Introduction to Medical Imaging

Linear System Theory

Klaus Mueller

Computer Science Department
Stony Brook University
Dimensions

1D signal $f(x)$

2D signal $f(x, y)$

3D signal $f(x, y, z)$

2D signal, shown as height field

4D signal $f(x, y, z, t=\text{time})$

example: 3D heart in motion
**Even / Odd Functions**

Signal is even if \( s(-x) = s(x) \)
- denote as \( s_e \)

\[
\int_{-\infty}^{\infty} s_e(x)dx = 2 \int_{0}^{\infty} s_e(x)dx
\]

Signal is odd if \( s(-x) = -s(x) \)
- denote as \( s_o \)

\[
\int_{-\infty}^{\infty} s_o(x)dx = 0
\]

Can write any signal as a sum of its even and odd part:

\[
s(x) = \left[ \frac{s(x)}{2} + \frac{s(-x)}{2} \right] + \left[ \frac{s(x)}{2} - \frac{s(-x)}{2} \right]
\]

\[
= s_e(x) + s_o(x)
\]
Periodic Signals

A signal is periodic if $s(x+X) = s(x)$
• we call $X$ the period of the signal
• if there is no such $X$ then the signal is aperiodic

Sinusoids are periodic functions
• sinusoids will play an important role in this course

Write as:
$$A \sin\left(\frac{2\pi x}{X} + \phi_x\right)$$
• where $\phi_x$ is the phase shift and $A$ is the amplitude

Sinusoids can combine
• they can also occur in higher dimensions:
A complex number $c$ has a real and an imaginary part:

- $c = Re\{c\} + i \ Im\{c\}$ (cartesian representation)
- here, $i$ always denotes the complex part

We can also use a polar representation:

$$A_c = \sqrt{Re\{c\}^2 + Im\{c\}^2}$$

$$\varphi_c = \tan^{-1}\left(\frac{Re\{c\}}{Im\{c\}}\right)$$

Now think of $c$ as a periodic signal $s(x)$:

- then the pointer $(A_c, \varphi_c)$ rotates with period $X$, that is, it completes one rotation after each integer multiple of $X$
- if there is a phase shift $\varphi_x$ then the pointer simply is already located at $(A, \varphi_x)$ when $x=0$
- considering $c$ a 2D vector: $Re\{c\} = A_c \cos(\varphi_c)$ and $Im\{c\} = A_c \sin(\varphi_c)$
Important Signals (1)

Exponential $\exp$

$\exp(ax) = e^{ax}$

- when $a > 0$ then $\exp$ increases with increasing $x$
- when $a < 0$ then $\exp$ approximates 0 with increasing $x$

Complex exponential / sinusoid:

$A e^{i(2\pi kx + \phi)} = A(\cos(2\pi kx + \phi) + i \sin(2\pi kx + \phi))$

As before

- the $\cos$ term is the signal’s real part
- the $\sin$ term is the signal’s imaginary part
- $A$ is the amplitude, $\phi$ the phase shift, $k$ determines the frequency
Important Signals (2)

Rectangular function:

\[
\Pi\left(\frac{x}{2L}\right) = 1 \quad \text{for} \quad |x| < L
\]

\[
= \frac{1}{2} \quad \text{for} \quad |x| = L
\]

\[
= 0 \quad \text{for} \quad |x| > L
\]

Step function:

\[
u(x-x_0) = 0 \quad \text{for} \quad x < x_0
\]

\[
= \frac{1}{2} \quad \text{for} \quad x = x_0
\]

\[= 1 \quad \text{for} \quad x > x_0\]
Important Signals (3)

Triangular function:

\[ Tri\left(\frac{x}{2L}\right) = 1 - \frac{|x|}{L} \quad \text{for} \quad |x| < L \]
\[ = 0 \quad \text{for} \quad |x| > L \]

Normalized Gaussian:

\[ G_n(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]

- \( \mu \) is the mean
- \( \sigma \) is the standard deviation

- normalized means that the integral for all \( x \) is 0
**Sinc function:**

$$sinc(x) = \frac{\sin(\pi x)}{\pi x}$$

- $sinc(0) = 1$ (L’Hopital’s rule)

**Dirac impulse:**

$$\delta(x - x_0) = 0 \quad \text{for } x \neq x_0$$

$$\int_{-\infty}^{+\infty} \delta(x - x_0) dx = 1$$

- an important property is its sifting property:

$$\int_{-\infty}^{+\infty} s(x) \delta(x - x_0) dx = s(x_0)$$

a “needle” spike of infinite height at $x=x_0$
System response $L$:

$$s_o = L\{s_i\}$$

• might be a function of time $t$ or space $x$

$$s_o(t) = L\{s_i(t)\} \quad \text{or} \quad s_o(x) = L\{s_i(x)\}$$

Finding the mathematical relationship between in- and output is called *modeling*.

Linear systems fulfill *superposition principle*:

$$L\{c_1s_1 + c_2s_2\} = c_1L\{s_1\} + c_2L\{s_2\} \quad \forall c_1, c_2 \in \mathbb{R}$$

where $s_1, s_2$ are arbitrary signals

• for example, consider an amplifier with gain $A$:

$$L\{c_1s_1 + c_2s_2\} = A(c_1s_1 + c_2s_2)$$

$$= c_1As_1 + c_2As_2 = c_1L\{s_1\} + c_2L\{s_2\}$$
Linear Systems (2)

An example for a non-linear system:

\[ L\{c_1s_1 + c_2s_2\} = (c_1s_1 + c_2s_2)^2 \]

\[ \neq (c_1s_1)^2 + (c_2s_2)^2 \]

Time-invariance (shift-invariance = LSI):

- properties of \( L \) do not change over time (spatial position), that is:

\[ s_o(x) = L\{s_i(x)\} \quad \text{then} \quad s_o(x - X) = L\{s_i(x - X)\} \]
Impulse Response (1)

A system’s response to a Dirac impulse is called *impulse response* $h$:

Start with:

$$s_i(x) = \int_{-\infty}^{+\infty} s_i(\xi)\delta(x - \xi) d\xi$$

Then write:

$$s_o(x) = L\{s_i\} = \int_{-\infty}^{+\infty} s_i(\xi)L\{\delta(x - \xi)\} d\xi = \int_{-\infty}^{+\infty} s_i(\xi)h(x - \xi) d\xi$$
Impulse Response (2)

In practice we use non-causal impulse responses

• appear symmetric in their waveform
The expression

\[ s_o(x) = \int_{-\infty}^{+\infty} s_i(\xi)h(x-\xi)\,d\xi = s_i * h \]

is called *convolution*, defined as:

\[ s_1(x) * s_2(x) = \int_{-\infty}^{+\infty} s_1(\xi)s_2(x-\xi)\,d\xi = \int_{-\infty}^{+\infty} s_1(x-\xi)s_2(\xi)\,d\xi \]

Procedure:

for each \( x \) do:

1: mirror \( s_2 \) about \( \xi = 0 \) (change \( \xi \) to \( -\xi \))
2: translate mirrored \( s_2 \) by \( \xi = x \)
3: multiply \( s_1 \) and mirrored \( s_2 \)
4: integrate the resulting signal

See next slides for an example and detailed explanation…
Example $x=0.7$:

$$s_1(x) * s_2(x) = \int_{-\infty}^{+\infty} s_1(\xi)s_2(x - \xi)d\xi$$

1: mirror $s_2$ about $\xi=0$

2: translate mirrored $s_2$ by $\xi=0.7$

3: integrate over all $\xi$

4: write integration result at $x=0.7$
Animated gifs:
- red, blue: convolved signals
- green: convolution result

Two boxes

Two gaussians
Convolution: Detailed Explanation

Mirroring:
- when you take a function $f(t)$ and mirror it about the y-axis then you get a new function $f''(t) = f(-t)$

For convolution:
- you have two functions: $f_1(t)$ and $f_2(t)$
- you would like to compute:

$$f(x) = \int_{-\infty}^{+\infty} f_1(t) f_2(x-t) dt$$

- but in this form: $t$ increases in $f_1$ and decreases in $f_2$, which is not convenient
- to fix this, you mirror $f_2(x-t)$ into $f_2''(t-x) = f_2(-(x-t))$
- now the convolution writes:

$$f(x) = \int_{-\infty}^{+\infty} f_1(t) f_2''(-(x-t)) dt = \int_{-\infty}^{+\infty} f_1(t) f_2''(t-x) dt$$

- at this point you need $f_2''(t)$ which is obtained by mirroring $f_2(t)$: $f_2''(t) = f_2(-t)$
- now you can do the intuitive right-sliding of $f_2''$ for growing $x$
Convolution Properties

Also defined for multi-dimensional signals:

\[
s_1(x, y) * s_2(x, y) = \int_{-\infty}^{+\infty} s_1(x - \xi, y - \zeta)s_2(\xi, \zeta)d\xi d\zeta
\]

Some important properties:

- **commutativity:**
  \[s_1 * s_2 = s_2 * s_1\]

- **associativity:**
  \[(s_1 * s_2) * s_2 = s_1 * (s_2 * s_3)\]

- **distributivity:**
  \[s_1 * (s_2 + s_3) = s_1 * s_2 + s_1 * s_3\]
Typically, signals are only available in discrete form

- reconstruction into a continuous signal (for visualization, etc) occurs by overlapping point spread functions (see previous lecture)
- but all computer processing (convolution and others) is done on the discrete representations
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![Diagram showing discrete signals and their convolution process]
Now assume the input is a complex sinusoid with $Ae^{2\pi ikx}$ then:

$$s_0(x) = \int_{-\infty}^{+\infty} Ae^{2\pi ik(x-\xi)} h(\xi) d\xi$$

$$= Ae^{2\pi ikx} \int_{-\infty}^{+\infty} e^{-2\pi ik\xi} h(\xi) d\xi$$

$$= Ae^{2\pi ikx} H = S_i H$$

$H$ is called the Fourier Transform of $h(x)$:

$$H = \int_{-\infty}^{+\infty} e^{-2\pi ik\xi} h(\xi) d\xi$$

- $H$ is also often called the transfer function or filter
- the Fourier transform will be discussed in detail shortly

for now, assume $\phi=0$
In essence, we have now two alternative representations:

- determine the effect of $L$ on $s_i$ by convolution with $h$: $s_i * h$
- determine the effect of $L$ on $s_i$ by multiplication with $H$: $S_i \cdot H$

\[ s_i * h \leftrightarrow S_i \cdot H \]

Since convolution is expensive for wide $h$, the multiplication may be cheaper

- but we need to perform the Fourier transforms of $s_i$ and $h$
- in fact, there is a “sweetspot”
- more later…
Recall the factor $k$ in the complex sinusoid:

$$A e^{i(2\pi kx + \phi)} = A \cos(2\pi kx + \phi) + i \sin(2\pi kx + \phi)$$

- as $k$ increases, so does the frequency of the oscillation

- note: the higher $k$, the higher the signal resolution, that is, one can represent smaller signal details (signals that vary more quickly)
Any periodic signal can be created by a combination of weighted and shifted sinusoids at different frequencies:

\[ s_o(x) = \int_{-\infty}^{+\infty} A_k \cos(2\pi kx + \phi_k) + i \sin(2\pi kx + \phi_k) \, dk \]

\[ = \int_{-\infty}^{+\infty} A_k e^{i(2\pi k x + \phi_k)} \, dk = \int_{-\infty}^{+\infty} A_k e^{i\phi_k} e^{i2\pi kx} \, dk \]

\[ = \int_{-\infty}^{+\infty} S_i(k) e^{2\pi ikx} \, dk \]

- \( A_k \) is the amplitude and \( \phi_k \) is the phase shift.

Incorporating the transfer function, now one for each \( k \):

\[ s_o(x) = \int_{-\infty}^{+\infty} S_i(k) e^{2\pi ikx} H(k) \, dk \]