Strange Effects

Ever tried to reduce the size of an image and you got this?

We call this effect ‘aliasing’

Better

But what you really wanted is this:

We call this ‘anti-aliasing’

Why Is This Happening?

The smaller image resolution cannot represent the image detail captured at the higher resolution

• skipping this small detail leads to these undesired artifacts
Overview

So how do we get the nice image?

For this you need to understand:
- Fourier theory
- Sampling theory
- Digital filters

Don’t be scared, we’ll cover these topics gently

Periodic Signals

A signal is periodic if \( s(t+T) = s(t) \)
- we call \( T \) the period of the signal
- if there is no such \( T \) then the signal is aperiodic

Sinusoids are periodic functions
- sinusoids play an important role

Write as:

\[
A \sin\left(\frac{2\pi}{T} t + \phi_t \right)
\]
- where \( \phi_t \) is the phase shift and \( A \) is the amplitude

Alternatively:

\[
A \sin(2\pi f t + \phi_t) = A \sin(\omega t + \phi_t)
\]
- where \( f = 1/T \) is the frequency
- we may write \( \omega = 2\pi f \)

Fourier Theory

Jean Baptiste Joseph Fourier (1768-1830)

His idea (1807):
- Any periodic function can be rewritten as a weighted sum of sines and cosines of different frequencies.

Don’t believe it?
- neither did Lagrange, Laplace, Poisson and other major mathematicians of his time
- in fact, the theory was not translated into English until 1878

But it’s true!
- It is called the Fourier Series

Example

Consider the function:

\[ g(t) = \sin(2\pi f t) + (1/3)\sin(2\pi(3f) t) \]
Consider the function:

\[ g(t) = \sin(2\pi f t) + \frac{1}{3}\sin(2\pi (3f) t) \]

Further Example (1)

Further Example (2)

Further Example (3)
The Importance of the Frequency Spectrum

We observe:
• oscillations of different frequencies add to form the signal
• there is a characteristic frequency spectrum to any signal
• sharp edges can only be represented (generated) by high frequencies

The DC Component

The first component of the spectrum is the *signal average* DC

The Math…

The example just seen has the following Fourier Series:

\[ s(t) = \sum_{k=1}^{\infty} \frac{1}{k} \sin(2\pi kt) \]

• most of the time the phase is not interesting, so we shall omit it

We can convert any discrete signal into its Fourier Series (and back)
• this is called the Fourier Transform (Inverse Fourier Transform)
Fourier Transform of Discrete Signals: DFT

Discrete Fourier Transform (DFT)

- assumes that the signal is discrete and finite

\[ S(k) = \sum_{n=0}^{N-1} s(n)e^{-\frac{i2\pi kn}{N}} \]
\[ s(n) = \frac{1}{N} \sum_{n=0}^{N-1} S(k)e^{\frac{i2\pi kn}{N}} \]

- we have \( N \) samples, from which we can calculate \( N \) frequencies
- the frequency spectrum is discrete and it is periodic in \( N \)

Images are discrete signals

- so their frequency spectra are finite and periodic (see last slide)
- and therefore they have an upper limit (a maximum frequency)

Images are also finite (in size)

- the DFT assumes that they are also periodic
- as odd as this may sound, this is the underlying assumption

Therefore:

- frequency spectra are finite and periodic
- images are also finite and periodic

Keep this in mind for now

- it will help explain the strange resizing effects presented before

Now, What About the Complex Exponential...

It is Fourier’s way to encode phase and amplitude into one representation

- to understand it better, let’s first review complex numbers
- and then see what it means in the Fourier context

Note, we only discuss this to illustrate the full picture

- essential for this class is only to know the concept of frequency spectrum discussed thus far

Recall: Complex Numbers

A complex number \( c \) has a real and an imaginary part:

- \( c = Re\{c\} + i \text{Im}\{c\} \) (cartesian representation) \( i = \sqrt{-1} \)
- here, \( i \) always denotes the complex part

We can also use a polar representation:

\[ A_c = \sqrt{\text{Re}\{c\}^2 + \text{Im}\{c\}^2} \]
\[ \varphi_c = \tan^{-1}\left( \frac{\text{Im}\{c\}}{\text{Re}\{c\}} \right) \]
**Application: Complex Sinusoids**

Exponential $\exp$  

$$\exp(\alpha x) = e^{\alpha x}$$  

- when $\alpha > 0$ then $\exp$ increases with increasing $x$  
- when $\alpha < 0$ then $\exp$ approximates 0 with increasing $x$

Complex exponential / sinusoid:  

$$A_k e^{(2\pi kt + \phi)} = A_k \cos(2\pi kt + \phi) + i \sin(2\pi kt + \phi)$$

As before  

- the $\cos$ term is the signal’s real part  
- the $\sin$ term is the signal’s imaginary part  
- $A$ is the amplitude, $\phi$ the phase shift, $k$ determines the frequency

**Two-Dimensional Fourier Spectrum**

- $u$-axis  
- $v$-axis  

**Some Example Spectra**

- Spectrum along $u$ determines detail along spatial $x$  
- Spectrum along $v$ determines detail along spatial $y$

**Effects of Missing Spectra Portions: Axial**

(a) Spectrum along $u$ determines detail along spatial $x$  
(b) Spectrum along $v$ determines detail along spatial $y$
Effects of Missing Spectra Portions: Radial

(a) Lower frequencies (close to origin) give overall structure
(b) Higher frequencies (periphery) give detail (sharp edges)

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The Math… 2D DFT

The 2D transform:
\[ S(k, l) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} s(n, m) e^{-i2\pi(\frac{kn}{NM}+\frac{lm}{NM})} \]
\[ s(n, m) = \frac{1}{NM} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} S(k, l) e^{i2\pi(\frac{kn}{NM}+\frac{lm}{NM})} \]

Separability:
\[ S(k, l) = \frac{1}{NM} \sum_{m=0}^{M-1} e^{i2\pi \frac{km}{M}} P(k, m) \quad \text{where} \quad P(k, m) = \sum_{n=0}^{N-1} s(n, m) e^{-i2\pi \frac{kn}{N}} \]
\[ s(n, m) = \frac{1}{NM} \sum_{l=0}^{M-1} e^{i2\pi \frac{lm}{M}} p(n, l) \quad \text{where} \quad p(n, l) = \sum_{k=0}^{N-1} S(n, m) e^{-i2\pi \frac{kn}{N}} \]

• if M=N, complexity is \(2 \cdot O(2N^3)\)

Fast Fourier Transform (FFT)

Recursively breaks up the FT sum into odd and even terms:
\[ S(k) = \sum_{n=0}^{N-1} s(n) e^{-i2\pi \frac{kn}{N}} = \sum_{n=0}^{N/2-1} s(2n) e^{-i2\pi \frac{2n}{N}} + \sum_{n=0}^{N/2-1} s(2n+1) e^{-i2\pi \frac{(2n+1)}{N}} \]
\[ = \sum_{n=0}^{N/2-1} s_{\text{even}}(n) e^{-i2\pi \frac{kn}{N}} + e^{-i2\pi \frac{kN}{N}} \sum_{n=0}^{N/2-1} s_{\text{odd}}(n) e^{-i2\pi \frac{kn}{N}} \]

Results in an \(O(n \cdot \log(n))\) algorithm (in 1D)
• \(O(n^2 \cdot \log(n))\) for 2D (and so on)

Fast Fourier Transform (FFT)

Gives rise to the well-known butterfly Divide + Conquer architecture
• invented by Cooley-Tuckey, 1965