Strange Effects
Ever tried to reduce the size of an image and you got this?

We call this effect ‘aliasing’

Better
But what you really wanted is this:

We call this ‘anti-aliasing’

Why Is This Happening?
The smaller image resolution cannot represent the image detail captured at the higher resolution
• skipping this small detail leads to these undesired artifacts
So how do we get the nice image? For this you need to understand:

- Fourier theory
- Sampling theory
- Digital filters

Don’t be scared, we’ll cover these topics gently

Periodic Signals

A signal is periodic if $s(t+T) = s(t)$

- we call $T$ the period of the signal
- if there is no such $T$ then the signal is aperiodic

Sinusoids are periodic functions

- sinusoids play an important role

Write as:

$$A \sin \left( \frac{2\pi}{T} t + \phi \right)$$

- where $\phi$ is the phase shift and $A$ is the amplitude

Alternatively:

$$A \sin (2\pi ft + \phi) = A \sin (\omega t + \phi)$$

- where $f = 1/T$ is the frequency
- we may write $\omega = 2\pi f$

Example

Consider the function:

$$g(t) = \sin(2\pi ft) + \frac{1}{3}\sin(2\pi (3f) t)$$

Fourier Theory

Jean Baptiste Joseph Fourier (1768-1830)

His idea (1807):

- Any periodic function can be rewritten as a weighted sum of sines and cosines of different frequencies.

Don’t believe it?

- neither did Lagrange, Laplace, Poisson and other major mathematicians of his time
- in fact, the theory was not translated into English until 1878

But it’s true!

- it is called the Fourier Series
Consider the function:

\[ g(t) = \sin(2\pi f t) + \frac{1}{3}\sin(2\pi (3f) t) \]

the function’s frequency spectrum

Further Example (1)

Further Example (2)

Further Example (3)
Further Example (4)

The Importance of the Frequency Spectrum

We observe:
- oscillations of different frequencies add to form the signal
- there is a characteristic frequency spectrum to any signal
- sharp edges can only be represented (generated) by high frequencies

The DC Component

The first component of the spectrum is the *signal average DC*

The Math…

The example just seen has the following Fourier Series:

\[ s(t) = \sum_{k=1}^{\infty} \frac{1}{k} \sin(2\pi kt) \]

- most of the time the phase is not interesting, so we shall omit it

In fact, this is an interesting series: the *sinc* function
- we shall see more of it soon

We can convert any discrete signal into its Fourier Series (and back)
- this is called the *Fourier Transform (Inverse Fourier Transform)*
Fourier Transform of Discrete Signals: DFT

Discrete Fourier Transform (DFT)
- assumes that the signal is discrete and finite
- we have $N$ samples, from which we can calculate $N$ frequencies
- the frequency spectrum is discrete and it is periodic in $N$

$$S(k) = \sum_{n=0}^{N-1} s(n)e^{-\frac{i2\pi kn}{N}} \quad s(n) = \frac{1}{N} \sum_{n=0}^{N-1} S(k)e^{\frac{i2\pi kn}{N}}$$

**Periodicity**

Images are discrete signals
- so their frequency spectra are finite and periodic (see last slide)
- and therefore they have an upper limit (a maximum frequency)

Images are also finite (in size)
- the DFT assumes that they are also periodic
- as odd as this may sound, this is the underlying assumption

Therefore:
- frequency spectra are finite and periodic
- images are also finite and periodic

Keep this in mind for now
- it will help explain the strange resizing effects presented before

**Recall: Complex Numbers**

A complex number $c$ has a real and an imaginary part:
- $c = Re\{c\} + i Im\{c\}$ (cartesian representation) $i = \sqrt{-1}$
- here, $i$ always denotes the complex part

We can also use a polar representation:

$$A_c = \sqrt{Re\{c\}^2 + Im\{c\}^2}$$

$$\varphi_c = \tan^{-1}\left(\frac{Im\{c\}}{Re\{c\}}\right)$$
**Application: Complex Sinusoids**

Exponential $\exp$

$$\exp(ax) = e^{ax}$$
- when $a > 0$ then $\exp$ increases with increasing $x$
- when $a < 0$ then $\exp$ approximates 0 with increasing $x$

Complex exponential / sinusoid:

$$A_x e^{(2\pi(ax + \varphi)} = A_x (\cos(2\pi kt + \varphi) + i \sin(2\pi kt + \varphi))$$

As before
- the $\cos$ term is the signal’s real part
- the $\sin$ term is the signal’s imaginary part
- $A$ is the amplitude, $\varphi$ the phase shift, $k$ determines the frequency

**Two-Dimensional Fourier Spectrum**

**Effects of Missing Spectra Portions: Axial**

(a) Spectrum along $u$ determines detail along spatial $x$
(b) Spectrum along $v$ determines detail along spatial $y$
Effects of Missing Spectra Portions: Radial

(a) Lower frequencies (close to origin) give overall structure
(b) Higher frequencies (periphery) give detail (sharp edges)

The Math... 2D DFT

The 2D transform:

\[ S(k, l) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} s(n, m) e^{-j2\pi(k\frac{m}{M} + l\frac{n}{N})} \]

\[ s(n, m) = \frac{1}{NM} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} S(k, l) e^{j2\pi(k\frac{m}{M} + l\frac{n}{N})} \]

Separability:

\[ S(k, l) = \frac{1}{NM} \sum_{m=0}^{M-1} e^{j2\pi l\frac{m}{M}} P(k, m) \quad \text{where} \quad P(k, m) = \sum_{n=0}^{N-1} s(n, m) e^{-j2\pi \frac{kn}{N}} \]

\[ s(n, m) = \frac{1}{NM} \sum_{l=0}^{L-1} e^{j2\pi k\frac{l}{M}} p(n, l) \quad \text{where} \quad p(n, l) = \sum_{k=0}^{K-1} S(n, m) e^{-j2\pi \frac{kn}{N}} \]

* if M=N, complexity is \( 2 \cdot O(2N^3) \)

Fast Fourier Transform (FFT)

Recursively breaks up the FT sum into odd and even terms:

\[ S(k) = \sum_{n=0}^{N-1} s(n)e^{-j2\pi \frac{kn}{N}} = \sum_{n=0}^{N/2-1} s(2n)e^{-j2\pi \frac{2kn}{N}} + \sum_{n=0}^{N/2-1} s(2n+1)e^{-j2\pi \frac{2kn+1}{N}} \]

\[ = \sum_{n=0}^{N/2-1} s_{even}(n)e^{-j2\pi \frac{kn}{N/2}} + e^{-j\pi k N/2} \sum_{n=0}^{N/2-1} s_{odd}(n)e^{-j2\pi \frac{kn}{N/2}} \]

Results in an \( O(n \cdot \log(n)) \) algorithm (in 1D)

- \( O(n^2 \cdot \log(n)) \) for 2D (and so on)

Fast Fourier Transform (FFT)

Gives rise to the well-known butterfly Divide + Conquer architecture

- invented by Cooley-Tuckey, 1965)