Introduction to Medical Imaging

Sampling

Klaus Mueller

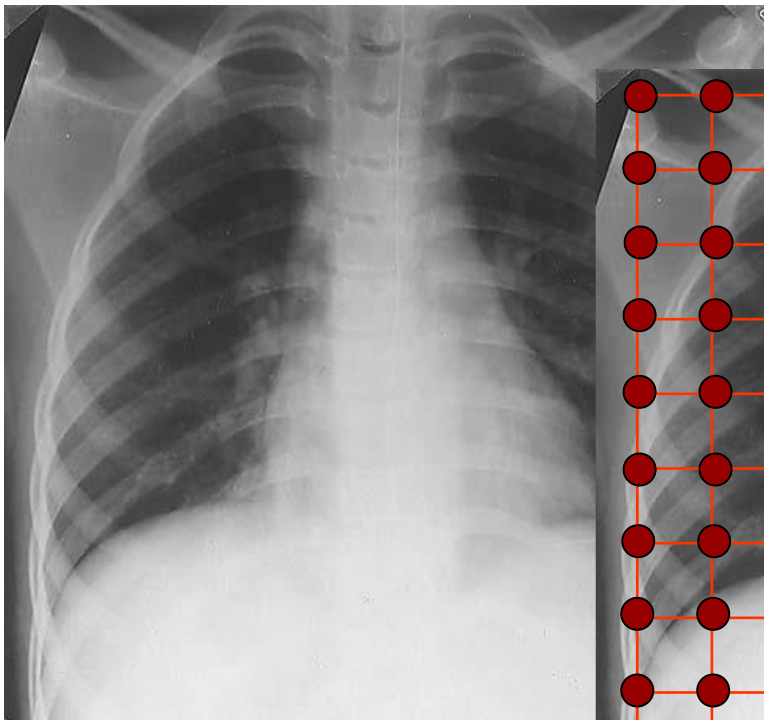
Computer Science Department

Stony Brook University

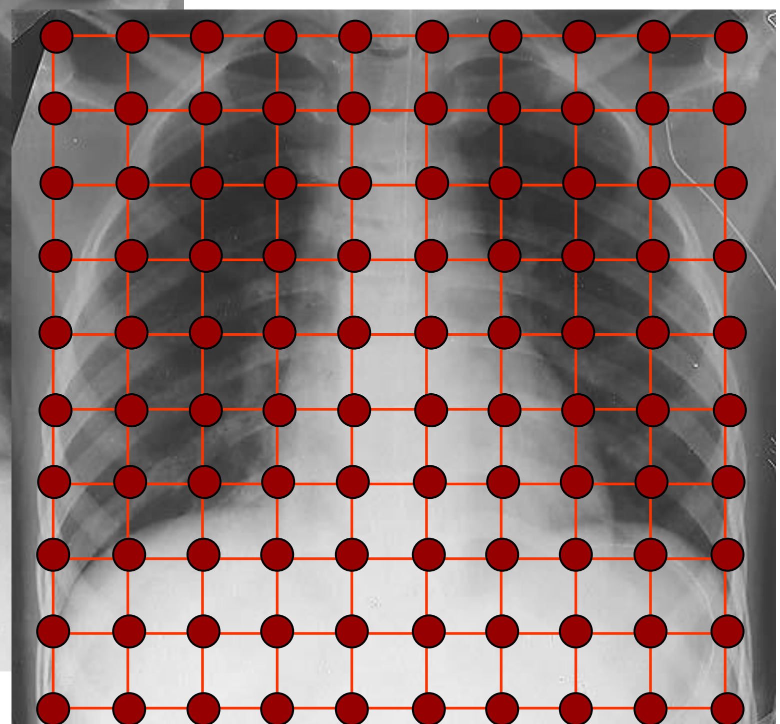
Introduction

Sampling is the process of discretizing a continuous function into an array/matrix of data points

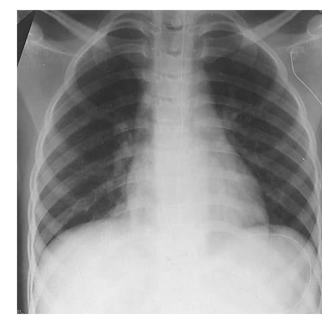
- the matrix values are some function of the sampled real-life object
- this function is given by the *sampling filter* (more to follow)



object



sampling the object



sampling result

Importance of the Fourier Domain

Visual artifacts are also often easier understood in the Fourier domain

We can use the Fourier domain to:

- gain insight into the spatial / temporal frequency content of the data (see last lecture)
- from this, gain insight into how much a continuous signal must be sampled when it is discretized
- design proper filters to avoid an important phenomenon: *aliasing*

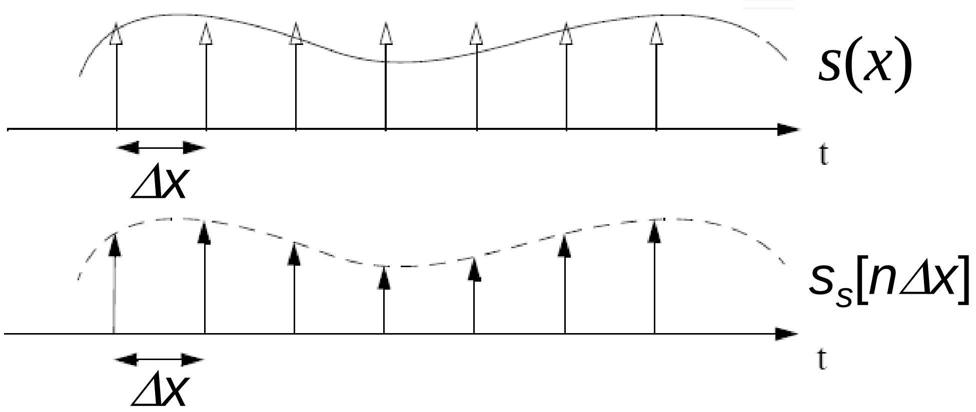
We usually do not use the Fourier domain to:

- perform the actual signal filtering, sampling, resampling, reconstruction (there are exceptions, however)
- these real operations are usually performed in the original signal domain (spatial, temporal)

Sampling: Spatial Domain

Definition:

- a continuous signal $s(x)$ is measured at fixed instances spaced apart by an interval Δx
- the data points so obtained form a discrete signal $s_s[n\Delta x] = s_s(n\Delta x)$
- here, Δx is called the *sampling period (distance)*, and $K = 1/\Delta x$ the *sampling frequency*



Sampling is the multiplication of the signal with an impulse train:

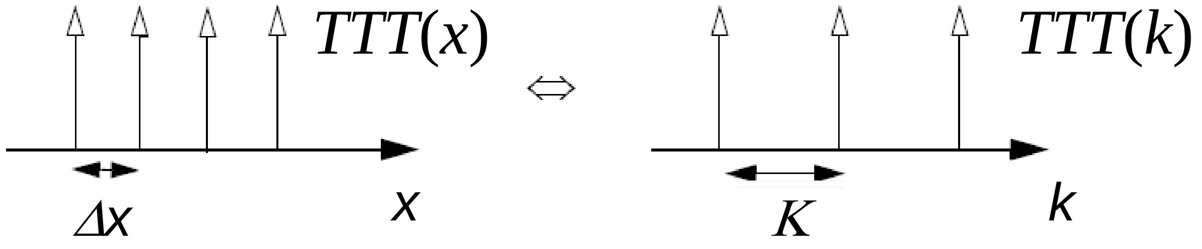
$$s_s(x) = s(x) \cdot \text{TTT}(x)$$

$$\text{TTT}(x) = \sum_{n=-\infty}^{+\infty} \delta(x - n\Delta x), \quad \text{TTT}(x) \text{ is the comb function}$$

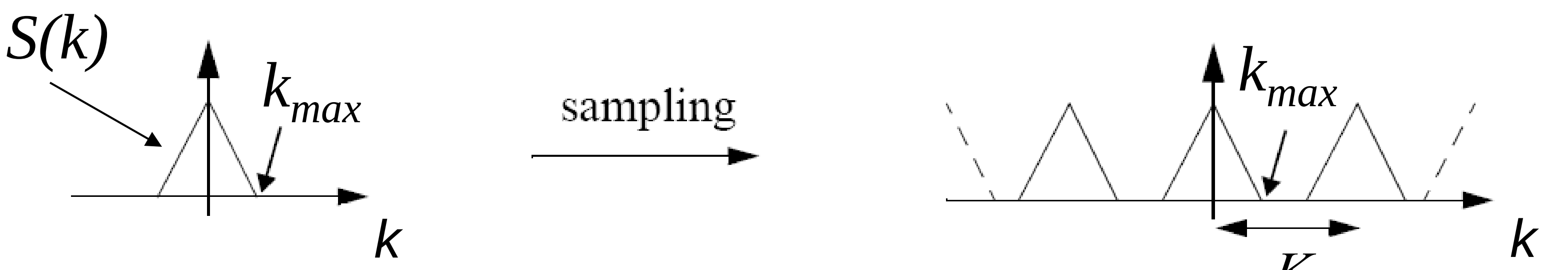
Sampling: Frequency Domain

Using the convolution theorem of the Fourier transform:

$$S_s(k) = S(k) * F\{\text{TTT}(x)\}, \quad \text{where } F\{\text{TTT}(x)\} = K \sum_{l=-\infty}^{+\infty} \delta(k - lK)$$



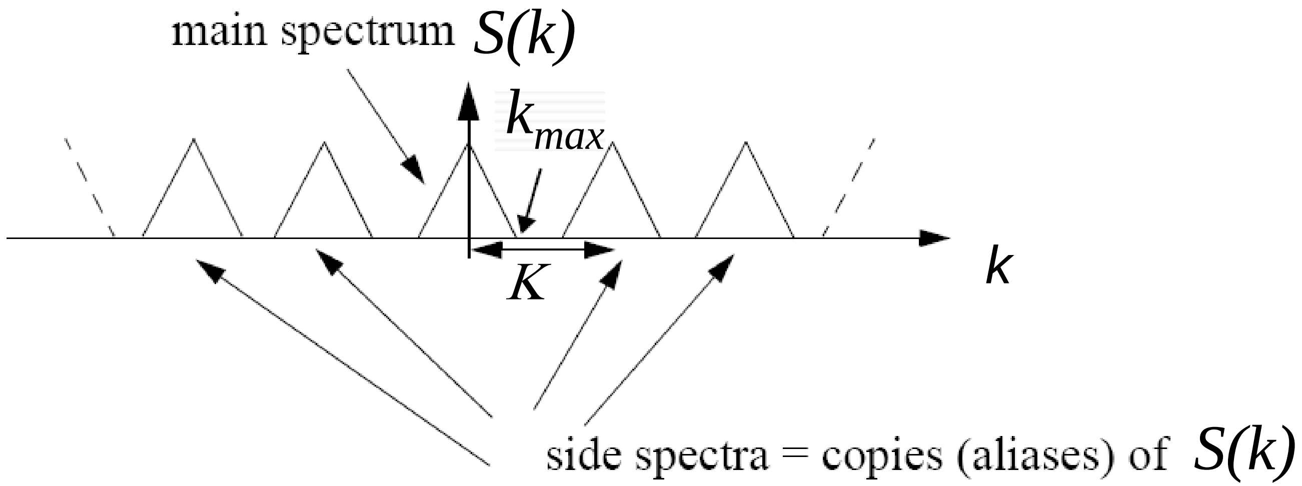
- the smaller Δx the wider K (recall the Fourier scaling theorem)
- sampling (the convolution of $TTT(k)$ and $S(k)$) replicates the signal spectrum $S(k)$ at integer multiples of sampling frequency K



- k_{max} is maximum frequency occuring in the signal

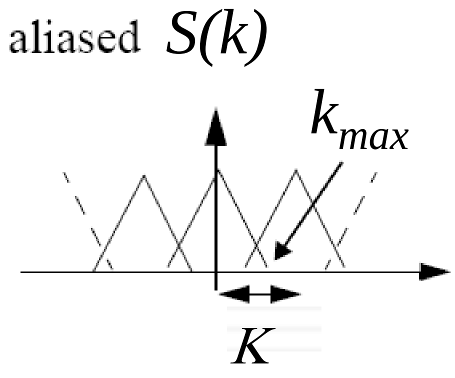
Aliasing

Terminology:



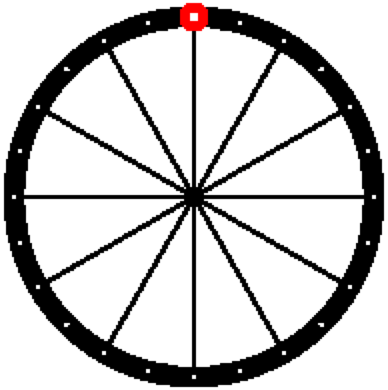
However, if we choose $K < 2\,k_{max}$ the aliases overlap and we get *aliasing*

- what does aliasing look like?
- let's see some examples



Aliasing: A Commonly Observed Phenomenon

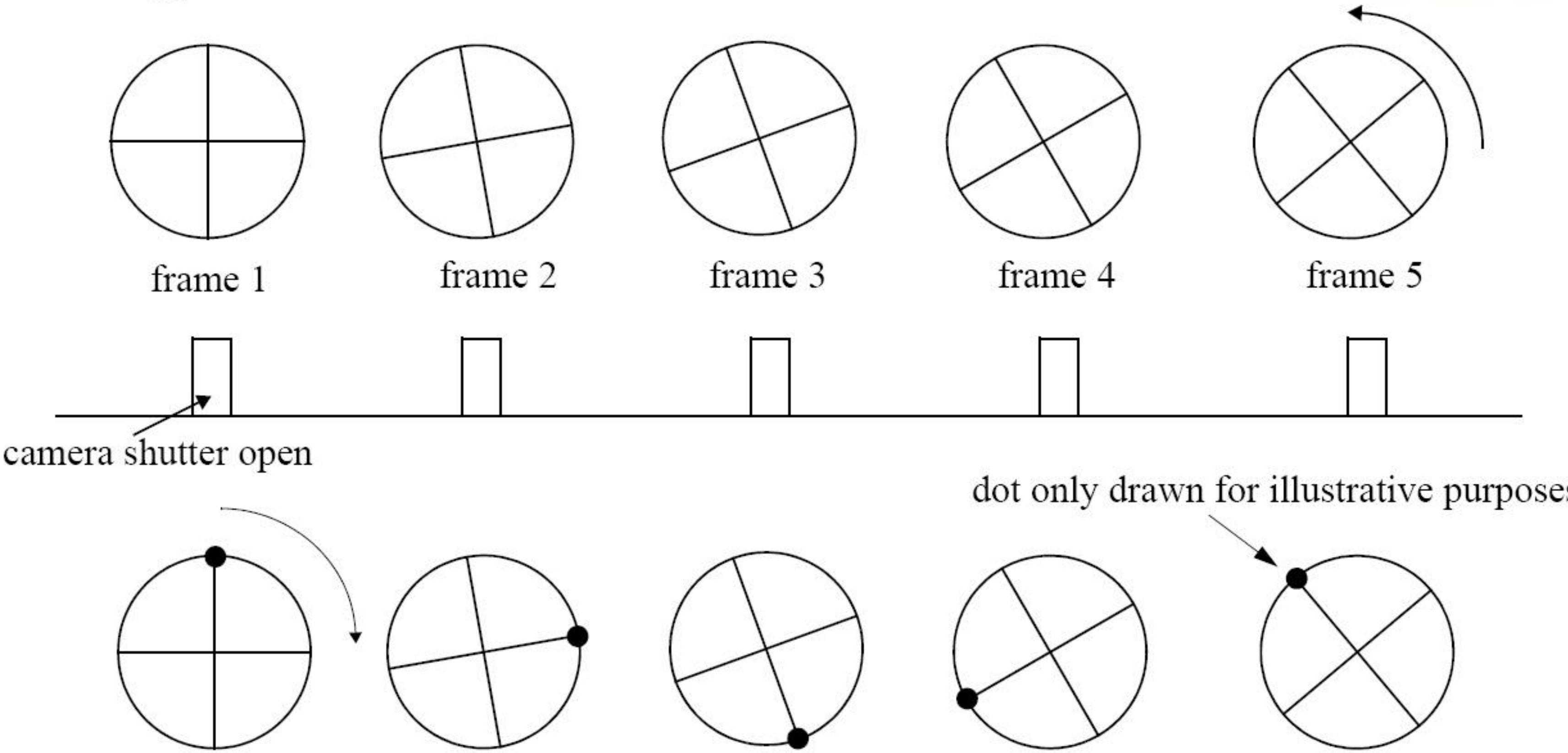
Ever wondered about the wagon wheels in old Western movies:



Aliasing: A Commonly Observed Phenomenon

- Wagon wheel in old Western movies:

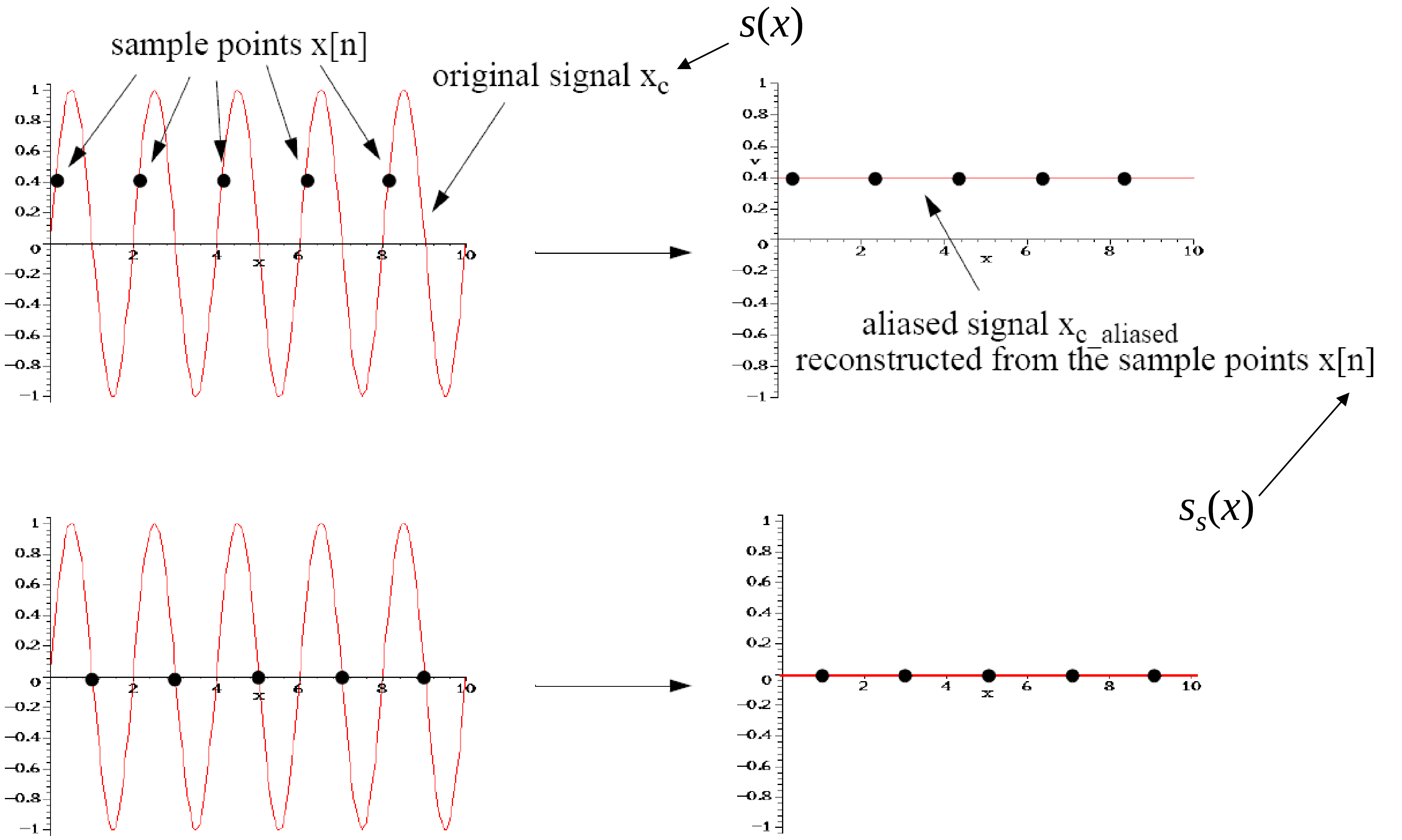
wheel appears to turn counter-clockwise at a slow rate...



but in reality turns clockwise at a much faster rate...

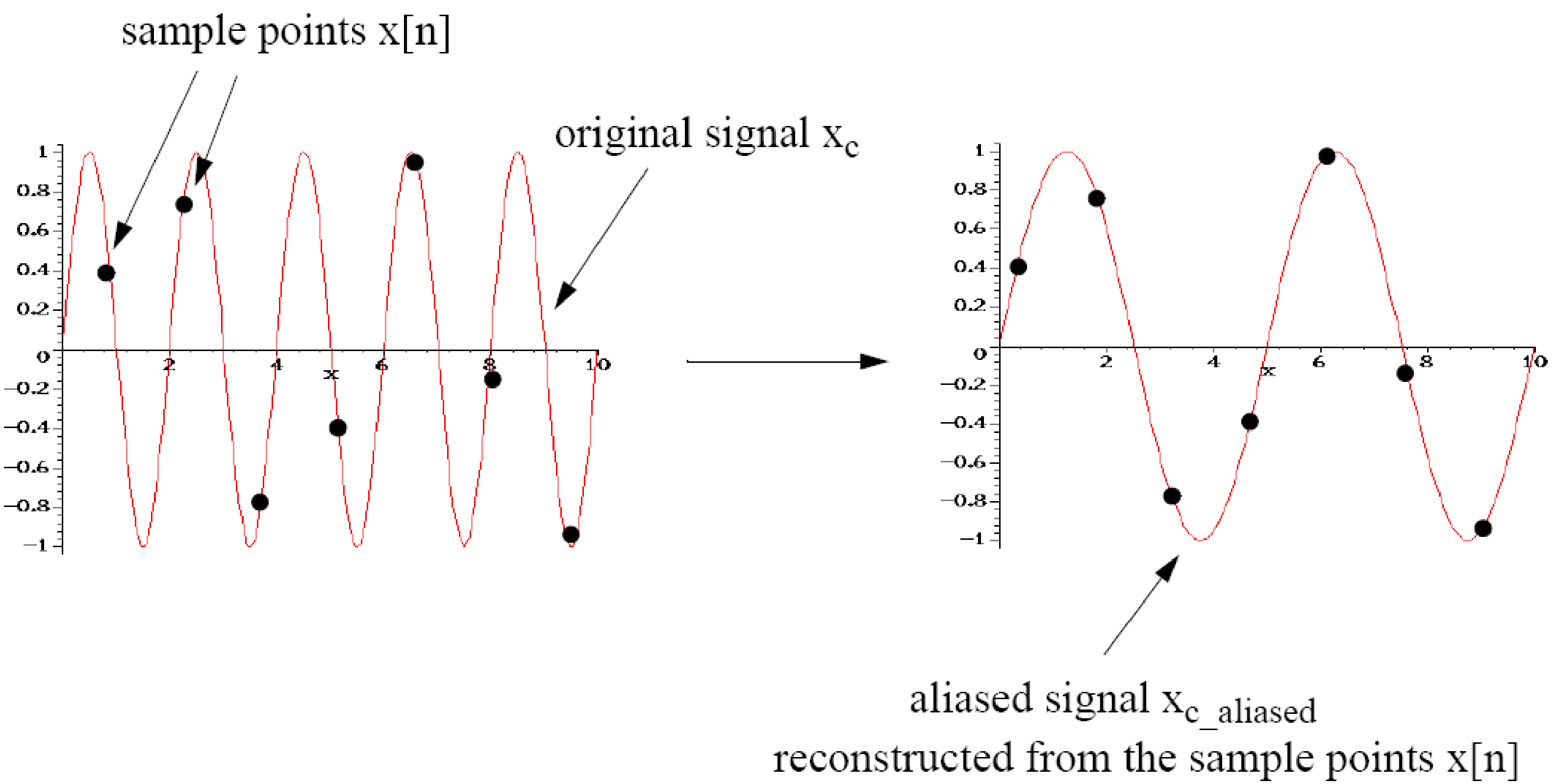
Aliasing: A More Analytical Example (1)

- Frequency of original signal: 0.5 (oscillations per time unit)
- Sampling frequency: 0.5 (samples per time unit) → original signal can not be recovered



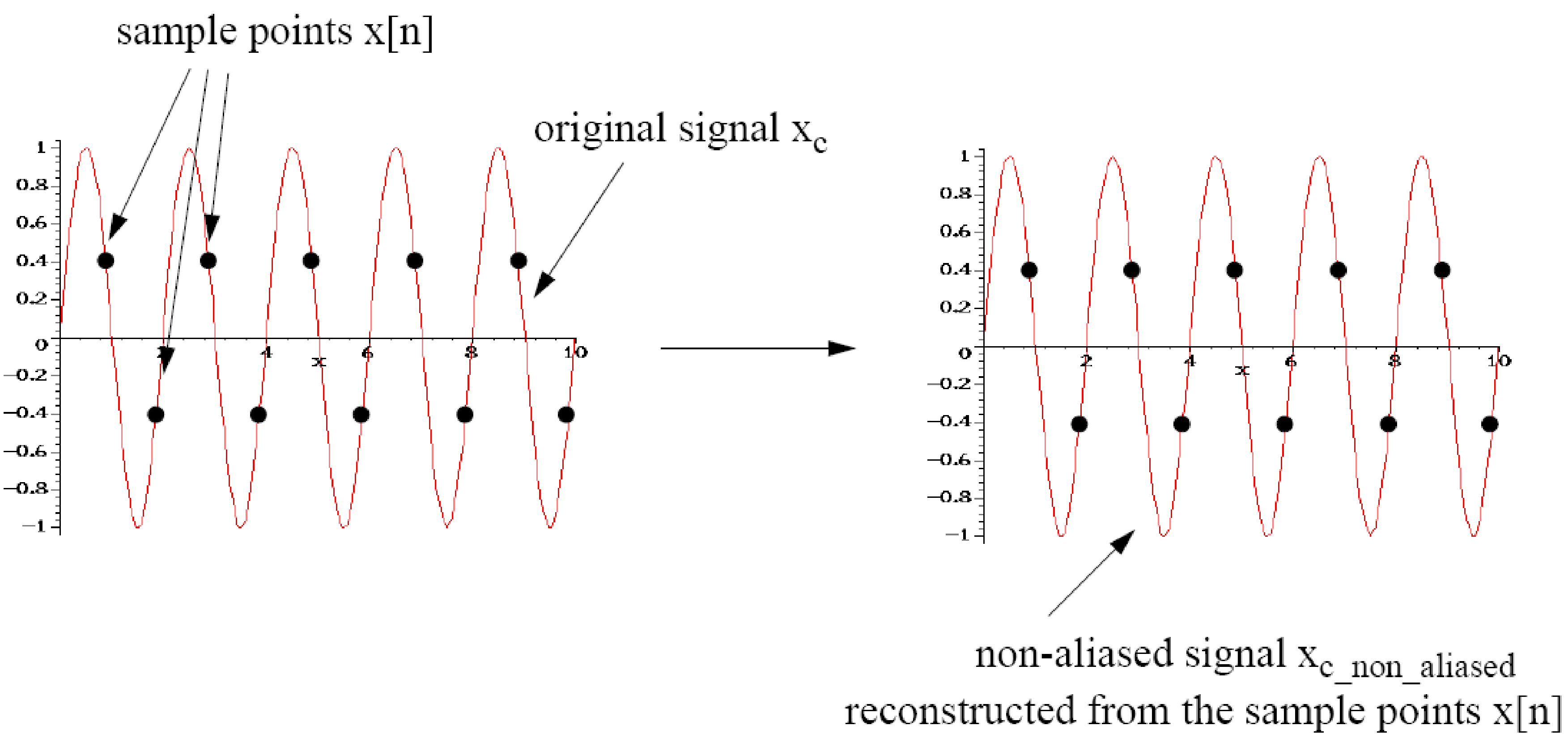
Aliasing: A More Analytical Example (2)

- Frequency of original signal: 0.5 (oscillations per time unit)
- Sampling frequency: 0.7 (samples per time unit)
- Looking at the sample points x[n], they appear to originate from a sine wave $x_{c\_aliased}$ of much lower frequency → again, the original sine wave is lost and can not be recovered



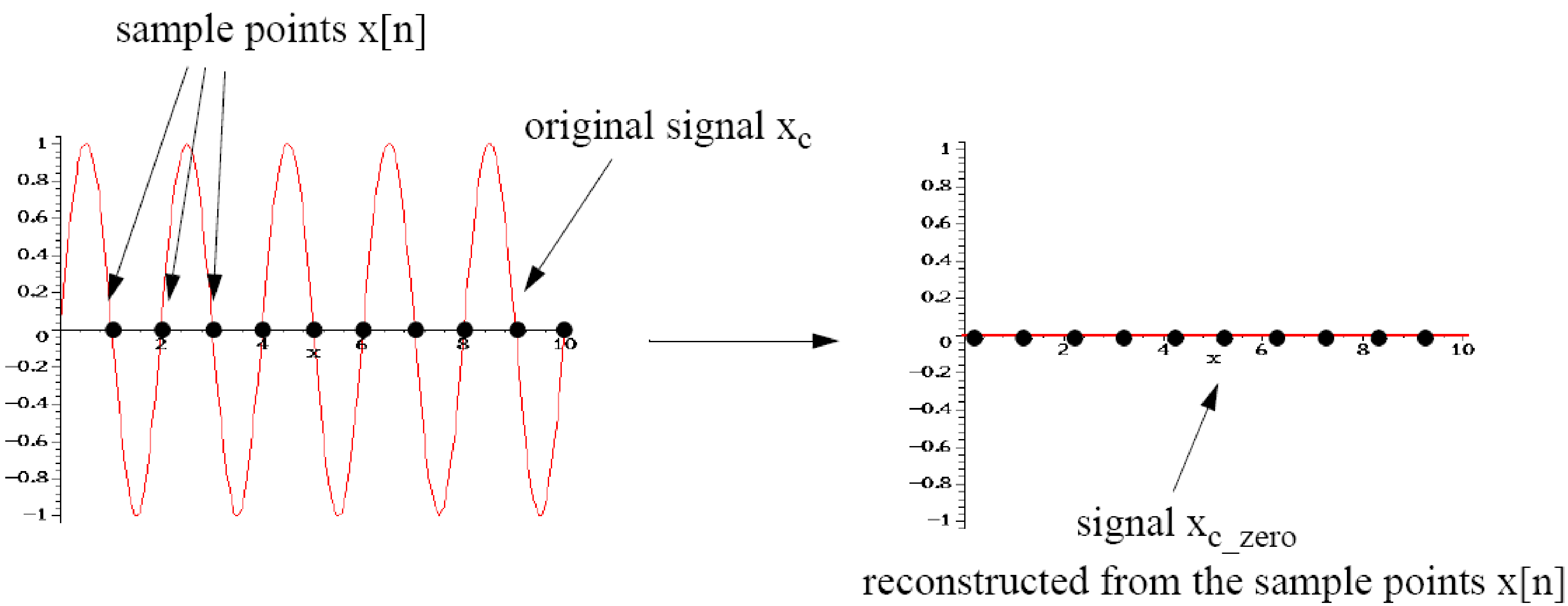
Aliasing: A More Analytical Example (3)

- Frequency of original signal: 0.5 (oscillations per time unit)
- Sampling frequency: 1.0 (sample per time unit) → original signal can be recovered
- We learn that we need to sample each oscillation period twice for good reconstruction



Aliasing: A More Analytical Example (4)

- In practice, it is best to use more than 2 samples per oscillation period
  - else one may get wrong reconstructions for some special sample alignments



- Thus, to be on the safe side:
  - sample each oscillation period more than twice
- Next: a closer look onto the whole process

Aliasing: Prevention

So must choose:

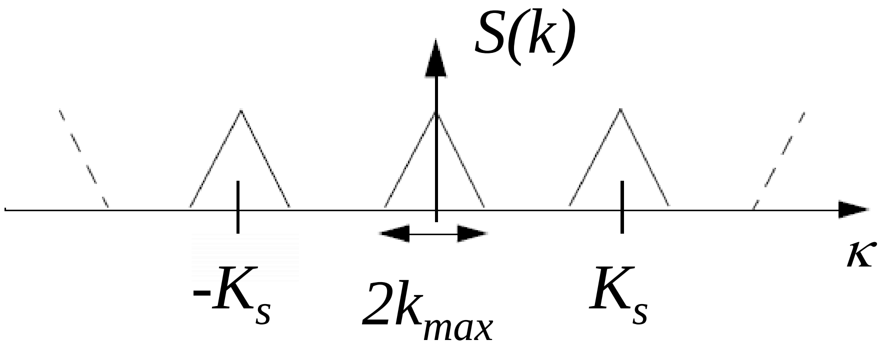
$$K > K_s = 2 \cdot k_{\max}, \quad K_s \text{ is the } \textit{Nyquist rate}$$

In other words:

- the samples only uniquely define the signal if:

$$S(k) = 0 \qquad \forall |k| > k_{\max}$$

$$\frac{1}{\Delta x} > 2k_{\max} = K_s$$



- this assumes that the signal is band-limited ($S(k)$=0 above K_s

Anti-Aliasing

Usually signals are not band-limited

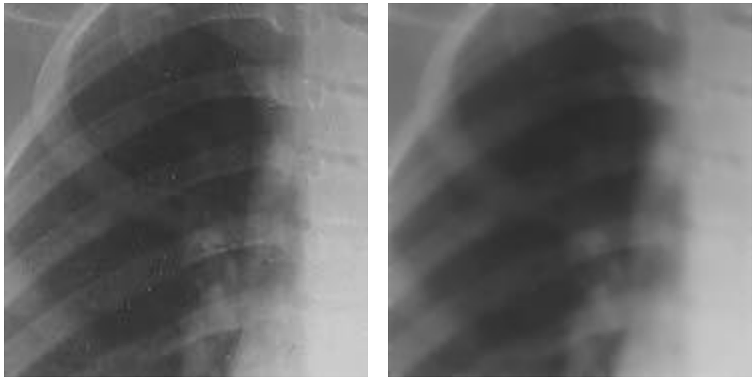
- recall the infinite spectrum of a sharp edge (for example: a bone)

To prevent the inevitable aliasing we must perform anti-aliasing before sampling the signal

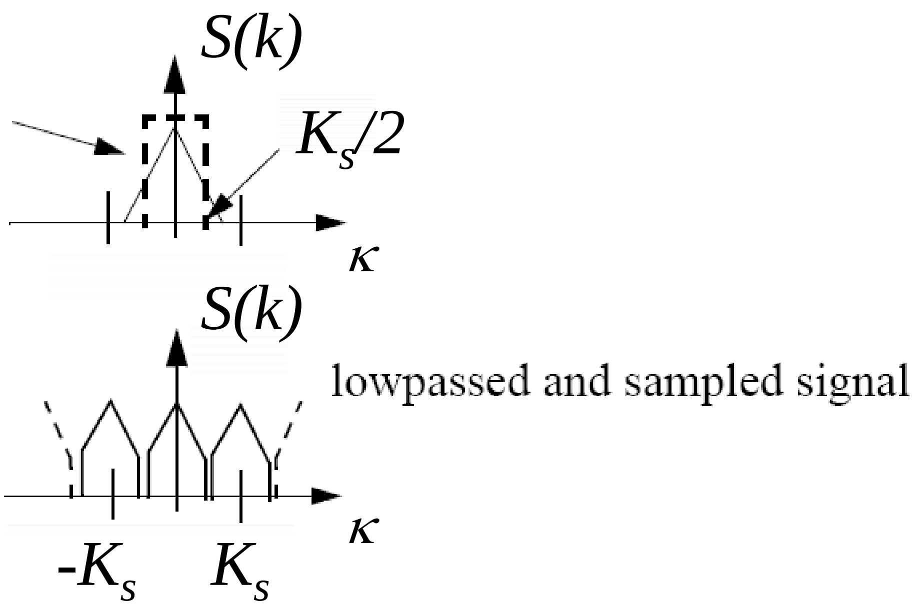
- for example: when digitizing a radiograph of a bone or a chest

Anti-aliasing is done by low-pass filtering (blurring)

- band-limit the signal *prior* to sampling
- we shall see later, how

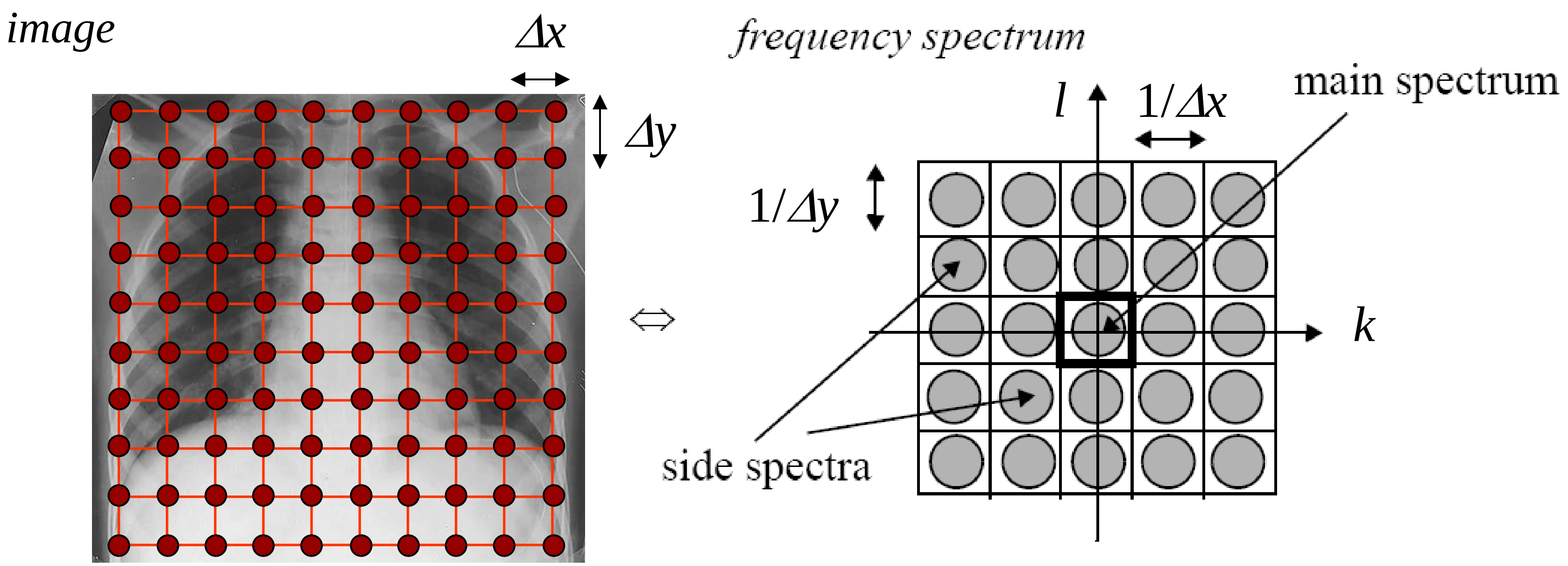


original blurred



Higher Dimensions

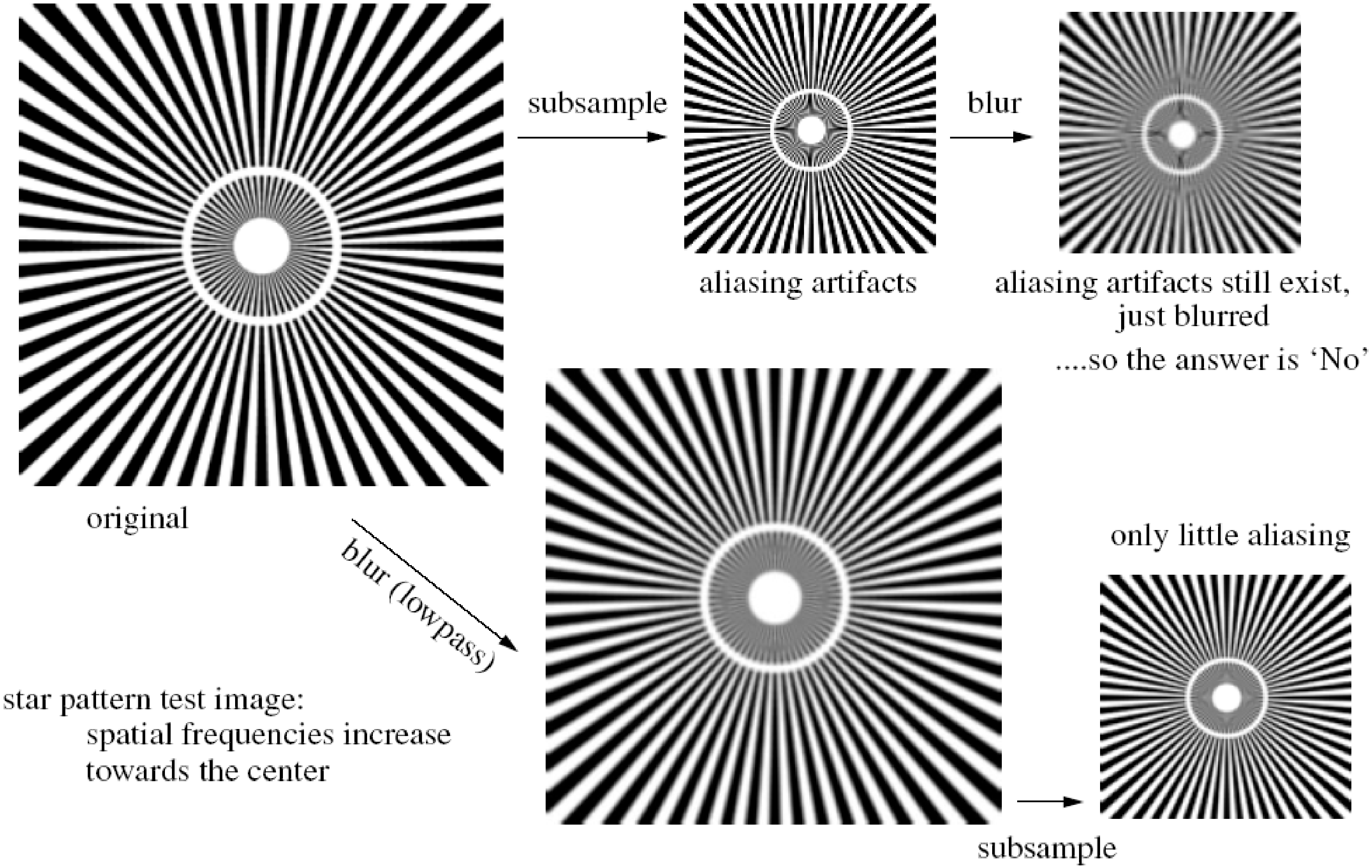
All of these concepts readily extend to higher dimensions



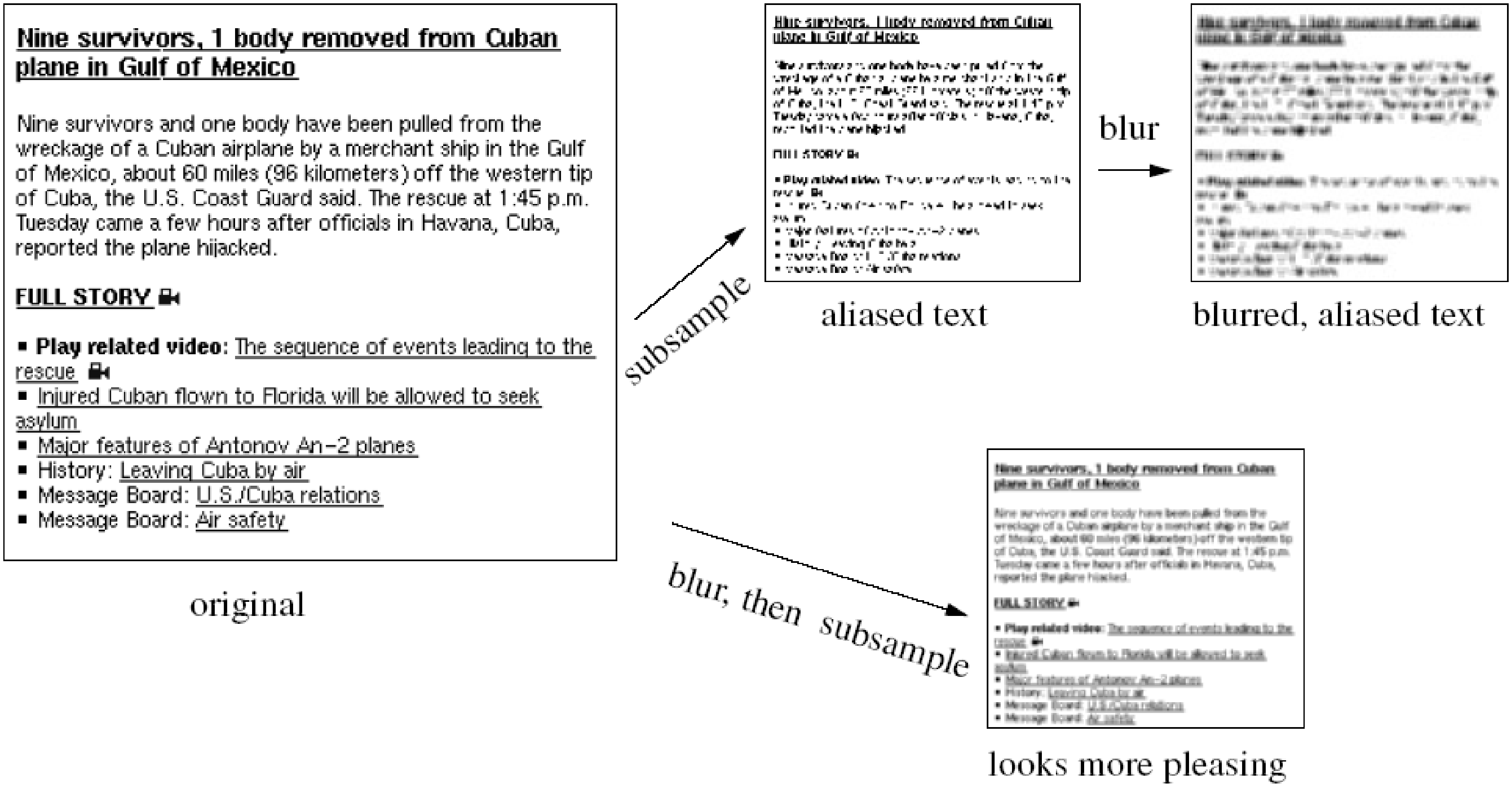
Main spectrum ($S(k,l)$) must fit into the center box to prevent overlap with side-spectra (and aliasing)

$$\frac{1}{\Delta x} > 2 \cdot k_{x\max} \qquad \frac{1}{\Delta y} > 2 \cdot k_{y\max}$$

Anti-Aliasing: Practical Examples (1)



Anti-Aliasing: Practical Examples (2)

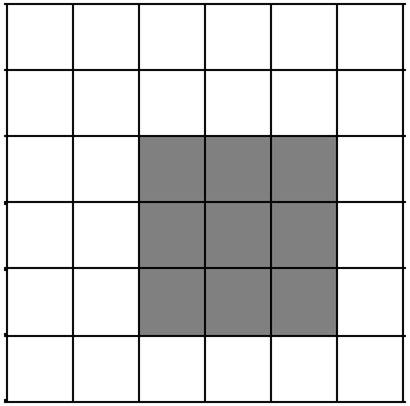


We observe: Anti-aliasing (i.e., blurring, lowpassing) must be applied before sampling

Image Representation

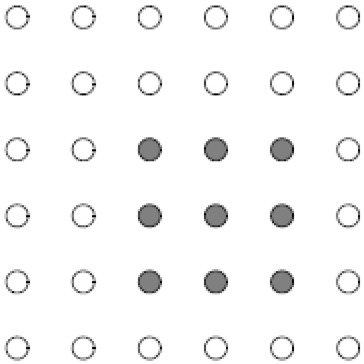
We know that a discrete image is a matrix of pixels

- do keep this in mind, however:



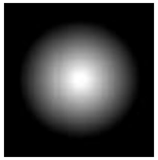
an image is NOT a matrix of solid squares

rather, each pixel is a Dirac impulse, with the pixel's value as its height

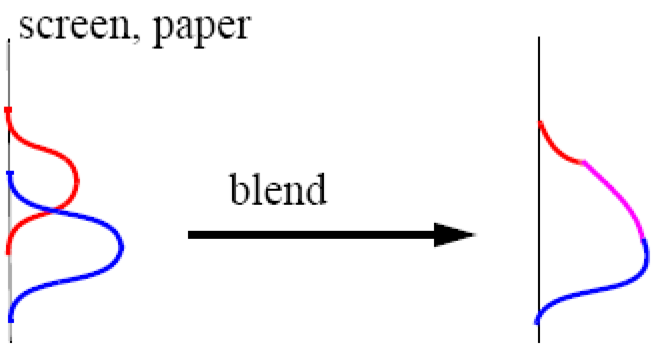


So, why do we not see isolated dots on the screen or paper?

- a monitor or printer "splats" the pixels onto the screen or paper.
- each pixels assumes the shape of a Gaussian



- the Gaussians blend together and form a continuous image

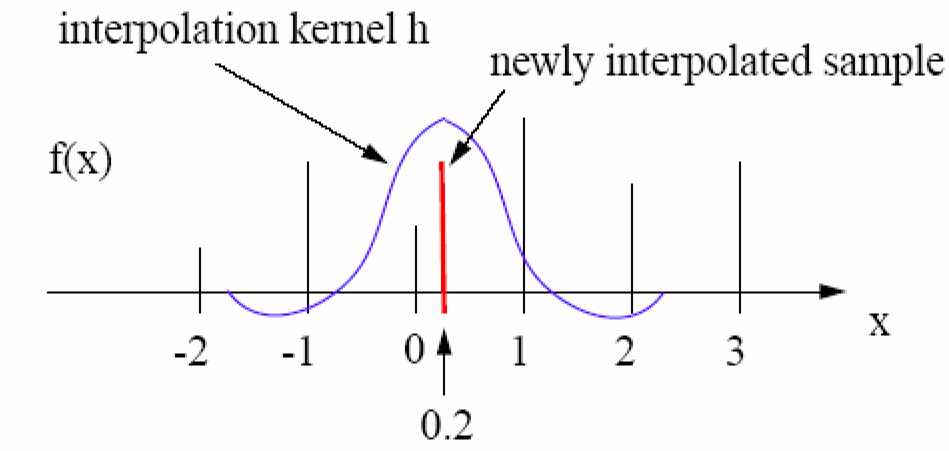


Interpolation

Often we want to estimate the formerly continuous function from the discretized function represented by the matrix of sample points

This is done via *interpolation*

Concept:

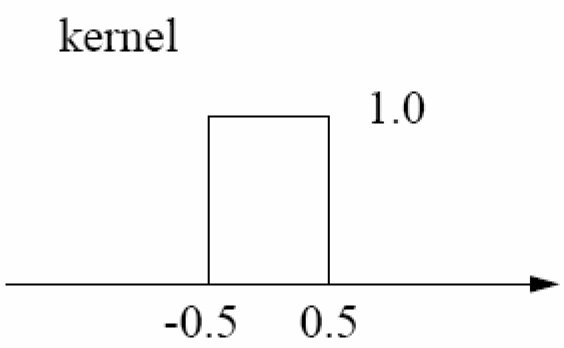


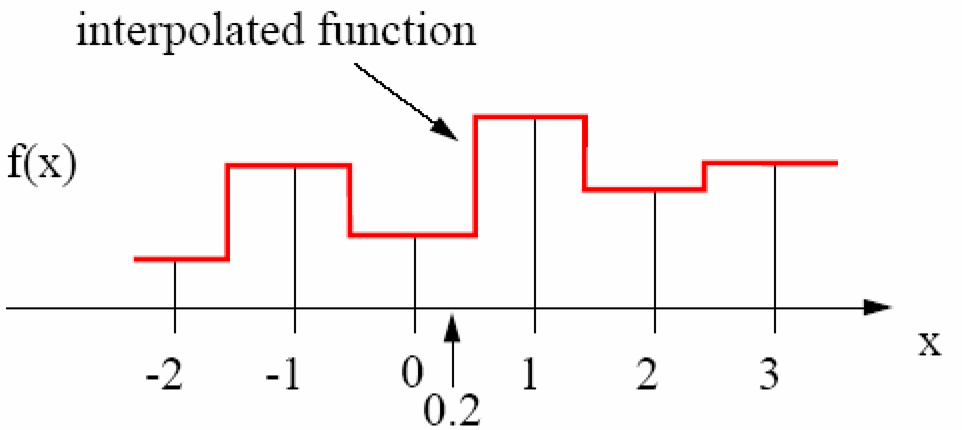
- center the interpolation kernel (filter) *h* at the sample position and superimpose it onto the grid
- multiply the values of the grid samples with the kernel value at the superimposed position
- add all the products → this gives the value of the newly interpolated sample
- in the shown case:

$$f(0.2) = h(-0.2)\, f(0) + h(-1.2)\, f(-1) + h(0.8)\, f(1) + h(1.8)\, f(2)$$

Interpolation Kernels (1)

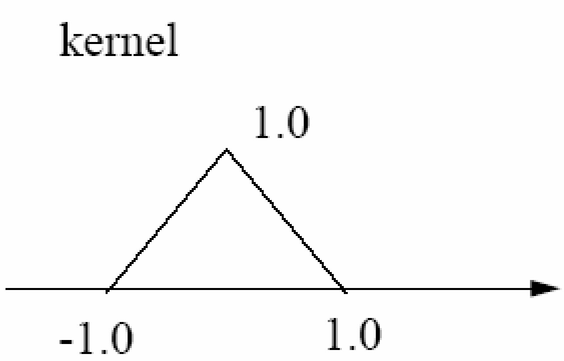
- Nearest Neighbor:

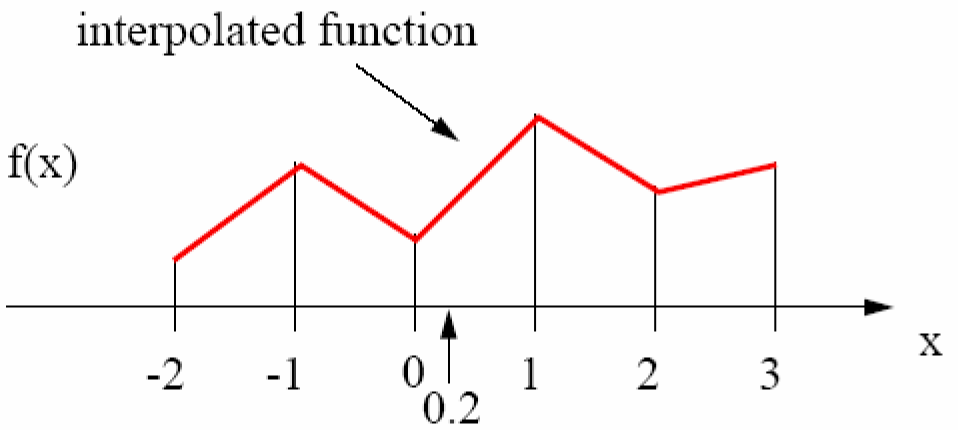




- simply pick the value of the nearest grid point: f(0.2) = f(trunc(0.2+0.5) = f(round(0.2))

- Linear filter:

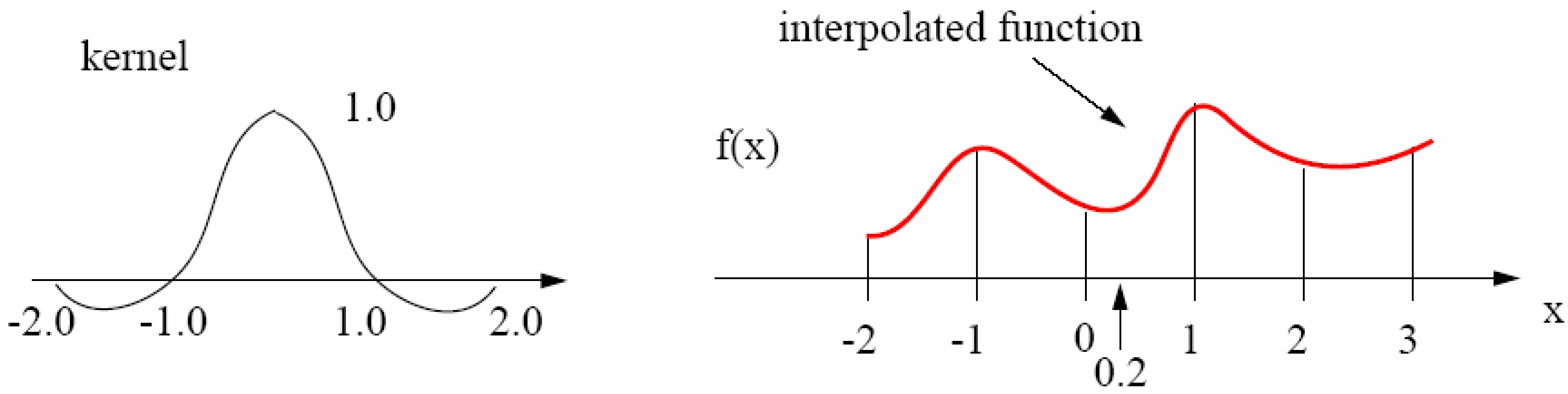




- use a linear combination of the two neighboring grid values: $f(0.2) = 0.2 \cdot f(1) + 0.8 \cdot f(0)$

Interpolation Kernels (2)

- Cubic filter:



An additional popular filter is the Gaussian function

Discussion:

- nearest neighbor is fastest to compute (just one add), gives sharp edges, but sometimes jagged lines
- linear interpolation takes 2 mults and 1 add and gives a piecewise smooth function
- cubic filter takes 4 mults and 3 adds, but gives an overall smooth interpolated function
- linear interpolation is most popular in many application

Interpolation in Higher Dimensions

- All interpolation kernels shown here are separable

  $h(x, y) = h(x) \cdot h(y)$ and $h(x, y, z) = h(x) \cdot h(y) \cdot h(z)$

- Linear interpolation

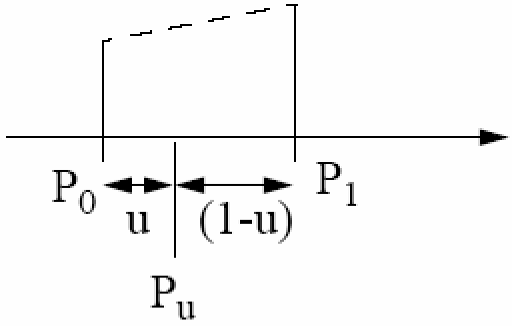
  assume: grid distance = 1.0

  P_u is the location of the sample value

  P_0 and P_1 are neighboring grid points

  then: $u = P_u - P_0$

  $f(x) = f(P_u) = (1 - u) \cdot f(P_0) + u \cdot f(P_1)$

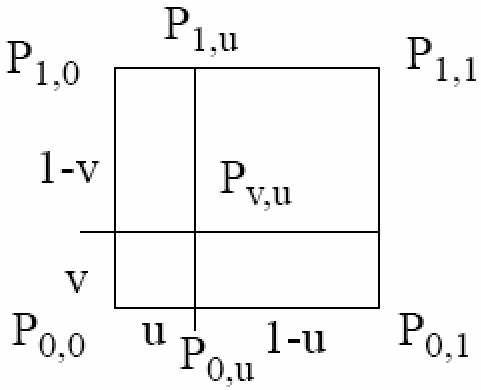


- Bilinear interpolation

  $f(P_{0,u}) = (1 - u) \cdot f(P_{0,0}) + u \cdot f(P_{0,1})$

  $f(P_{1,u}) = (1 - u) \cdot f(P_{1,0}) + u \cdot f(P_{1,1})$

  $f(P_{v,u}) = (1 - v) \cdot f(P_{0,u}) + v \cdot f(P_{1,u})$

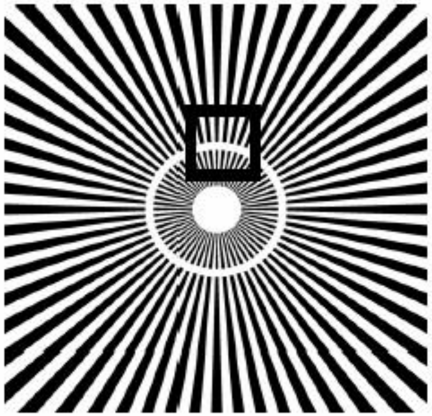


$\rightarrow f(x, y) = f(P_{v,u}) = (1\text{-}v)(1\text{-}u) f(P_{0,0}) + (1\text{-}v) u f(P_{0,1}) + v (1\text{-}u) f(P_{1,0}) + v u f(P_{1,1})$

Interpolation Quality

Example:

- resampling of a portion of the star image onto a high resolution grid
- magnification factor ~20



Computation of the Fourier Transform

The analytical form of the Fourier transform (and its laws) is convenient for theoretical, fundamental considerations

- examples: filter design, sampling rates, image resolutions

But in practical applications (for example, low-passing and other filtering) we require a means to compute a discretized signal's Fourier transform:

$$S(m\Delta k_x, n\Delta k_y) = \sum_{q=0}^{N-1}\sum_{p=0}^{M-1} s(p\Delta x, p\Delta y) e^{-2\pi i(\frac{mp}{M}+\frac{nq}{N})}$$

$$s(p\Delta x, q\Delta y) = \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} S(m\Delta k_x, n\Delta k_y) e^{2\pi i(\frac{mp}{M}+\frac{nq}{N})}$$

Assume $M=N$, then this is an $O(N^4)$ algorithm

- the Fast Fourier Transform (FFT) brings this down to $O(N^2 \log N)$