Introduction to Medical Imaging

Linear System Theory

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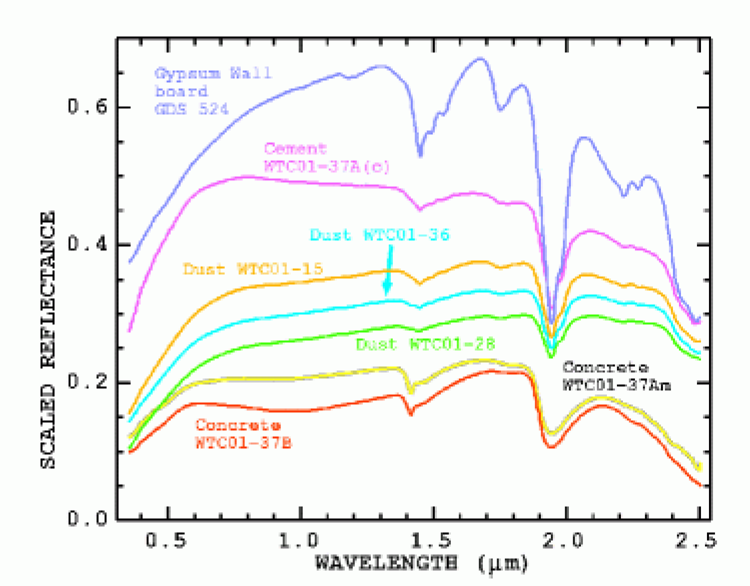
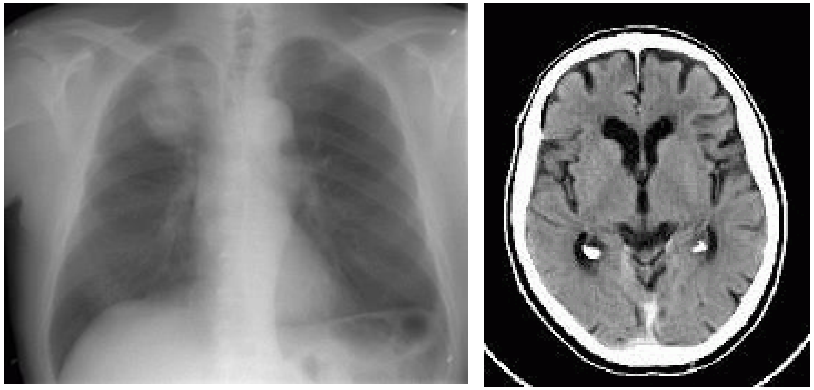
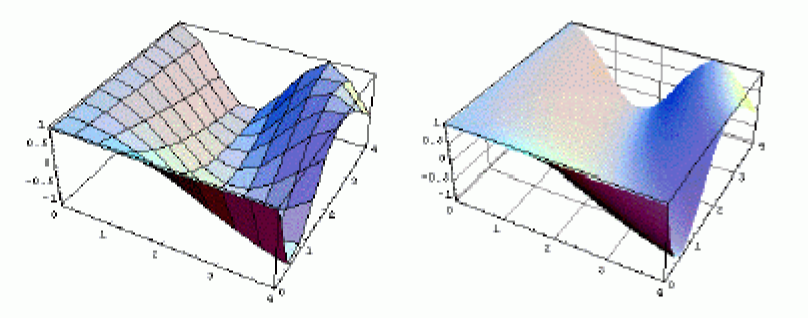
Dimensions



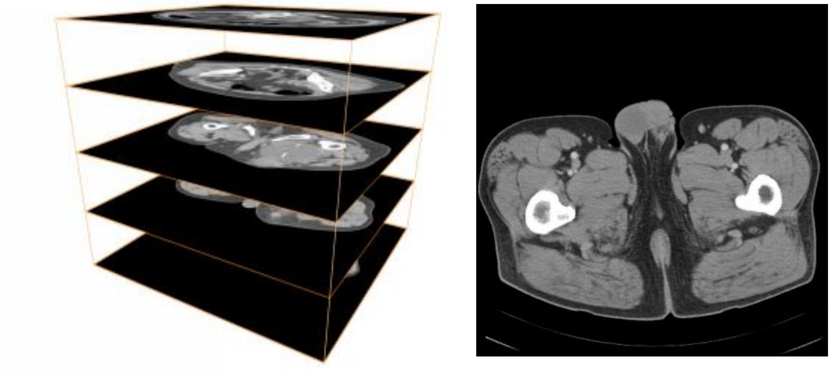
1D signal $f(x)$



2D signal $f(x, y)$



2D signal, shown as height field



3D signal $f(x, y, z)$

4D signal $f(x, y, z, t\text{=time})$
example: 3D heart in motion

Even / Odd Functions

Signal is even if *s*(-*x*) = *s*(*x*)

- denote as s_e

$$\int_{-\infty}^{+\infty} s_e(x)dx = 2\int_{0}^{+\infty} s_e(x)dx$$

Signal is odd if *s*(-*x*) = -*s*(*x*)

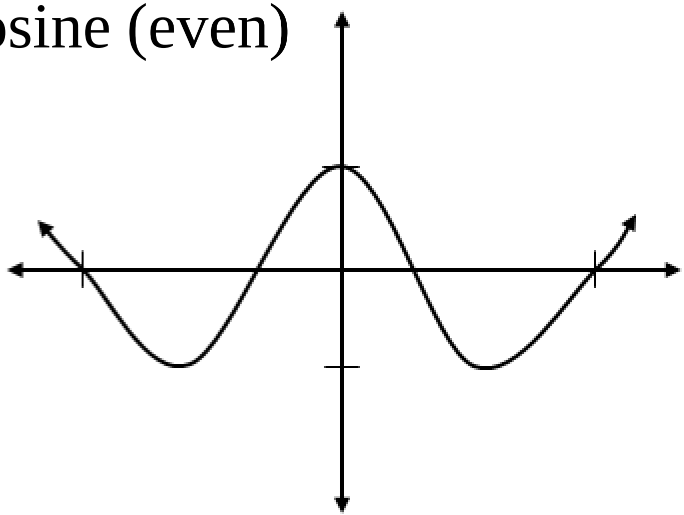
- denote as s_o

$$\int_{-\infty}^{+\infty} s_o(x)dx = 0$$

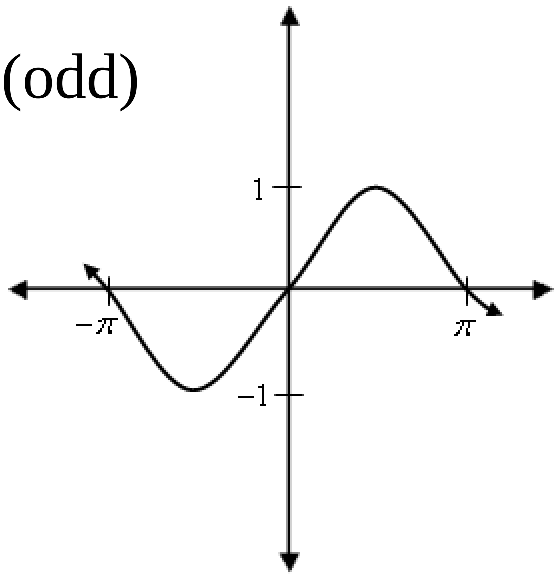
Can write any signal as a sum of its even and odd part:

$$s(x) = [\frac{s(x)}{2} + \frac{s(-x)}{2}] + [\frac{s(x)}{2} - \frac{s(-x)}{2}]$$
$$= s_e(x) + s_o(x)$$

cosine (even)



sine (odd)



Periodic Signals

A signal is periodic if *s*(*x*+*X*) = *s*(*x*)

- we call *X* the period of the signal
- if there is no such *X* then the signal is aperiodic

Sinusoids are periodic functions

- sinosoids will play an important role in this course

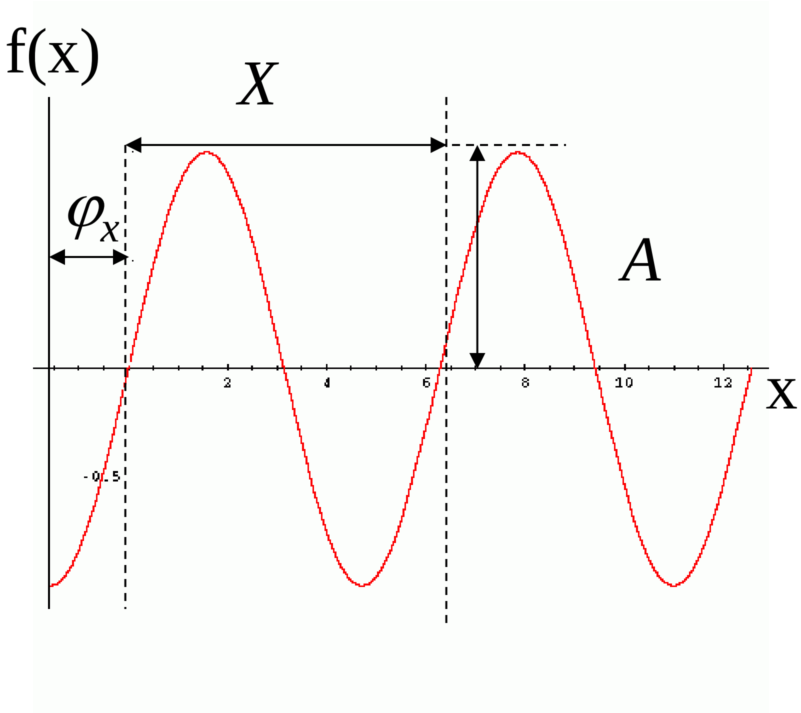
Write as:

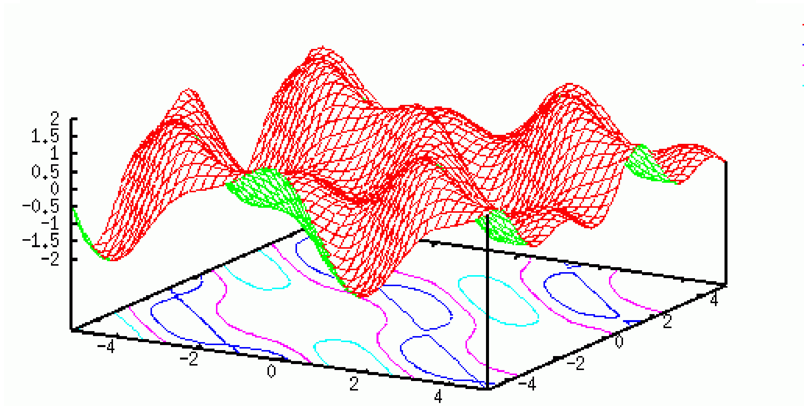
$$A\sin\left(\frac{2\pi x}{X} + \varphi_x\right)$$

- where φ_x is the phase shift and *A* is the amplitude

Sinusoids can combine

- they can also occur in higher dimensions:





Complex Numbers

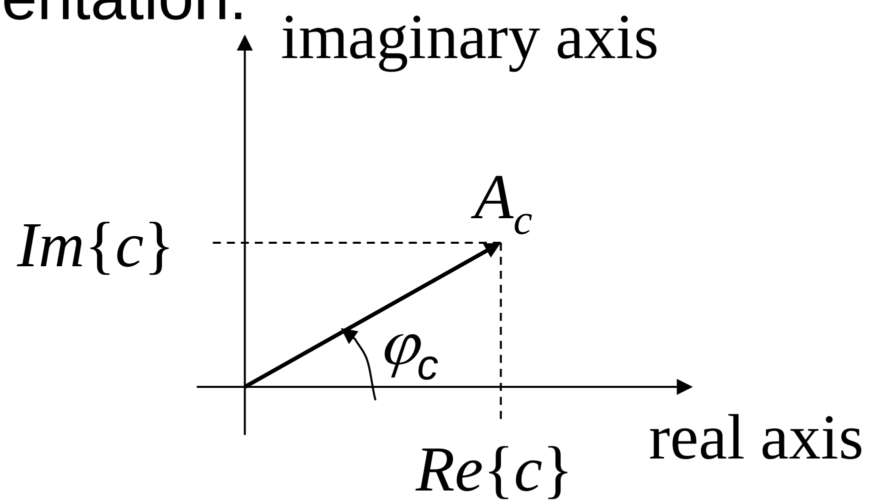
A complex number *c* has a real and and an imaginary part:

- $c = Re\{c\} + i\, Im\{c\}$ (cartesian representation) $i = \sqrt{-1}$
- here, *i* always denotes the complex part

We can also use a polar representation:

$$A_c = \sqrt{\mathrm{Re}\{c\}^2 + \mathrm{Im}\{c\}^2}$$

$$\varphi_c = \tan^{-1}\left(\frac{\mathrm{Re}\{c\}}{\mathrm{Im}\{c\}}\right)$$



Now think of *c* as a periodic signal $s(x)$:

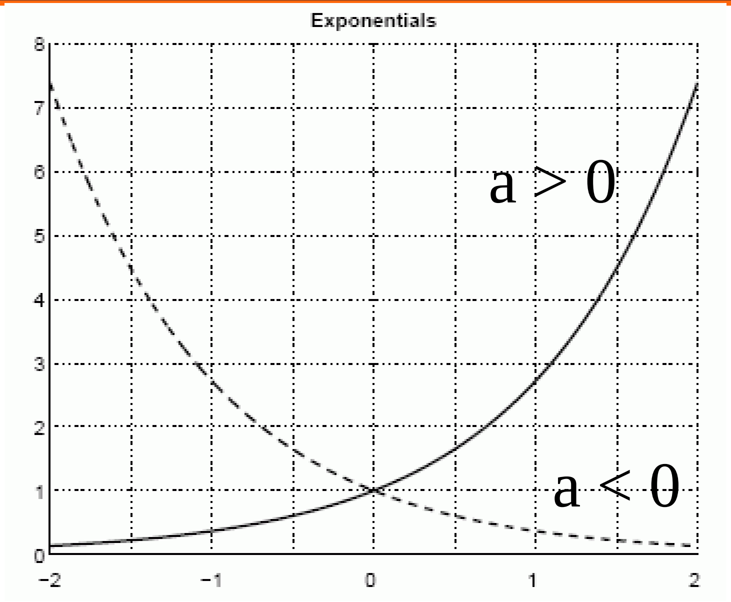
- then the pointer (A_c, φ_c) rotates with period *X*, that is, it completes one rotation after each integer multiple of *X*
- if there is a phase shift φ_x then the pointer simply is already located at (A, φ_x) when $x=0$
- considering *c* a 2D vector: $Re\{c\} = A_c cos(\varphi_c)$ and $Im\{c\} = A_c\, sin(\varphi_c)$

Important Signals (1)

Exponential *exp*

$$\exp(ax) = e^{ax}$$

- when $a > 0$ then *exp* increases with increasing x
- when $a < 0$ then *exp* approximates 0 with increasing x

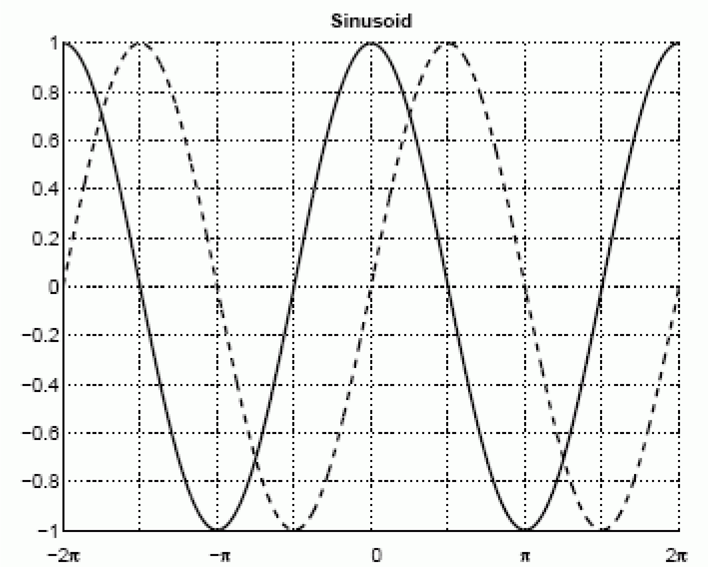


Complex exponential / sinusoid:

$$Ae^{i(2\pi kx+\phi)} = A(\cos(2\pi kx+\phi)+i\sin(2\pi kx+\phi))$$

As before

- the *cos* term is the signal's real part
- the *sin* term is the signal's imaginary part
- A is the amplitude, φ the phase shift, k determines the frequency



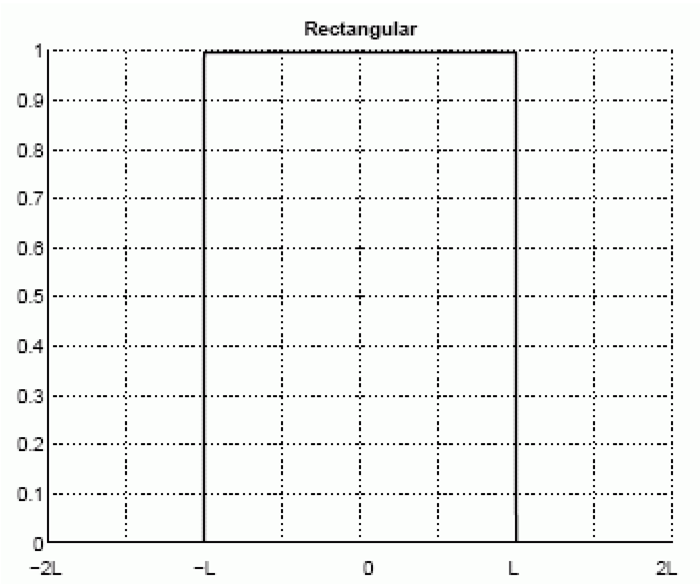
Important Signals (2)

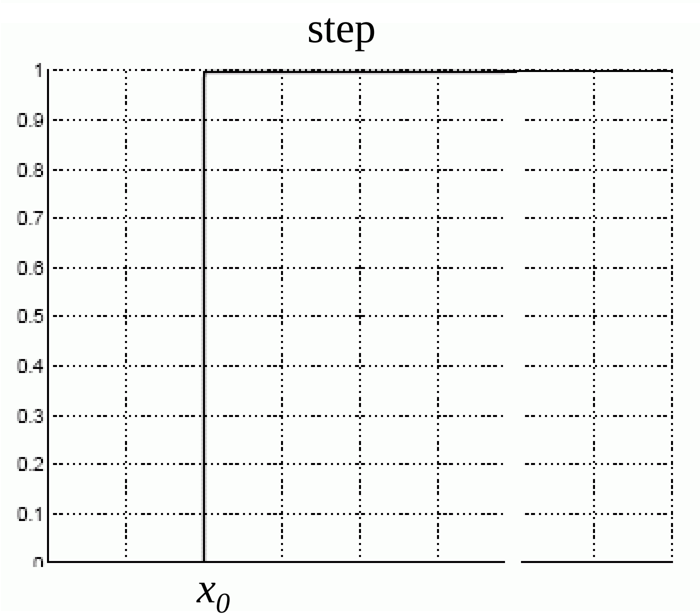
Rectangular function:

$$\Pi\left(\frac{x}{2L}\right) = 1 \quad \text{for} \quad |x| < L$$

$$= \frac{1}{2} \quad \text{for} \quad |x| = L$$

$$= 0 \quad \text{for} \quad |x| > L$$





Step function:

$$u(x - x_0) = 0 \quad \text{for} \quad x < x_0$$

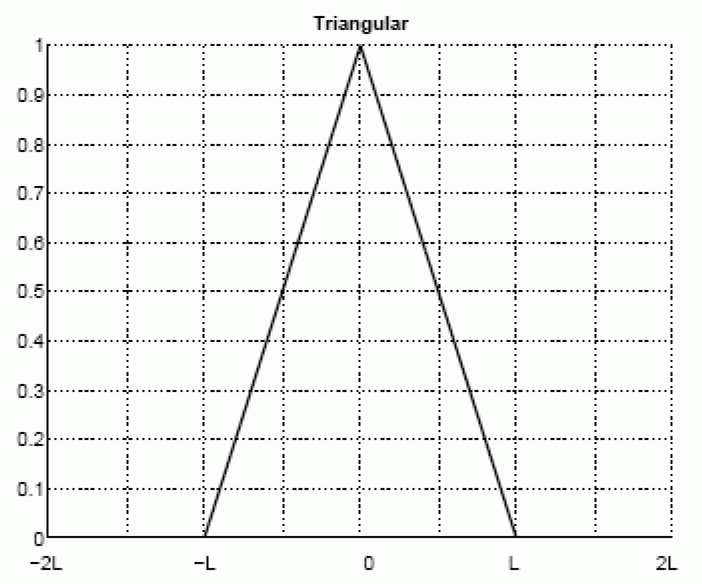
$$= \frac{1}{2} \quad \text{for} \quad x = x_0$$

$$= 1 \quad \text{for} \quad x > x_0$$

Important Signals (3)

Triangular function:

$$Tri\left(\frac{x}{2L}\right) = 1 - \frac{|x|}{L} \quad \text{for} \quad |x| < L$$
$$= 0 \quad \text{for} \quad |x| > L$$



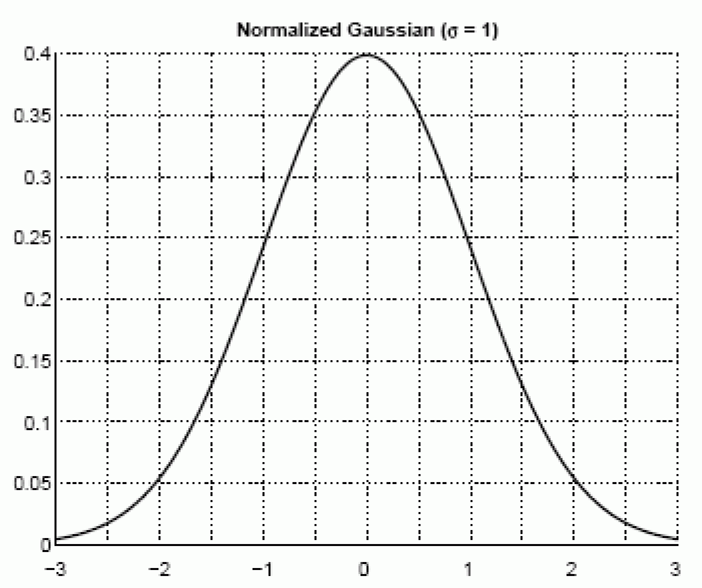
Normalized Gaussian:

$$G_n(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\eta)^2}{2\sigma^2}}$$

μ is the mean

σ is the standard deviation

- normalized means that the integral for all *x* is 0

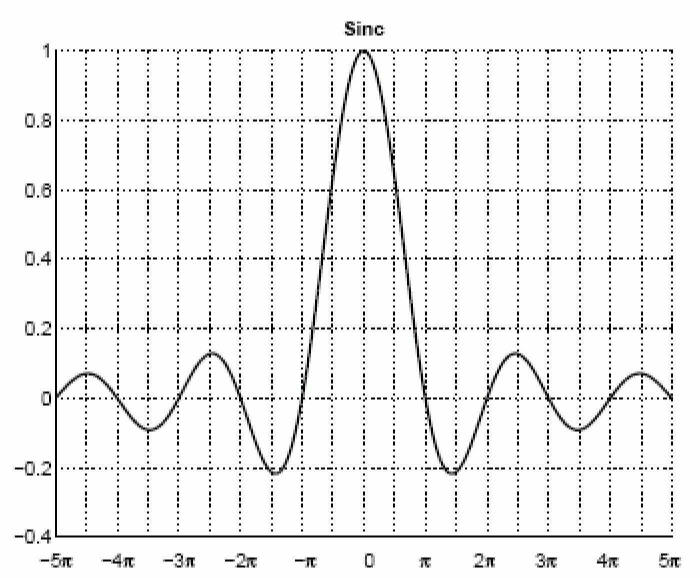


Important Signals (4)

Sinc function:

$$sinc(x) = \frac{\sin(\pi x)}{\pi x}$$

- sinc(0) = 1 (L'Hopital's rule)



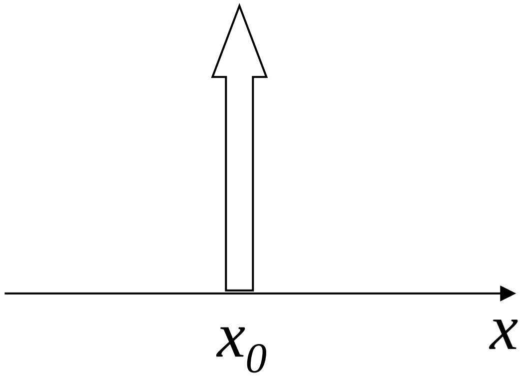
Dirac impulse:

$$\delta(x - x_0) = 0 \quad \text{for } x \neq x_0$$

$$\int_{-\infty}^{+\infty} \delta(x - x_0)\,dx = 1$$

- an important property is its sifting property:

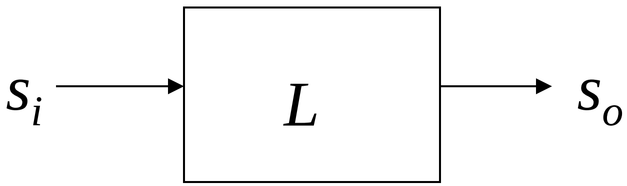
$$\int_{-\infty}^{+\infty} s(x)\delta(x - x_0)\,dx = s(x_0)$$



a "needle" spike of infinite height at $x = x_0$

Linear Systems (1)

System response *L:*



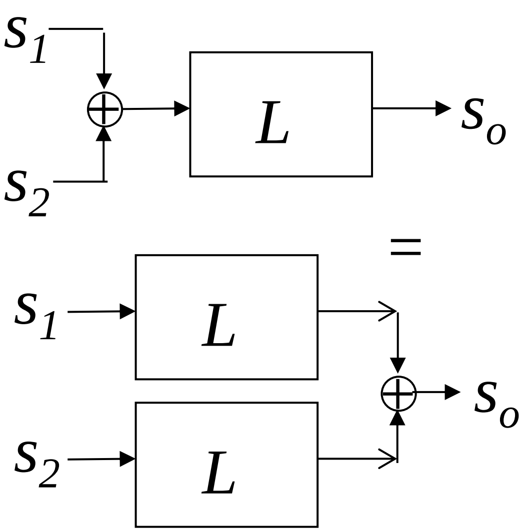
$$s_o = L\{s_i\}$$

- might be a function of time *t* or space *x*

$$s_o(t) = L\{s_i(t)\} \quad \text{or} \quad s_o(x) = L\{s_i(x)\}$$

Finding the mathematical relationship between in- and output is called *modeling*

Linear systems fulfill *superposition principle*:



$$L\{c_1 s_1 + c_2 s_2\} = c_1 L\{s_1\} + c_2 L\{s_2\} \quad \forall c_1, c_2 \in \Re$$

where s_1, s_2 are arbitrary signals

- for example, consider an amplifier with gain A:

$$L\{c_1 s_1 + c_2 s_2\} = A(c_1 s_1 + c_2 s_2)$$

$$= c_1 A s_1 + c_2 A s_2 = c_1 L\{s_1\} + c_2 L\{s_2\}$$

Linear Systems (2)

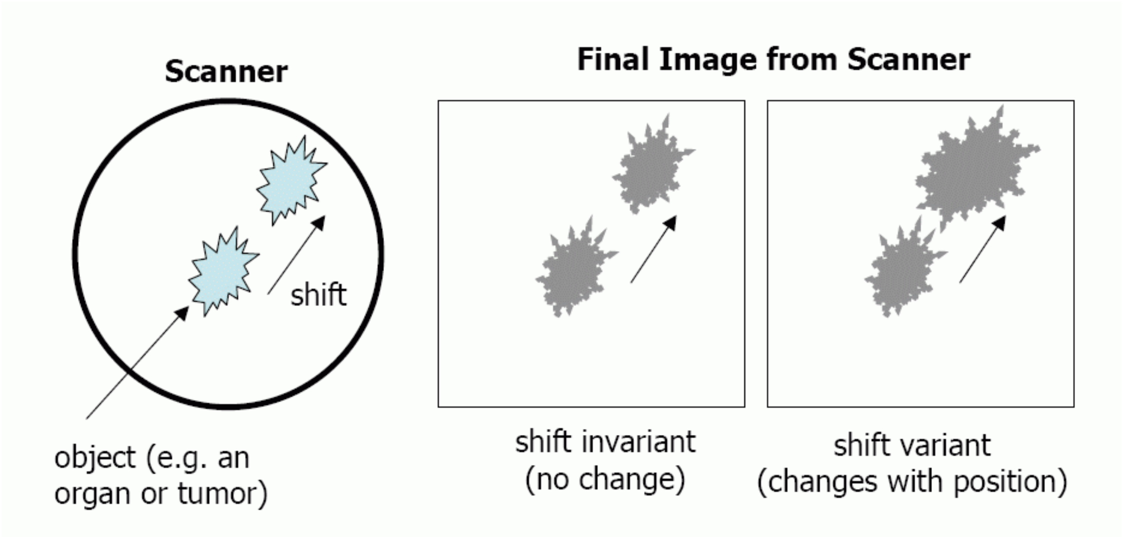
An example for a non-linear system:

$$L\{c_1 s_1 + c_2 s_2\} = (c_1 s_1 + c_2 s_2)^2$$
$$\neq (c_1 s_1)^2 + (c_2 s_2)^2$$

Time-invariance (shift-invariance = LSI):

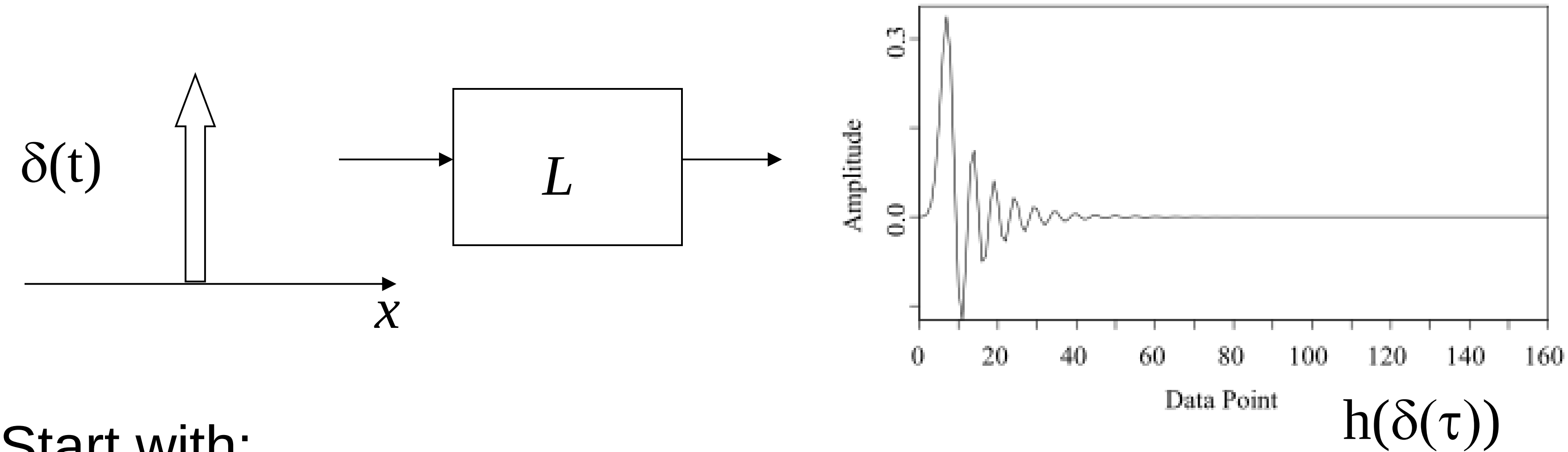
- properties of *L* do not change over time (spatial position), that is:

$$s_o(x) = L\{s_i(x)\} \ \text{ then } \ s_o(x - X) = L\{s_i(x - X)\}$$



Impulse Response (1)

A system's response to a Dirac impulse is called *impulse response h*:



Start with:

$$s_i(x) = \int_{-\infty}^{+\infty} s_i(\xi)\delta(x-\xi)d\xi$$

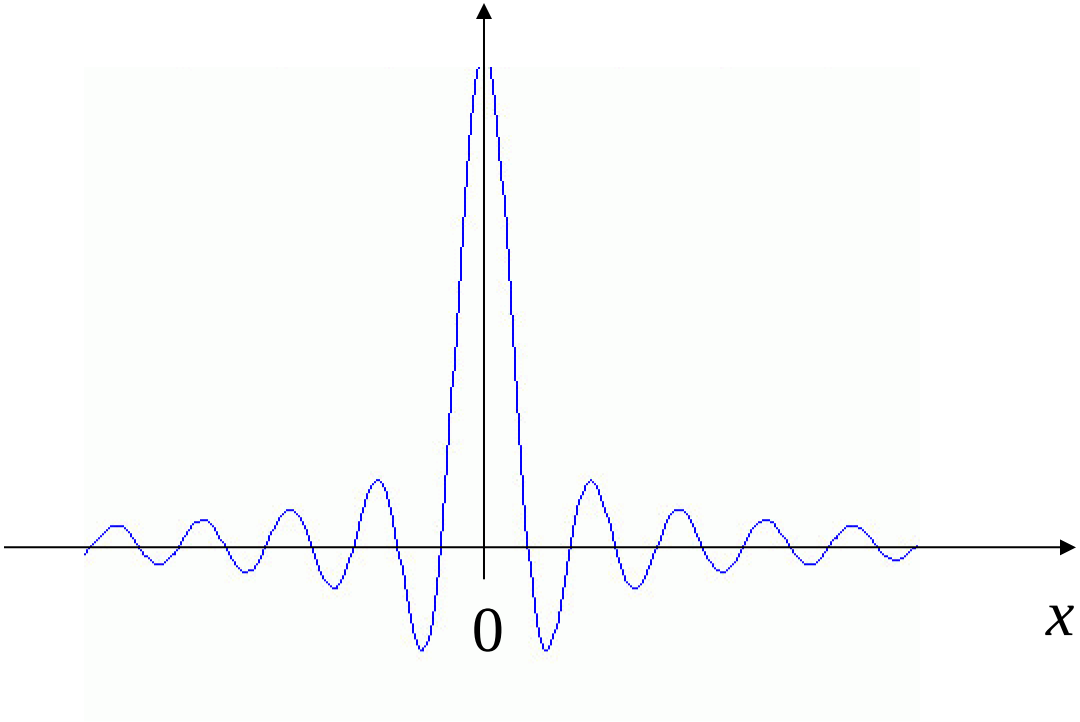
Then write:

$$s_o(x) = L\{s_i\} = \int_{-\infty}^{+\infty} s_i(\xi)L\{\delta(x-\xi)\}d\xi = \int_{-\infty}^{+\infty} s_i(\xi)h(x-\xi)d\xi$$

Impulse Response (2)

In practice we use non-causal impulse responses

- appear symmetric in their waveform



Convolution

The expression

$$s_o(x) = \int_{-\infty}^{+\infty} s_i(\xi)h(x-\xi)d\xi = s_i * h$$

is called *convolution*, defined as:

$$s_1(x) * s_2(x) = \int_{-\infty}^{+\infty} s_1(\xi)s_2(x-\xi)d\xi = \int_{-\infty}^{+\infty} s_1(x-\xi)s_2(\xi)d\xi$$

Procedure:

for each *x* do:

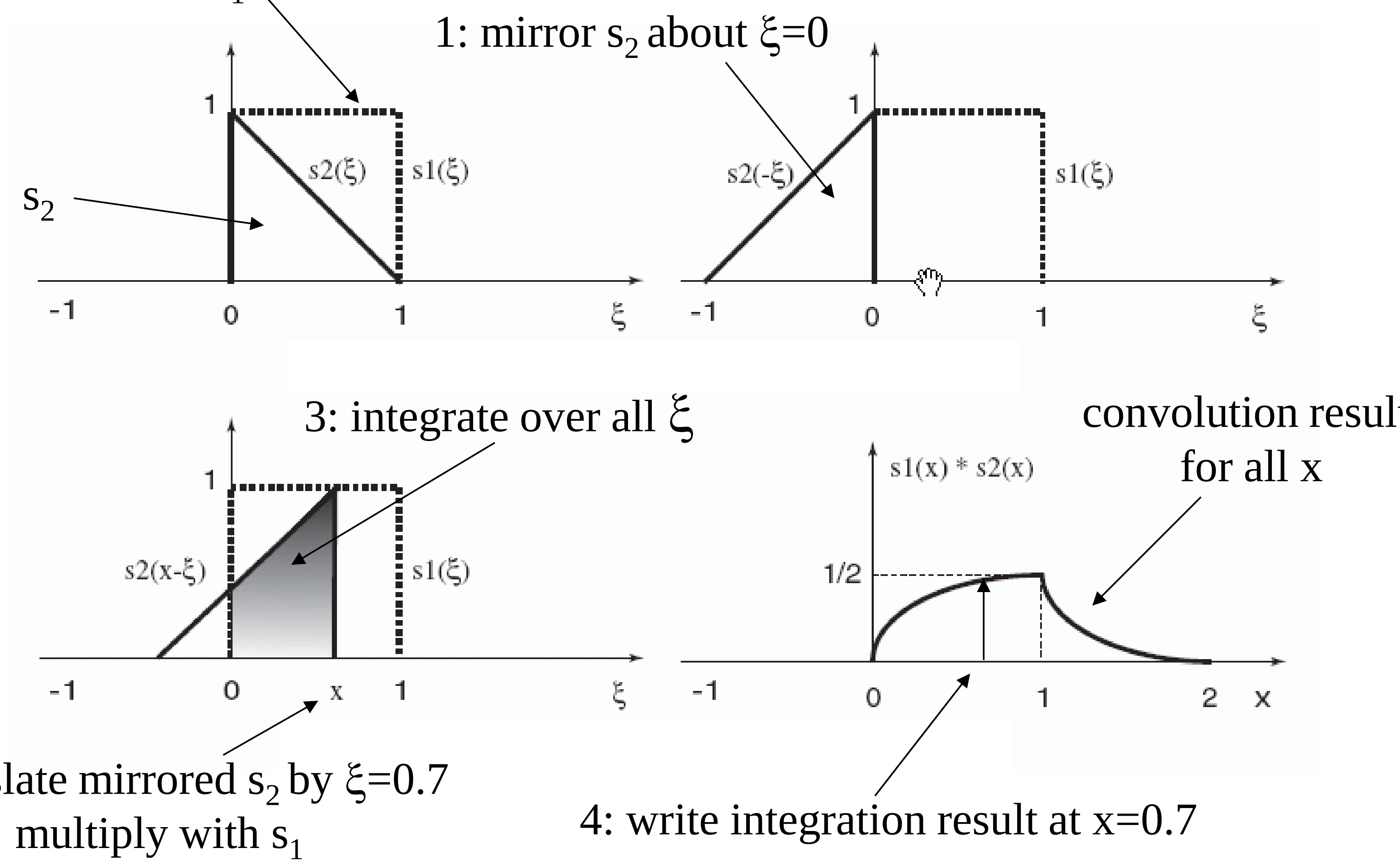
1: mirror s_2 about $\xi = 0$ (change ξ το $-\xi$)
2: translate mirrored s_2 by $\xi = x$
3: multiply s_1 and mirrored s_2
4: integrate the resulting signal

See next slides for an example and detailed explanation…

Convolution: Example

Example x=0.7:

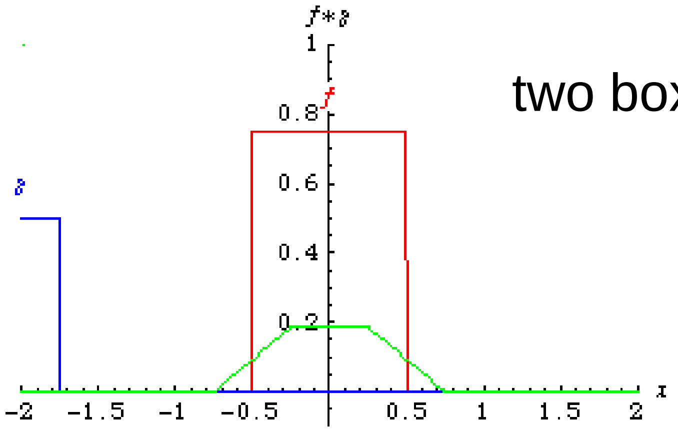
$$s_1(x) * s_2(x) = \int_{-\infty}^{+\infty} s_1(\xi) s_2(x-\xi) d\xi$$



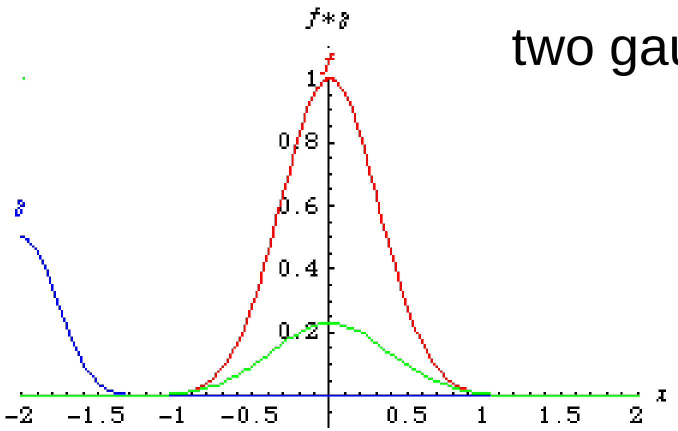
Convolution: More Examples

Animated gifs:

- red, blue: convolved signals
- green: convolution result



two boxes

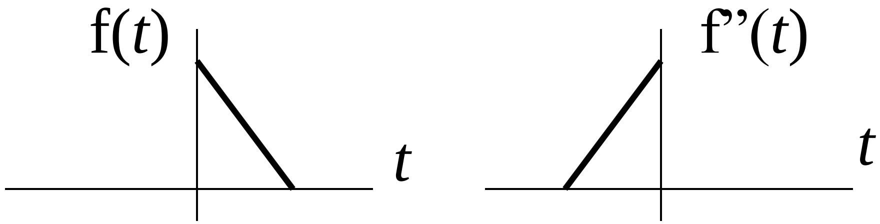


two gaussians

Convolution: Detailed Explanation

Mirroring:

- when you take a function f(*t*) and mirror it about the y-axis then you get a new function f”(*t*) = f(-*t*)



For convolution:

- you have two functions: $f_1(t)$ and $f_2(t)$
- you would like to compute:

$$f(x) = \int_{-\infty}^{+\infty} f_1(t) f_2(x-t)\,dt$$

- but in this form: *t* increases in f_1 and decreases in f_2, which is not convenient
- to fix this, you mirror $f_2(x-t)$ into $f_2''(t-x) = f_2(-(x-t))$
- now the convolution writes:

$$f(x) = \int_{-\infty}^{+\infty} f_1(t) f_2''(-(x-t))\,dt = \int_{-\infty}^{+\infty} f_1(t) f_2''(t-x)\,dt$$

- at this point you need $f_2''(t)$ which is obtained by mirroring $f_2(t)$: $f_2''(t) = f_2(-t)$
- now you can do the intuitive right-sliding of f_2“ for growing x

Convolution Properties

Also defined for multi-dimensional signals:

$$s_1(x,y) * s_2(x,y) = \int_{-\infty}^{+\infty} s_1(x-\xi, y-\zeta) s_2(\xi,\zeta) d\xi d\zeta$$

Some important properties:

- commutativity:

$$s_1 * s_2 = s_2 * s_1$$

- associativity:

$$(s_1 * s_2) * s_2 = s_1 * (s_2 * s_3)$$

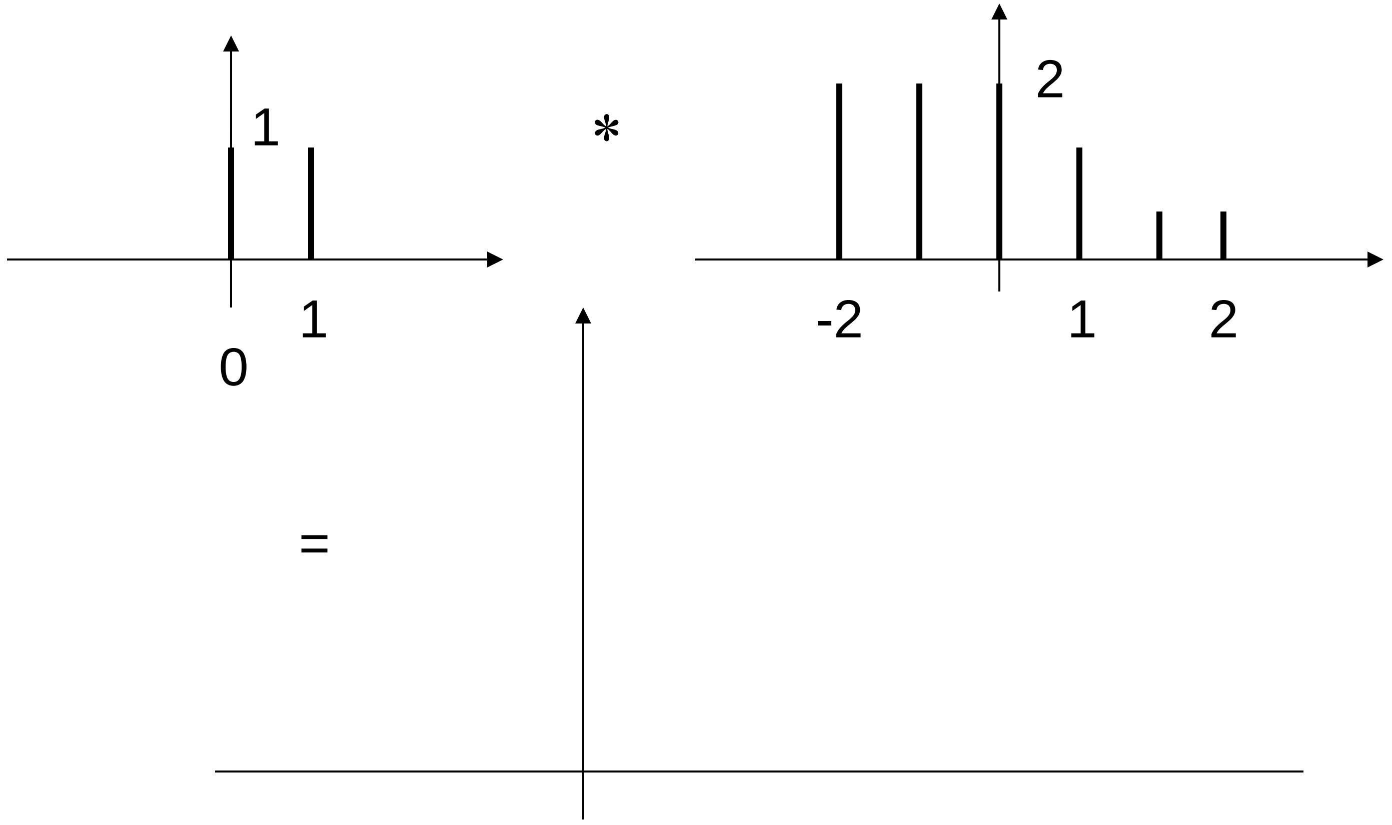
- distributivity:

$$s_1 * (s_2 + s_3) = s_1 * s_2 + s_1 * s_3$$

Discrete Signals

Typically, signals are only available in discrete form

- reconstruction into a continuous signal (for visualization, etc) occurs by overlapping point spread functions (see previous lecture)
- but all computer processing (convolution and others) is done on the discrete representations



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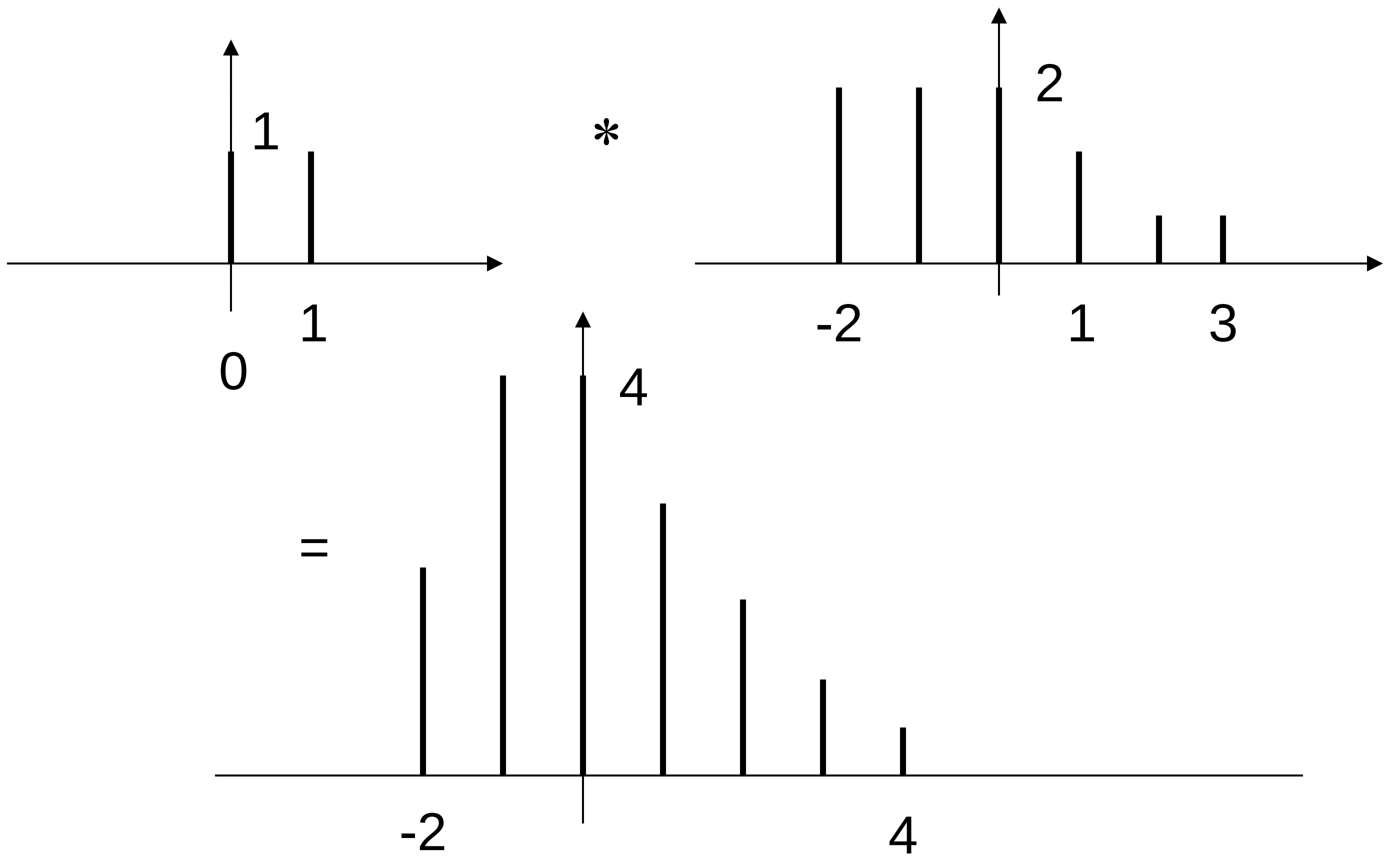


LSI System Response (1)

Now assume the input is a complex sinusoid with $Ae^{2\pi ikx}$ then:

for now, assume $\varphi=0$

$$s_0(x) = \int_{-\infty}^{+\infty} Ae^{2\pi ik(x-\xi)}h(\xi)d\xi$$

$$= Ae^{2\pi ikx}\int_{-\infty}^{+\infty} e^{-2\pi ik\xi}h(\xi)d\xi$$

$$= Ae^{2\pi ikx}H = S_iH$$

H is called the *Fourier Transform* of *h*(*x*):

$$H = \int_{-\infty}^{+\infty} e^{-2\pi ik\xi}h(\xi)d\xi$$

- *H* is also often called the *transfer function* or *filter*
- the Fourier transform will be discussed in detail shortly

LSI System Response (2)

H scales, and maybe phase-shifts, the input sinusoid S_i

In essence, we have now two alternative representations:

- determine the effect of *L* on s_i by convolution with *h*: $s_i * h$
- determine the effect of *L* on s_i by multiplication with *H*: $S_i \cdot H$

$$s_i * h \leftrightarrow S_i \cdot H$$

Since convolution is expensive for wide *h*, the multiplication may be cheaper

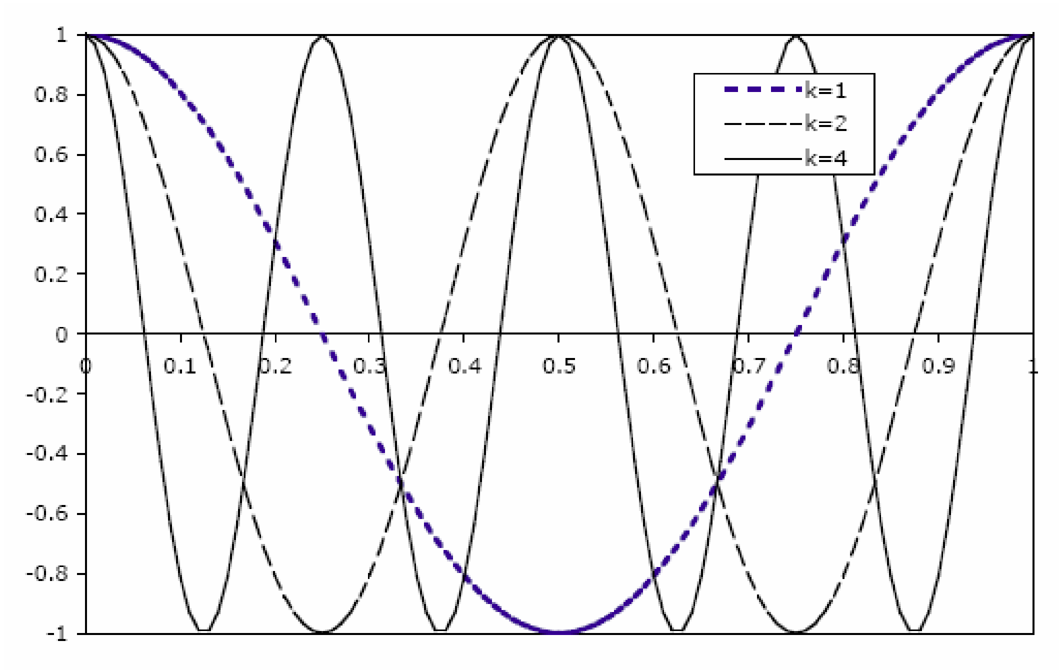
- but we need to perform the Fourier transforms of s_i and *h*
- in fact, there is a “sweetspot”
- more later…

Complex Sinusoids Revisited (1)

Recall the factor *k* in the complex sinusoid:

$$Ae^{i(2\pi kx+\phi)} = A\cos(2\pi kx+\phi) + i\sin(2\pi kx+\phi)$$

- as *k* increases, so does the frequency of the oscillation



- note: the higher *k*, the higher the signal resolution, that is, one can represent smaller signal details (signals that vary more quickly)

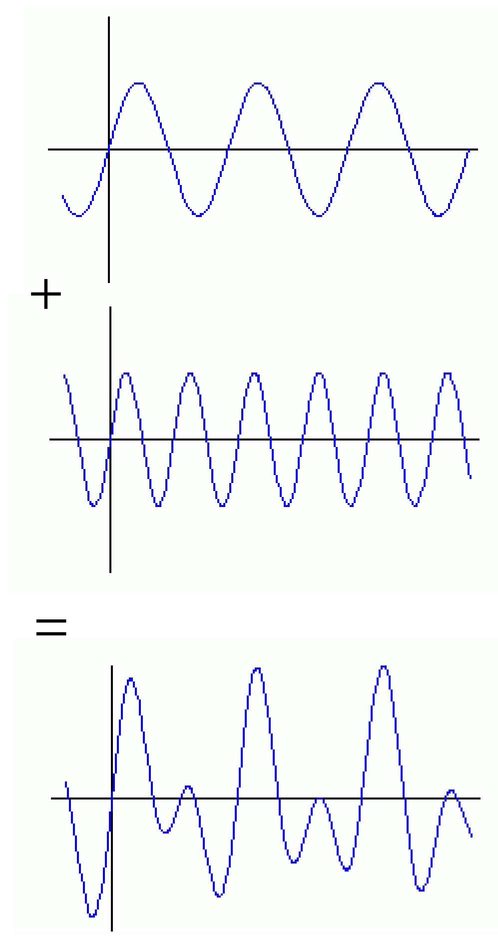
Signal Synthesis With Sinusoids

Any periodic signal can be created by a combination of weighted and shifted sinusoids at different frequencies

$$s_o(x) = \int_{-\infty}^{+\infty} A_k \cos(2\pi kx + \phi_k) + i\sin(2\pi kx + \phi_k)\,dk$$

$$= \int_{-\infty}^{+\infty} A_k e^{i(2\pi kx + \phi_k)}\,dk = \int_{-\infty}^{+\infty} A_k e^{i\phi_k} e^{i2\pi kx}\,dk$$

$$= \int_{-\infty}^{+\infty} S_i(k) e^{2\pi ikx}\,dk$$



- A_k is the amplitude and φ_k is the phase shift

Incorporating the transfer function, now one for each *k*:

$$s_0(x) = \int_{-\infty}^{+\infty} S_i(k) e^{2\pi ikx} H(k)\,dk$$