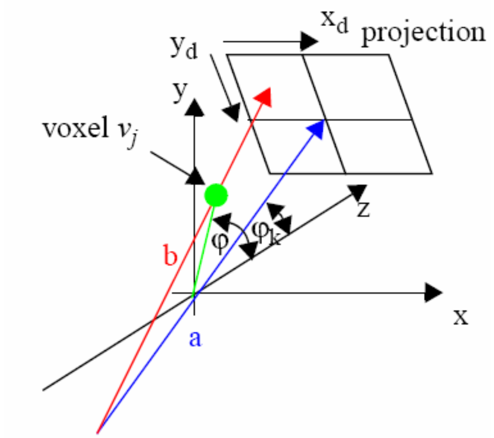
- it is quite complicated
- underestimates the redundancy of data
- typically in cone beam, the data are highly redundant in approximation

The Feldkamp-Davis-Kress Algorithm

Approximate cone-beam algorithm

Works well for smaller cone-beam angles

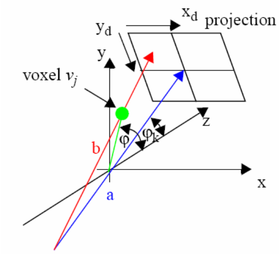
Widely in use



The Feldkamp Algorithm: Details

for each projection

- weight pixels by *a/b*
- ramp-filter each column (along y_d direct.)
- for each grid voxel v_j
  - project v_j onto image along cone-rays
  - interpolate voxel update dv_j
  - weight dv_j by depth factor c_j: $dv_j = dv_j \cdot c_j$
  - add result to grid voxel: $v_j = v_j + dv_j$



$$c_j = \frac{a^2}{\left(a + \sqrt{v_{jy}^2 + v_{jz}^2}\cos(\varphi - \varphi_k)\right)^2}$$