Introduction to Medical Imaging

Cone-Beam CT

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Available cone-beam reconstruction methods:
  • exact
  • approximate

Our discussion:
  • exact (now)
  • approximate (next)

The Radon transform and its inverse are important mechanisms to understand cone-beam CT
Cone-Beam Transform

\[ D \mu(\vec{a}(t), \vec{\beta}) = \int_0^\infty \mu(\vec{a}(t) + s\vec{\beta}) \, ds, \quad (\vec{a}, \vec{\beta}) \in \Gamma \times S^2 \]

\( \vec{a}(t) \) is the source position along trajectory \( \Gamma \)

\( \vec{\beta} \) the unit vector pointing along a particular x-ray beam

The cone-beam transform reflects the data acquisition process of measuring line integrals of the attenuation coefficient \( \mu \).

from: Dr. Günter Lauritsch, Siemens
2D Radon Transform

The analytical approach of reconstruction by projections has to be done in the context of the Radon transform $\mathcal{R}$

$$\mathcal{R} \mu(\rho, \theta) = \int d^2 r \, \delta(\vec{r} \cdot \vec{\theta} - \rho) \cdot \mu(\vec{r}) = \int_{-\infty}^{+\infty} dl \, \mu(\rho \cdot \vec{\theta} + l \cdot \vec{\theta}_\perp)$$

Thus in the 2D case the Radon transform $\mathcal{R} \mu$ is identical to the measured cone beam transform $D\mu$

$$\left. D\mu(\vec{a}, \vec{\theta}_\perp) \right|_{\vec{a} \cdot \vec{\theta} = \rho} = \mathcal{R} \mu(\rho, \theta)$$

with projection angle $\theta$.

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from: Dr. Günter Lauritsch, Siemens
In three dimensions the Radon transform $\mathcal{R}$ is a plane integral

$$\mathcal{R} \mu(\rho, \vec{\theta}) = \int d^3r \, \delta(\vec{r} \cdot \vec{\theta} - \rho) \cdot \mu(\vec{r}) = \int_{-\infty}^{+\infty} dl_1 \int_{-\infty}^{+\infty} dl_2 \, \mu(\rho \cdot \vec{\theta} + l_1 \cdot \vec{\theta}_{\perp,1} + l_2 \cdot \vec{\theta}_{\perp,2})$$

which is a severe complication compared to the 2D case. As we will see the link to the measured cone beam transform $D\mu$ is not trivial.
Fourier-Slice Theorem in 2D

\[ F_\rho \mathcal{R} \mu(\rho, \bar{\theta}) = (F_2 \mu)(\omega_\rho \cdot \bar{\theta}) \]

The radial 1D Fourier transform \( F_\rho \) of the Radon transform \( \mathcal{R} \mu \) along \( \bar{\theta} \) is equal to the 2D Fourier transform \( F_2 \) of the object \( \mu \) along \( \bar{\theta} \) perpendicular to the direction of the projection.

from: Dr. Günter Lauritsch, Siemens
Fourier-Slice Theorem in 3D

\[ F_\rho \mathcal{R}_\mu(\rho, \hat{\theta}) = (F_3 \mu)(\omega_\rho \cdot \hat{\theta}) \]

The radial 1D Fourier transform \( F_\rho \) of the Radon transform \( \mathcal{R}_\mu \) along \( \hat{\theta} \) is equal to the 3D Fourier transform \( F_3 \) of the object \( \mu \) along \( \hat{\theta} \) perpendicular to the direction of the projection.
In 2D:

- use 2D inversion formula: the filtered backprojection procedure
- we have seen a spatial technique, only performing filtering in the frequency domain (in a polar grid)
- but may also interpolate the polar grid in the frequency domain and invert the resulting cartesian lattice
- employ linogram techniques for the latter (see later)

In 3D:

- use 3D inversion formula: not nearly as straightforward than 2D inversion
- full frequency-space methods also exist
- more details next (on all)
The basic 3D inversion filtered backprojection formula, due to Natterer (1986):

\[
f(x) = \frac{-1}{8\pi^2} \int_{S^2} \frac{\partial^2}{\partial \rho^2} R f(|\rho|\theta) \, d\theta.
\]

- \(\theta\) is the angle, a unit vector on a unit sphere
- \(x, \rho\) are object and Radon space coordinates, resp.: \(|\rho| = x \cdot \theta\)
- involves a 2\textsuperscript{nd} derivative of the 3D Radon transform
- the second derivative operator can be treated as a convolution kernel

Some manipulations can reduce the second derivative to a first derivative, along with convolution operators

\[
f(x) = \frac{1}{2} \int_{S^2} \frac{-1}{4\pi^2} \frac{\partial^2}{\partial \rho^2} R f(|\rho|\theta) \, d\theta = \frac{1}{2} \int_{S^2} \frac{-1}{2\pi^2 \rho^2} \ast \frac{\partial}{\partial \rho} \left[ \frac{1}{2\pi^2 \rho} \ast R f(|\rho|\theta) \right] \, d\theta
\]

- many different variants have been proposed
  - for example: Kudo/Saito (1990), Smith (1985)
Grangeat’s Algorithm

Phase 1:
- from cone-beam data to derivatives of Radon data

Phase 2:
- from derivatives of Radon data to reconstructed 3D object

There are many ways to achieve Phase 2
- direct, $O(N^5)$
- a two-step procedure, $O(N^4)$ [Marr et al, 1981]
- a Fourier method, $O(N^3 \log N)$, [Axelsson/Danielsson, 1994]
- a divide-and-conquer strategy, $O(N^3 \log N)$ [Basu/Bresler, 2002]
- we shall discuss the first three here

But first let us see how Radon data are generated from cone-beam data
Transforming Cone-Beam to Radon Data

from Axelsson/Danielsson
Transforming Cone-Beam to Radon Data

\[
\frac{d}{d\rho} \left[ \mathcal{R} f(\rho) \right] = \int_{-\pi/2}^{\pi/2} \int_0^\infty \frac{d}{d\rho} f(\rho, r, \gamma) r \, dr \, d\gamma = \frac{d}{d\kappa} \int_{-\pi/2}^{\pi/2} X f(\rho, \gamma) \frac{1}{\cos \gamma} \, d\gamma
\]

\[
= \frac{SC}{\cos^2 \beta} \frac{d}{ds} \int_{-\infty}^{\infty} \frac{1}{SA} X f(\rho, t) \, dt.
\]

Strategy:

- weigh detector data with a factor 1/SA
- integrate along all intersections (lines) between the detector plane and the required Radon planes
  - there are \( N^2 \) such lines (N lines and N rotations)
- take the derivative in the s-direction (in the detector plane perpendicular to t)
- weight the 2D data set resulting from a single source position by the factor \( SC / \cos^2 \beta \)

The order of these operations can be switched since they are all linear (Grangeat swapped the order of operation 2 and 3)
Radon Data to Object: Direct Method

There are $O(N^3)$ data points in Radon (derivative) space

Each is due to a plane integral

The direct method simply inserts the plane data into the object space, one by one

- this is basically the expansion of a point into a plane, defined by $(\theta, \rho)$
- this gives rise to an $O(N^5)$ algorithm
Radon Data to Object: Two-Step Method

from Axelsson/Danielsson
Each vertical plane holds all Radon points due to plane integrals of perpendicularly intersecting planes
  • filtered backprojection reduces the plane integrals to line integrals, confined to horizontal planes

The horizontal planes are then reconstructed with another filtered backprojection

Each such operation is $O(N^3)$ and there are $O(N)$ of them, resulting in a complexity of $O(N^4)$
Radon Data to Object: Fourier Space Approach

from Axelsson/Danielsson
Takes advantage of the $O(N \log N)$ complexity of the FFT at various steps

It also uses linograms [Edholm/Herman, 1987] to reduce 2D interpolation to 1D interpolation

The complexity is then $O(N^3 \log N)$
Long Object Problem

- Reconstruction of an ROI should be feasible from projection data restricted to the ROI and some surrounding.
- The basic 3D Radon inversion formula does not fulfill this request.

from: Dr. Günter Lauritsch, Siemens
Tuy's Sufficiency Condition

To reconstruct a point $x$ of the object any plane containing $x$ must have at least one non tangential intersection point with the source trajectory.

from: Dr. Günter Lauritsch, Siemens
Concept of PI-Lines

For a point $x$ on a PI line any plane containing $x$ has at least one intersection point with the PI segment associated with the PI line.

The PI segment is that portion of the source trajectory needed for reconstructing the point $x$.

from: Dr. Günter Lauritsch, Siemens
Examples of Complete Trajectories

- Spiral (helix)
- Saddle
- Two orthogonal (tilted) circles
- Circle and line

from: Dr. Günter Lauritsch, Siemens
A prominent example of an incomplete trajectory

- Due to incomplete data sampling cone artifacts show up at sharp z-edges of objects with high contrast.
- Almost horizontal rays (or integration planes) are missing to distinguish between the members of the object stack.

Thorax simulation study.
Coronal slice. C=0, W=200

from: Dr. Günter Lauritsch, Siemens
3D Radon Data Acquired by a Circular Trajectory

By a circular source trajectory a donut shaped region is acquired in 3D Radon space. At the z-axis a cone-like region is missing.

from: Dr. Günter Lauritsch, Siemens
The naive application of the 3D Radon inversion formula is prohibitive due to

- long object problem
- enormous computational expense

Simplifications have to be found to end up in an efficient and numerically stable reconstruction algorithm preferably in a shift-invariant 1D-filtered backprojection algorithm.

Utilization of redundant data is obscure. Ideally, redundancy in collected Radon planes has to be considered. However, this approach is suboptimal because:

- it is quite complicated
- underestimates the redundancy of data
- typically in cone beam, the data are highly redundant in approximation
A typical reconstruction algorithm is Filtered Backprojection.
Feldkamp-Davis-Kress (FDK) Cone-beam reconstruction
Filtered projection data

\[ \hat{P}_\phi(Y, Z) = \frac{D}{\sqrt{D^2 + Y^2 + Z^2}} P_\phi(Y, Z) * * g(Y) \]

circular pre-weighting

projection data

ramp filter
FDK: Backprojection

\[
\hat{P}_\phi (r) = \hat{P}_\phi (Y(r), Z(r)), \quad Y(r) = \frac{r \cdot y_\phi}{d + r \cdot x_\phi} D, \quad Z(r) = \frac{r \cdot z_\phi}{d + r \cdot x_\phi} D
\]

voxel \rightarrow projection mapping  

projection coordinates of mapped voxel
FDK: Accumulation, Depth-Weighting

\[ f(r) = \frac{1}{4\pi^2} \int_0^{2\pi} \frac{d^2}{(d + r \cdot x_\phi)^2} \hat{P}_\phi(r) d\phi \]

reconstructed voxel

accumulation for all projections

depth-weighting