Introduction to Medical Imaging

Lecture 3: Linear System Theory

Klaus Mueller

Computer Science Department
Stony Brook University

**Even / Odd Functions**

Signal is even if $s(-x) = s(x)$
- denote as $s_e$

$$
\int_{-\infty}^{\infty} s_e(x)dx = 2 \int_{0}^{\infty} s_e(x)dx
$$

Signal is odd if $s(-x) = -s(x)$
- denote as $s_0$

$$
\int_{-\infty}^{\infty} s_0(x)dx = 0
$$

Can write any signal as a sum of its even and odd part:

$$
s(x) = \frac{1}{2} \left[ s(x) + s(-x) \right] + \frac{1}{2} \left[ s(x) - s(-x) \right] = s_e(x) + s_0(x)
$$

**Periodic Signals**

A signal is periodic if $s(x+X) = s(x)$
- we call $X$ the period of the signal
- if there is no such $X$ then the signal is aperiodic

Sinusoids are periodic functions
- sinusoids will play an important role in this course

Write as:

$$
A \sin\left(\frac{2\pi x}{X} + \phi_x\right)
$$

- where $\phi_x$ is the phase shift and $A$ is the amplitude

Sinusoids can combine
- they can also occur in higher dimensions:
Complex Numbers

A complex number $c$ has a real and an imaginary part:

- $c = \text{Re}(c) + i \text{Im}(c)$ (cartesian representation)
  
where, $i$ always denotes the complex part

We can also use a polar representation:

$$A_c = \sqrt{\text{Re}(c)^2 + \text{Im}(c)^2}$$

$$\varphi_c = \tan^{-1}\left(\frac{\text{Im}(c)}{\text{Re}(c)}\right)$$

Now think of $c$ as a periodic signal $s(x)$:

- then the pointer $(A_c, \varphi_c)$ rotates with period $X$, that is, it completes one rotation after each integer multiple of $X$
- if there is a phase shift $\varphi_x$ then the pointer simply is already located at $(A_c, \varphi_c)$ when $x=0$
- considering $c$ a 2D vector: $\text{Re}(c) = A_c \cos(\varphi_c)$ and $\text{Im}(c) = A_c \sin(\varphi_c)$

Important Signals (1)

Exponential $\exp$

$\exp(ax) = e^{ax}$

- when $a > 0$ then $\exp$ increases with increasing $x$
- when $a < 0$ then $\exp$ approximates 0 with increasing $x$

Complex exponential / sinusoid:

$A e^{(2\pi kx + \phi)} = A(\cos(2\pi kx + \phi) + i \sin(2\pi kx + \phi))$

As before

- the $\cos$ term is the signal’s real part
- the $\sin$ term is the signal’s imaginary part
- $A$ is the amplitude, $\varphi$ the phase shift, $k$ determines the frequency

Important Signals (2)

Rectangular function:

$$\Pi\left(\frac{x}{2L}\right) = \begin{cases} 1 & \text{for } |x| < L \\ \frac{1}{2} & \text{for } |x| = L \\ 0 & \text{for } |x| > L \end{cases}$$

Step function:

$$u(x - x_0) = \begin{cases} 0 & \text{for } x < x_0 \\ \frac{1}{2} & \text{for } x = x_0 \\ 1 & \text{for } x > x_0 \end{cases}$$

Important Signals (3)

Triangular function:

$$\text{Tri}\left(\frac{x}{2L}\right) = 1 - \left|\frac{x}{L}\right|$$

- 1 for $|x| < L$
- 0 for $|x| > L$

Normalized Gaussian:

$$G_n(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- $\mu$ is the mean
- $\sigma$ is the standard deviation

- normalized means that the integral for all $x$ is 0
**Sinc function:**

\[
sinc(x) = \frac{\sin(\pi x)}{\pi x}
\]

- \(sinc(0) = 1\) (L'Hopital's rule)

**Dirac impulse:**

\[
\delta(x - x_0) = 0 \quad \text{for} \quad x \neq x_0
\]

\[
\int_{-\infty}^{+\infty} \delta(x - x_0) \, dx = 1
\]

- an important property is its sifting property:

\[
\int_{-\infty}^{+\infty} s(x) \delta(x - x_0) \, dx = s(x_0)
\]

a “needle” spike of infinite height at \(x = x_0\)

**System response \(L\):**

\[
s_o = L\{s_i\}
\]

- might be a function of time \(t\) or space \(x\)

\[
s_o(t) = L\{s_i(t)\} \quad \text{or} \quad s_o(x) = L\{s_i(x)\}
\]

Finding the mathematical relationship between in- and output is called **modeling**

Linear systems fulfill **superposition principle**:

\[
L\{c_1s_1 + c_2s_2\} = c_1L\{s_1\} + c_2L\{s_2\} \quad \forall c_1, c_2 \in \mathbb{R}
\]

where \(s_1\), \(s_2\) are arbitrary signals

- for example, consider an amplifier with gain \(A\):

\[
L\{c_1s_1 + c_2s_2\} = A(c_1s_1 + c_2s_2)
\]

\[
= c_1As_1 + c_2As_2 = c_1L\{s_1\} + c_2L\{s_2\}
\]

**Time-invariance (shift-invariance = LSI):**

- properties of \(L\) do not change over time (spatial position), that is:

\[
s_o(x) = L\{s_i(x)\} \quad \text{then} \quad s_o(x - X) = L\{s_i(x - X)\}
\]

A system’s response to a Dirac impulse is called **impulse response** \(h\):

\[
\delta(t)
\]

Start with:

\[
s_i(x) = \int_{-\infty}^{+\infty} s_i(\xi)\delta(x - \xi) \, d\xi
\]

Then write:

\[
s_o(x) = L\{s_i\} = \int_{-\infty}^{+\infty} s_i(\xi)L\{\delta(x - \xi)\} \, d\xi = \int_{-\infty}^{+\infty} s_i(\xi)h(x - \xi) \, d\xi
\]
Impulse Response (2)

In practice we use non-causal impulse responses
- appear symmetric in their waveform

Convolution

The expression
\[ s_o(x) = \int_{-\infty}^{\infty} s_i(\xi) h(x-\xi) d\xi = s_i * h \]

is called convolution, defined as:
\[ s_i(x) * s_2(x) = \int_{-\infty}^{\infty} s_i(\xi) s_2(x-\xi) d\xi = \int_{-\infty}^{\infty} s_i(x-\xi) s_2(\xi) d\xi \]

Procedure:
for each \(x\) do:
1: mirror \(s_2\) about \(\xi = 0\) (change \(\xi \to -\xi\))
2: translate mirrored \(s_2\) by \(\xi = x\)
3: multiply \(s_1\) and mirrored \(s_2\)
4: integrate the resulting signal

See next slides for an example and detailed explanation…

Convolution: Example

Example \(x=0.7\):
\[ s_1(x) * s_2(x) = \int_{-\infty}^{\infty} s_1(\xi) s_2(x-\xi) d\xi \]

1: mirror \(s_2\) about \(\xi = 0\)
2: translate mirrored \(s_2\) by \(\xi = 0.7\)
multiply with \(s_1\)
3: integrate over all \(\xi\)
4: write integration result at \(x=0.7\)

Convolution: More Examples

Animated gifs:
- red, blue: convolved signals
- green: convolution result

Two boxes
Two gaussians
Convolution: Detailed Explanation

Mirroring:
• when you take a function \( f(t) \) and mirror it about the y-axis then you get a new function \( f'(t) = f(-t) \)

For convolution:
• you have two functions: \( f_1(t) \) and \( f_2(t) \)
• you would like to compute:
  \[
  f(x) = \int_{-\infty}^{\infty} f_1(t) f_2(x-t) dt
  \]
• but in this form: \( t \) increases in \( f_1 \) and decreases in \( f_2 \), which is not convenient
• to fix this, you mirror \( f_2(x-t) \) into \( f_2^*(t-x) = f_2(-x-t) \)
• now the convolution writes:
  \[
  f(x) = \int_{-\infty}^{\infty} f_1(t) f_2^*(t-x) dt = \int_{-\infty}^{\infty} f_1(t) f_2^*(t-x) dt
  \]
• at this point you need \( f_2^*(t) \) which is obtained by mirroring \( f_2(t) \): \( f_2^*(t) = f_2(-t) \)
• now you can do the intuitive right-sliding of \( f_2^* \) for growing \( x \)

Convolution Properties

Also defined for multi-dimensional signals:

\[
\begin{align*}
  s_1(x,y) * s_2(x,y) &= \int_{-\infty}^{+\infty} s_1(x-\xi,y-\zeta)s_2(\xi,\zeta)d\xi d\zeta
\end{align*}
\]

Some important properties:
• commutativity:
  \[
  s_1 * s_2 = s_2 * s_1
  \]
• associativity:
  \[
  (s_1 * s_2) * s_3 = s_1 * (s_2 * s_3)
  \]
• distributivity:
  \[
  s_1 * (s_2 + s_3) = s_1 * s_2 + s_1 * s_3
  \]

Discrete Signals

Typically, signals are only available in discrete form
• reconstruction into a continuous signal (for visualization, etc) occurs by overlapping point spread functions (see previous lecture)
• but all computer processing (convolution and others) is done on the discrete representations
**LSI System Response (1)**

Now assume the input is a complex sinusoid with $Ae^{2\pi ikx}$ then:

$$s_0(x) = \int_{-\infty}^{+\infty} Ae^{2\pi ik(x-\xi)} h(\xi) d\xi$$

for now, assume $\phi=0$

$$= Ae^{2\pi ikx} \int_{-\infty}^{+\infty} e^{-2\pi ik\xi} h(\xi) d\xi$$

$$= Ae^{2\pi ikx} H = S_i H$$

$H$ is called the **Fourier Transform** of $h(x)$:

$$H = \int_{-\infty}^{+\infty} e^{-2\pi ik\xi} h(\xi) d\xi$$

• $H$ is also often called the **transfer function** or **filter**
• the Fourier transform will be discussed in detail shortly

**Complex Sinusoids Revisited (1)**

Recall the factor $k$ in the complex sinusoid:

$$Ae^{(2\pi k\xi + \phi)} = A\cos(2\pi k\xi + \phi) + i\sin(2\pi k\xi + \phi)$$

• as $k$ increases, so does the frequency of the oscillation

• note: the higher $k$, the higher the signal resolution, that is, one can represent smaller signal details (signals that vary more quickly)

**Signal Synthesis With Sinusoids**

Any periodic signal can be created by a combination of weighted and shifted sinusoids at different frequencies

$$s_0(x) = \int_{-\infty}^{+\infty} A_k \cos(2\pi k x + \phi_k) + i\sin(2\pi k x + \phi_k) dk$$

$$= \int_{-\infty}^{+\infty} A_k e^{i(2\pi k x + \phi_k)} dk = \int_{-\infty}^{+\infty} A_k e^{i\phi_k} e^{2\pi k x} dk$$

$$= \int_{-\infty}^{+\infty} S_i(k) e^{2\pi ikx} dk$$

• $A_k$ is the amplitude and $\phi_k$ is the phase shift

Incorporating the transfer function, now one for each $k$:

$$s_0(x) = \int_{-\infty}^{+\infty} S_i(k) e^{2\pi ikx} H(k) dk$$

**LSI System Response (2)**

$H$ scales, and maybe phase-shifts, the input sinusoid $S_i$

In essence, we have now two alternative representations:

• determine the effect of $L$ on $s_i$ by convolution with $h$: $s_i * h$
• determine the effect of $L$ on $s_i$ by multiplication with $H$: $S_i \cdot H$

$$s_i * h \leftrightarrow S_i \cdot H$$

Since convolution is expensive for wide $h$, the multiplication may be cheaper

• but we need to perform the Fourier transforms of $s_i$ and $h$
• in fact, there is a “sweetspot”
• more later…