Introduction to Medical Imaging

Iterative Reconstruction Methods

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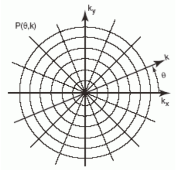
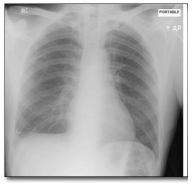
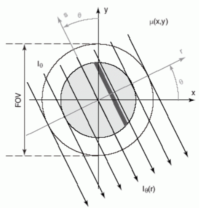
Ideal Assumptions



Dense and regular sampling of the Fourier domain → many projections



Noise free projections

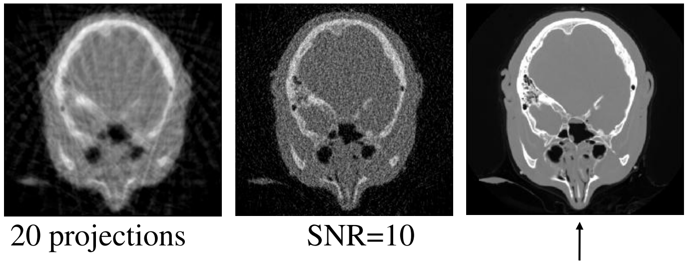


Straight rays

Non-Ideal Scenarios

Projections might be:

- sparse
- acquired over less than 180°
- noisy



20 projections SNR=10

low-dose CT high-dose CT

Rays might be non-linear (curved, refracted, scattered,…)

- for example: refraction in ultrasound imaging

Dealing With Non-Ideal Scenarios

Iterative methods are advantageous in these cases

They can handle:

- limited number of projections
- irregularly-spaced and -angled projections
- non-straight ray paths (example: refraction in ultrasound imaging)
- corrective measures during reconstruction (example: metal artifacts)
- presence of statistical (Poisson) noise and scatter (mainly in functional imaging: SPECT, PET)

Specifics

In medical imaging:

- M unknown voxels (depending on desired object resolution)
- N known measurements (pixels in the projection images)
- represent voxels and pixels as vectors V and P, respectively

$$w_{11}v_1 + w_{12}v_2 + ... w_{1M}v_M = p_1$$
$$w_{21}v_1 + w_{22}v_2 + ... w_{2M}v_M = p_2$$
$$....$$
$$w_{N1}v_1 + w_{N2}v_2 + ... w_{NM}v_M = p_N$$

- this gives rise to a system $W{\cdot}V{=}P$

Solving for *V*

The obvious solution is then:

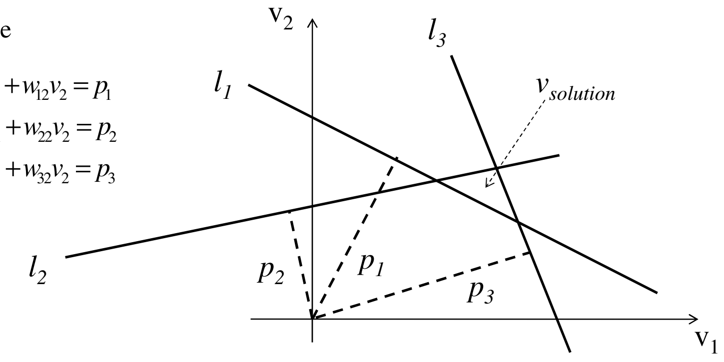
- compute $V = W^{-1} \cdot P$

The main problem with this direct approach:

- P is not be consistent due to noise → lines do not intersect in solution
- This turns $W{\cdot}V{=}P$ into an optimization problem

2D case

$$w_{11}v_1 + w_{12}v_2 = p_1$$
$$w_{21}v_1 + w_{22}v_2 = p_2$$
$$w_{31}v_1 + w_{32}v_2 = p_3$$



Optimization Algorithms

Algebraic methods

- Algebraic Reconstruction Technique (ART), SART, SIRT
- Projection Onto Convex Sets (POCS)

Sparse system solvers

- Gradient Descent (GD), Conjugate Gradients (CG)
- Gauss-Seidel

Statistical methods

- Expectation Maximization (EM)
- Maximum Likelihood Estimation (MLE)

All of these are *iterative* methods:

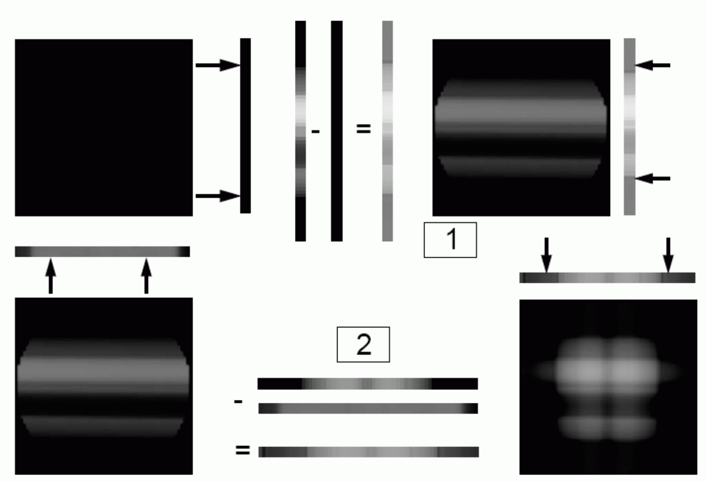
- predict → compare → correct → predict → compare → correct …

Big Picture: Iterative Reconstruction

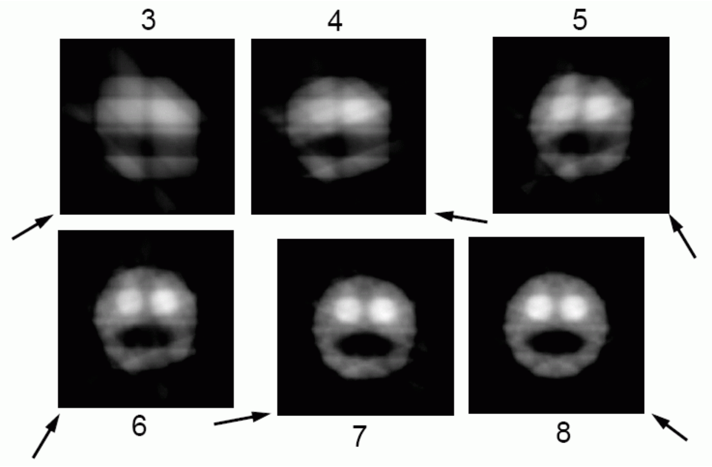
Before delving into details,

let’s see an iterative scheme at work

Iterative Reconstruction Demonstration: SART

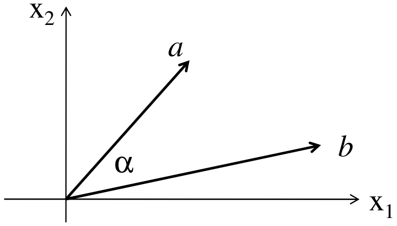


Iterative Reconstruction Demonstration: SART



Foundations: Vectors

Consider two vectors, *a* and *b*



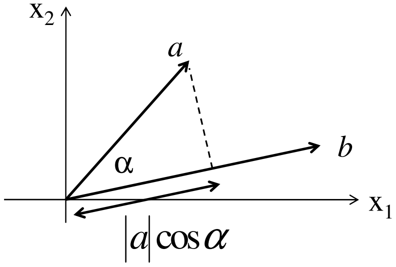
$$a = \vec{a} = [a_1\ a_2], \qquad |a| = \sqrt{a_1^2 + a_2^2}$$

$$b = \vec{b} = [b_1\ b_2], \qquad |b| = \sqrt{b_1^2 + b_2^2}$$

Foundations: Scalar Projection

Scalar projection of *a* onto *b*:

$$|a| \cos\alpha = a \cdot \frac{b}{|b|}$$



The *dot product*:

$$a \cdot b = \vec{a} \cdot \vec{b}^T = [a_1\ a_2] \cdot [b_1\ b_2]^T = a_1 b_1 + a_2 b_2$$

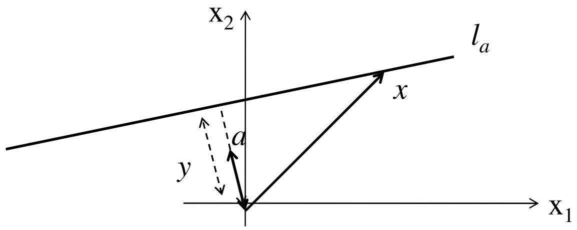
$$= |a| \cdot |b| \cos\alpha$$

→ the scalar projection is the dot product with */b/* =1 (unit vector)

$$|b| = \sqrt{b_1^2 + b_2^2} = 1$$

Foundations: Line Equation

$$a_1x_1 + a_2x_2 = y$$
$$|a| = \sqrt{a_1^2 + a_2^2} = 1$$



The vector *a* is the unit vector normal to the line l_a

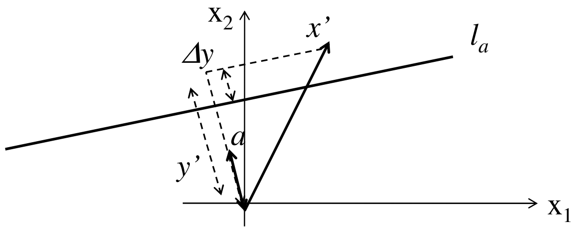
The length *y* is the perpendicular distance of l_a to the origin

For any point *x*:

- if *x* is on l_a then the scalar projection of *x* onto *a* will be:

$$x \cdot a = y$$

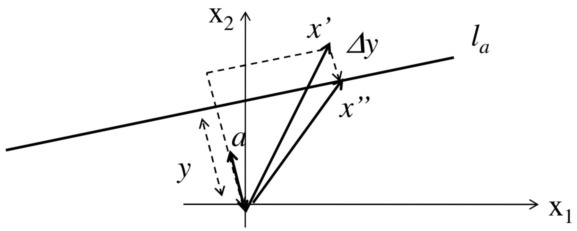
Foundations: Distance From Line



For any other point *x'* not on l_a the scalar projection of *x'* onto *a* will be:

$$x' \cdot a = y' = y + \Delta y$$

Foundations: Closest Point



The closest point to *x'* on l_a is *x''*, computed by:

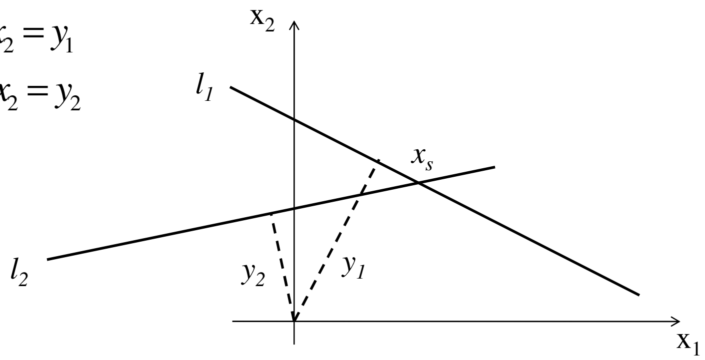
$$\begin{aligned} x'' &= x' - \Delta y \\ &= x' - (x' \cdot a - y) \\ &= x' + (y - x' \cdot a) \end{aligned}$$

Foundations: Solving an Equation System

Assume you have two equations to solve for solution point $x_s=(x_1,x_2)$

- the intersection of the two lines

$$a_{11}x_1 + a_{12}x_2 = y_1$$
$$a_{21}x_1 + a_{22}x_2 = y_2$$

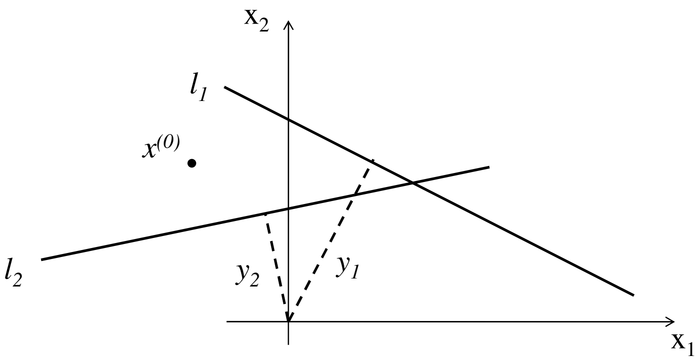


Foundations: Iterating to Solution

Of course, you could solve this equation via Gaussian elimination

- we shall take an iterative approach instead

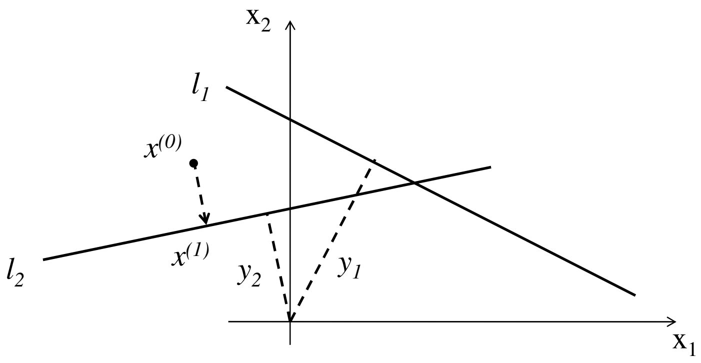
Start with some point $x^{(0)}=(x_1,x_2)$



Foundations: Iterating to Solution

Pick an equation (line, say l_2) and find the closest point to $x^{(0)}$

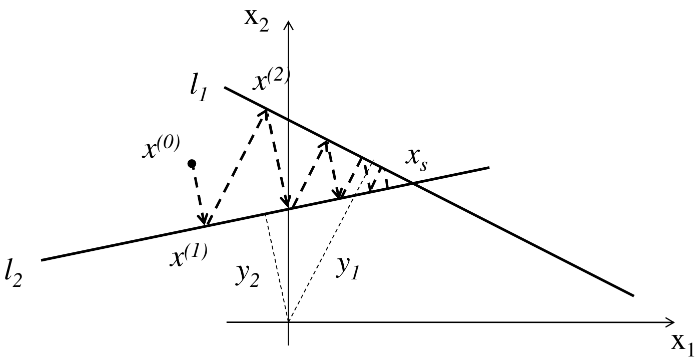
- use the approach outlined before
- this gives a new point $x^{(1)}$



Foundations: Iterating to Solution

Iteratively

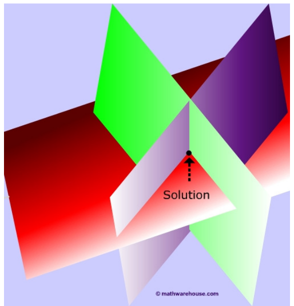
- pick alternate equations (lines) and project
- the solution will *converge* towards x_s
- the more iterations the closer the convergence



Foundations: Extension to Higher Dimensions

Three dimensions:

- 3 equations with 3 unknowns



N dimensions:

- *N* equations with *M* unknowns
- *M* can be less or greater than *N*
- inconsistent (most often) or not

Specifics to Medical Imaging

In medical imaging:

- *M* unknown voxels (depending on desired object resolution)
- *N* known measurements (pixels in the projection images)
- represent voxels and pixels as vectors *V* and *P*, respectively

$$w_{11}v_1 + w_{12}v_2 + \dots w_{1M}v_M = p_1$$
$$w_{21}v_1 + w_{22}v_2 + \dots w_{2M}v_M = p_2$$
$$\dots.$$
$$w_{N1}v_1 + w_{N2}v_2 + \dots w_{NM}v_M = p_N$$

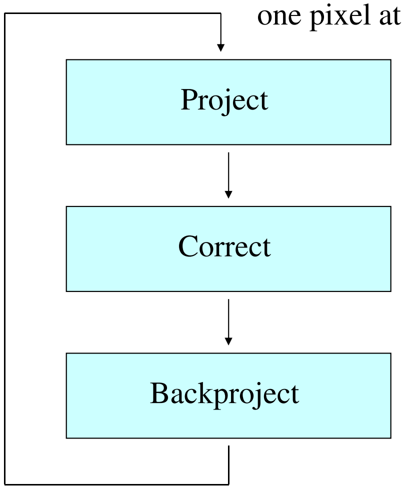
- this gives rise to a system $W \cdot V = P$

Iterate either by

- ray by ray (Algebraic Reconstruction Technique, ART)
- image by image (Simultaneous ART, SART)
- all data at once (SIRT)

Iterative Update Schedule: ART

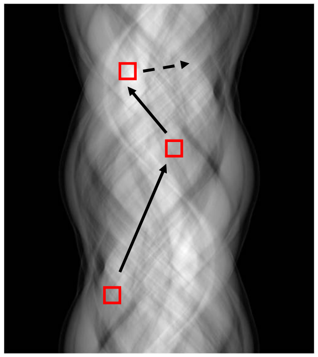
one pixel at a time



Project

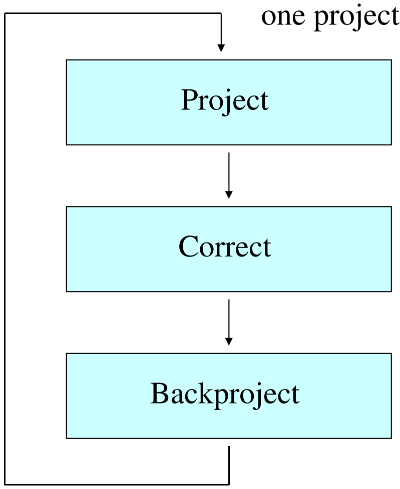
Correct

Backproject



Iterative Update Schedule: SART

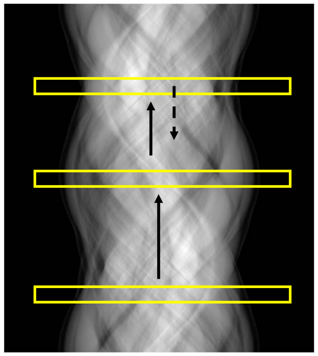
one projection at a time



Project

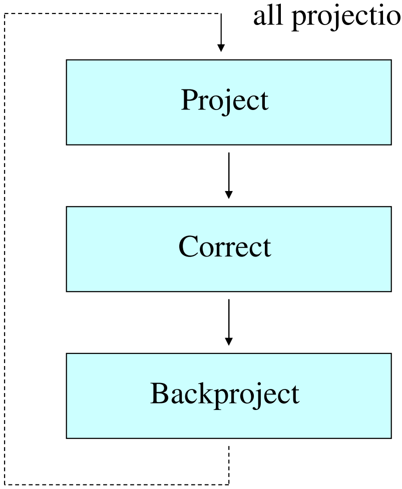
Correct

Backproject



Iterative Update Schedule: SIRT

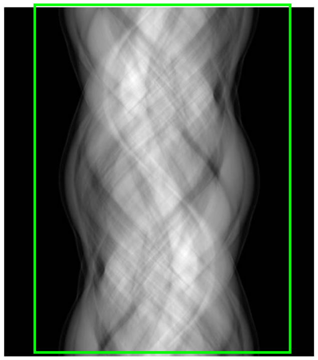
all projections



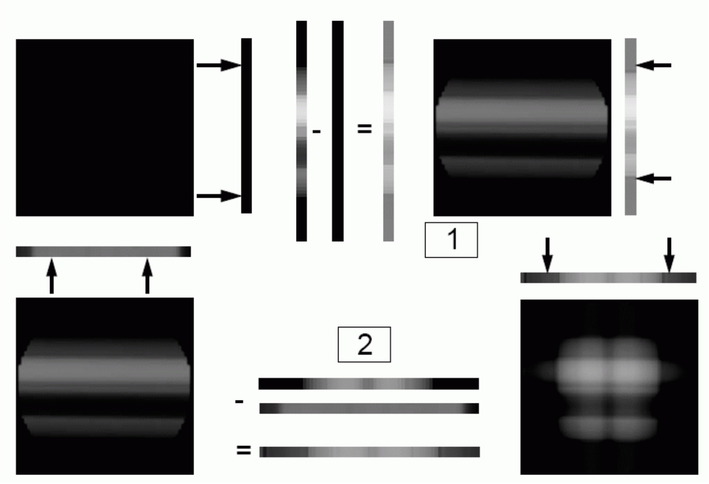
Project

Correct

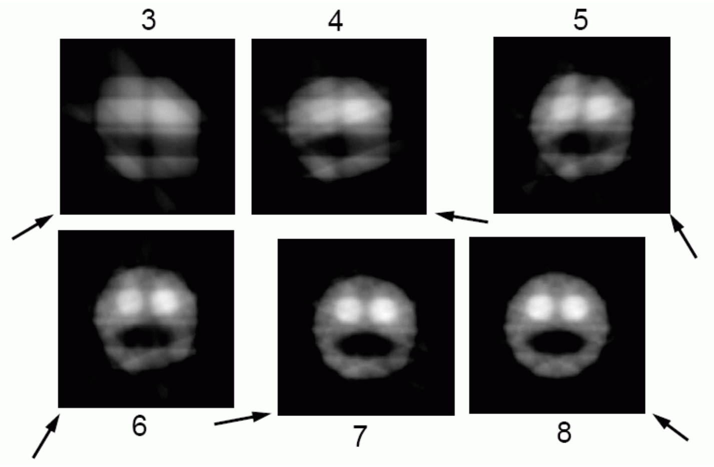
Backproject



Iterative Reconstruction Demonstration: SART



Iterative Reconstruction Demonstration: SART



SART

Iteratively solves *W·V=P*

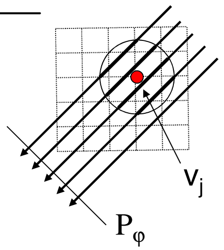
$$v_j^{k+1} = v_j^k + \lambda \frac{\sum_i \frac{p_i - \sum_j v_j^k w_{ij}}{\sum_j w_{ij}} w_{ij}}{\sum_i w_{ij}}$$

SART

Projection

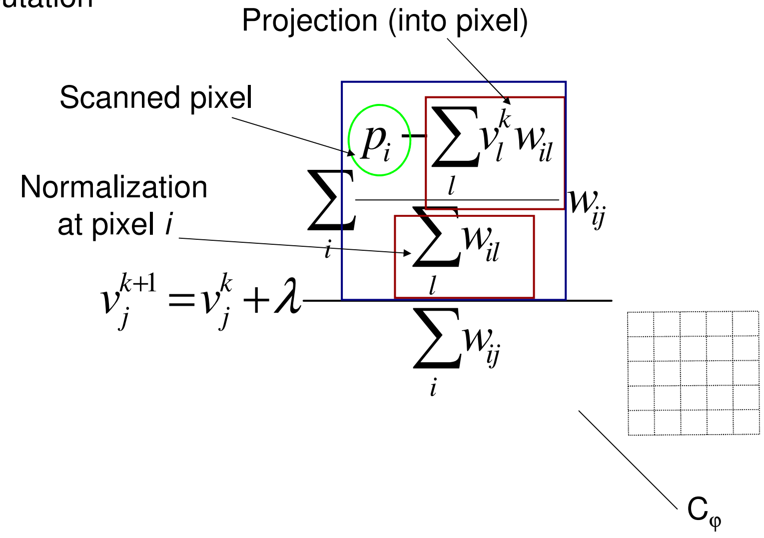
Projection (into pixel)

$$v_j^{k+1} = v_j^k + \lambda \frac{\sum_i \frac{p_i - \boxed{\sum_l v_l^k w_{il}}}{\boxed{\sum_l w_{il}}} w_{ij}}{\sum_i w_{ij}}$$



SART

Correction factor computation



Projection (into pixel)

Scanned pixel

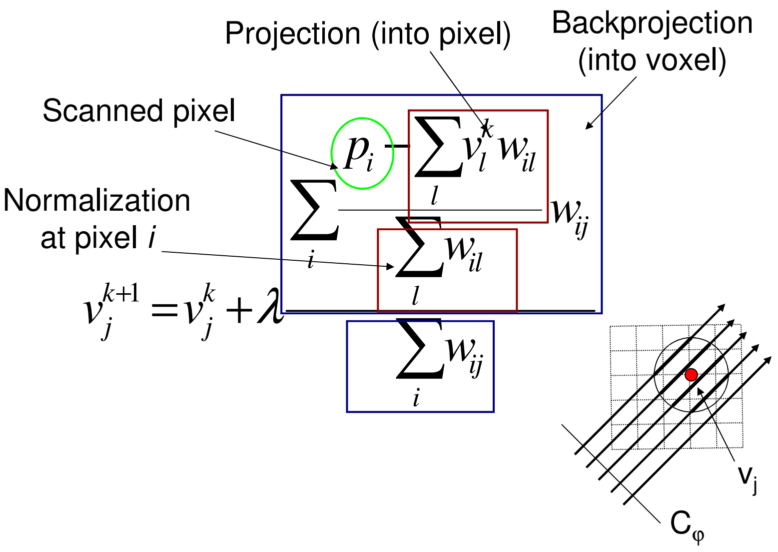
Normalization at pixel *i*

$$v_j^{k+1} = v_j^k + \lambda \frac{\sum_i \frac{p_i - \sum_l v_l^k w_{il}}{\sum_l w_{il}} w_{ij}}{\sum_i w_{ij}}$$

C_φ

SART

Backprojection



Projection (into pixel)

Backprojection (into voxel)

Scanned pixel

Normalization at pixel *i*

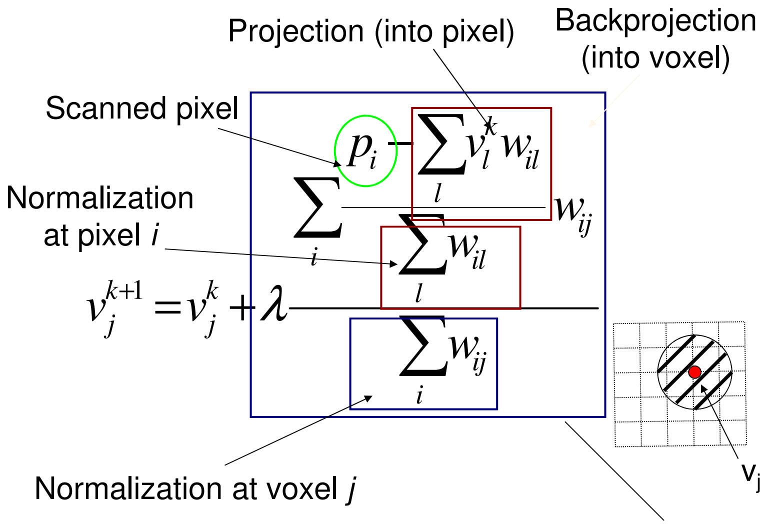
$$v_j^{k+1} = v_j^k + \lambda \frac{\sum_i \frac{p_i - \sum_l v_l^k w_{il}}{\sum_l w_{il}} w_{ij}}{\sum_i w_{ij}}$$

v_j

C_φ

SART

Voxel normalization



Projection (into pixel)

Backprojection (into voxel)

Scanned pixel

Normalization at pixel *i*

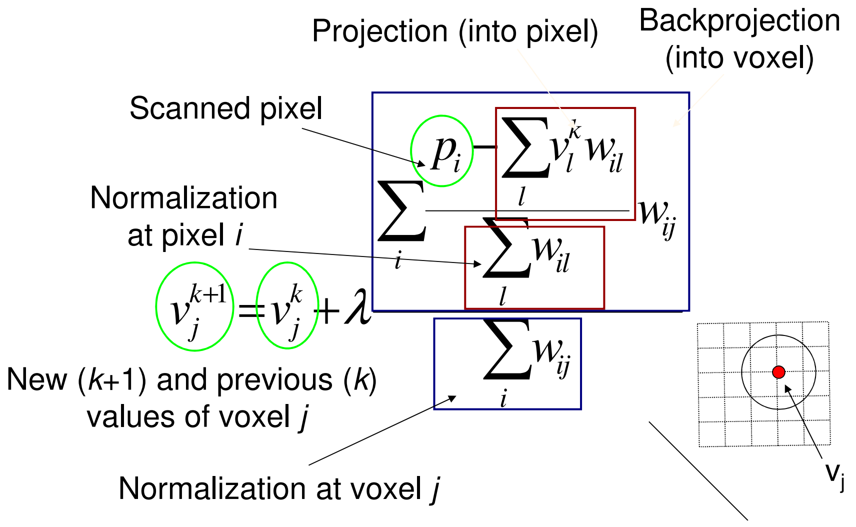
$$v_j^{k+1} = v_j^k + \lambda \frac{\sum_i \frac{p_i - \sum_l v_l^k w_{il}}{\sum_l w_{il}} w_{ij}}{\sum_i w_{ij}}$$

Normalization at voxel *j*

v_j

SART

Voxel update



Projection (into pixel)

Backprojection (into voxel)

Scanned pixel

Normalization at pixel *i*

$$v_j^{k+1} = v_j^k + \lambda \frac{\sum_i \frac{p_i - \sum_l v_l^k w_{il}}{\sum_l w_{il}} w_{ij}}{\sum_i w_{ij}}$$

New (*k*+1) and previous (*k*) values of voxel *j*

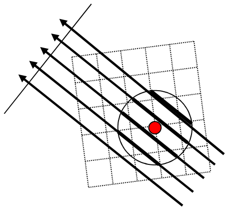
Normalization at voxel *j*

v_j

SART

Next projection

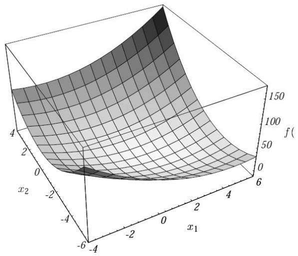
$$v_j^{k+1} = v_j^k + \lambda \frac{\sum_i \frac{p_i - \boxed{\sum_l v_l^k w_{il}}}{\boxed{\sum_l w_{il}}} w_{ij}}{\sum_i w_{ij}}$$



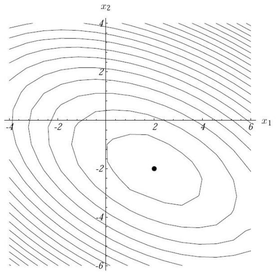
Gradient Descent

Quadratic form of a vector: $$f(x) = \frac{1}{2} x^T A x - b^T x + c$$

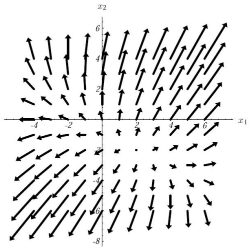
- this equation is minimized when $A \cdot x = b$
- this occurs when f'(x)=0
- thus, minimizing the quadratic form will solve the reconstruction problem



Graph plot



Contour plot



Gradient plot

Steepest Descent

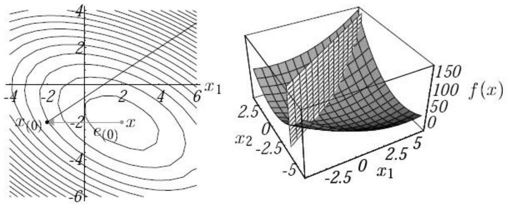
Start at an arbitrary point and slide down to the bottom of the parabola

- in practice this will be a hyper-parabola since *x, b* are high-dimensional
- choose the direction in which *f* decreases most quickly

$$-f'(x_{(i)}) = b - Ax_{(i)}$$

where $x_{(i)}$ is the current (predicted) solution

- similar to ART but now looks at all equations simultaneously

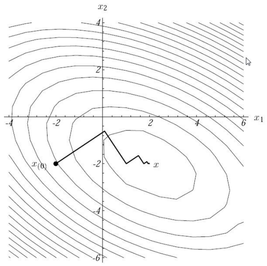


Figures from J. Shewchuk, UC Berkeley

Steepest Descent

Start at some initial guess $x_{(0)}$

- this will likely not find the solution
- need to follow $f'(x_{(0)})$ some ways and then change directions
- question is *where* do we change directions



Some basics:

- error: how far are we away from the solution

$$e_{(i)} = x_{(i)} - x$$

- residual: how far are we away from the correct value of *b*

$$r_{(i)} = b - Ax_{(i)}$$

$$r_{(i)} = Ae_{(i)}$$ *A* transforms *e* into the space of *b*

$$r_{(i)} = -f'(x_{(i)})$$

Steepest Descent

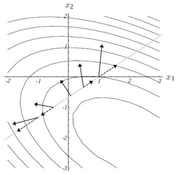
Finding the right place to turn directions is called *line search*

$$x_{(1)} = x_{(0)} + \alpha r_{(0)}$$

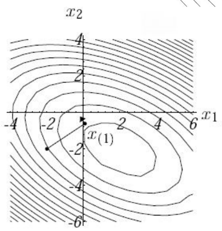
To find α we can use the following requirements:

- the new direction of *r* must be orthogonal to the previous:

$$r_{(1)}{}^{T} r_{(0)} = 0$$



- the residual at $x_{(1)}$ $\quad f'(x_{(1)}) = -r_{(1)}$



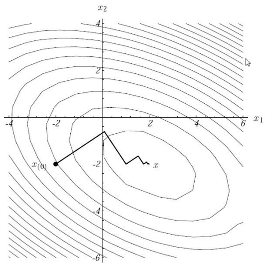
- after some math: $\alpha = \dfrac{r_{(0)}{}^{T} r_{(0)}}{r_{(0)}{}^{T} A r_{(0)}}$

Steepest Descent: Summary

$$r_{(i)} = b - A x_{(i)}$$

$$\alpha = \frac{r_{(i)}{}^{T} r_{(i)}}{r_{(i)}{}^{T} A r_{(i)}}$$

$$x_{(i+1)} = x_{(i)} + \alpha r_{(i)}$$



Shortcoming:

- sub-optimal since some directions might be taken more than once
- this can be fixed by the method of Conjugant Gradients

Conjugant Gradients

Picks a set of *orthogonal* search directions $d_{(0)}, d_{(1)}, d_{(2)}, \ldots$

- take exactly one step along each
- stop at exactly the right length for each to line up evenly with *x*

$$x_{(i+1)} = x_{(i)} + \alpha_{(i)} d_{(i)}$$

- to determine $\alpha_{(i)}$ use the fact that $e_{(i+1)}$ should be orthogonal to $d_{(i)}$

$$d_{(i)}{}^{T} e_{(i+1)} = 0$$

$$d_{(i)}{}^{T} (e_{(i)} + \alpha d_{(i)}) = 0$$

$$\alpha_{(i)} = \frac{d_{(i)}{}^{T} e_{(i)}}{d_{(i)}{}^{T} d_{(i)}}$$

- however, this requires knowledge of $e_{(i)}$ which we do not have

Conjugant Gradients

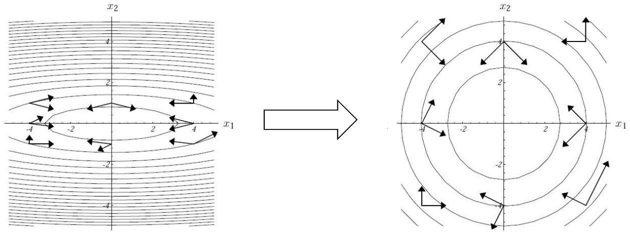
Solution:

- make the search direction *A*-orthogonal (or, *conjugate*)

$$\alpha_{(i)} = \frac{d_{(i)}{}^{T} A e_{(i)}}{d_{(i)}{}^{T} A d_{(i)}} = \frac{d_{(i)}{}^{T} r_{(i)}}{d_{(i)}{}^{T} A d_{(i)}}$$

- *A* transforms a coordinate system such that two vectors are orthogonal

$$d_{(i)}{}^{T} A d_{(j)} = 0 \quad i \neq j$$



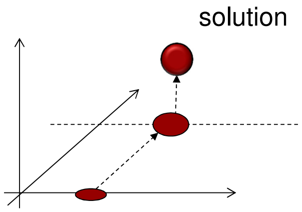
Conjugant Gradients

All directions taken are mutually orthogonal

- each new residual is orthogonal to all the previous residuals and search directions
- each new search direction is constructed (from the residual) to be *A*-orthogonal to all the previous residuals and search directions

Each new search direction adds a new dimension to the traversed sub-space

- the solution is a projection into the sub-space explored so far
- so after *n* steps the full space is built and the solution has been reached



Conjugant Gradients: Summary

$$d_{(0)} = r_{(0)} = b - Ax_{(0)},$$

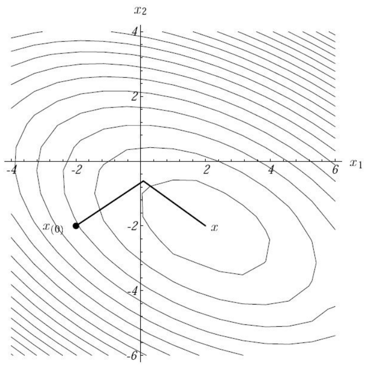
$$\alpha_{(i)} = \frac{r_{(i)}^T r_{(i)}}{d_{(i)}^T A d_{(i)}}$$

$$x_{(i+1)} = x_{(i)} + \alpha_{(i)} d_{(i)},$$

$$r_{(i+1)} = r_{(i)} - \alpha_{(i)} A d_{(i)},$$

$$\beta_{(i+1)} = \frac{r_{(i+1)}^T r_{(i+1)}}{r_{(i)}^T r_{(i)}},$$

$$d_{(i+1)} = r_{(i+1)} + \beta_{(i+1)} d_{(i)}.$$



Statistical Techniques

Algebraic/gradient methods do not model statistical effects in the underlying data

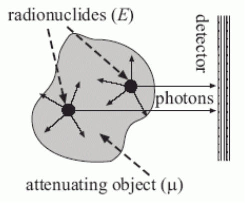
- this is OK for CT (within reason)

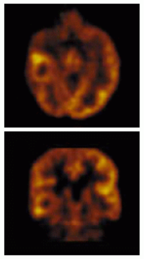
However, the emission of radiation from radionuclides is highly statistical

- the direction is chosen at random
- similar metabolic activities may not emit the same radiation
- not all radiation is actually collected (collimators reject many photons)
- in low-dose CT, noise is also a significant problem

Need a reconstruction method that can accounts for these statistical effects

- Maximum Likelihood – Expectation Maximization (ML-EM) is one such method





Foundations: The Poisson Distribution

Also called the *law of rare events*

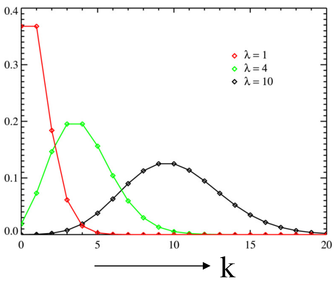
- it is the binomial distribution of *k* as the number of trials *n* goes to infinity

$$\lim_{n\to\infty} \Pr(X=k) = \lim_{n\to\infty} \binom{n}{k} p^k (1-p)^{n-k}$$

- with $p = \lambda / n$

$$f(k;\lambda) = \frac{e^{-\lambda}\lambda^k}{k!}$$

λ: expected number of events (the mean) in a given time interval

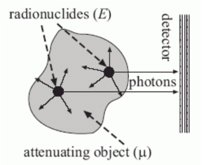


Some examples for Poisson-distributed events:

- the number of phone calls at a call center per minute
- the number of spelling errors a secretary makes while typing a single page
- the number of soldiers killed by horse-kicks each year in each corps in the Prussian cavalry
- the number of positron emissions in a radio nucleotide in PET and SPECT
- the number of annihilation events in PET and SPECT

Overall Concept of ML-EM

There are three types of variables



#1: The observed data y(d):

- the detector readings

#2: The unobserved (latent) data x(b):

- the photon emission activities in the pixels (the tissue), x(b)
- these give rise to the detector readings
- they follow a Poisson distribution

#3: The model parameters λ(b):

- these cause the emissions
- they are the metabolic activities (state) of interest
- the emissions only approximate those

→ they represent the expectations (means, λ) of the resulting Poisson distribution causing the readings at the detectors

---

Overall Concept of ML-EM

There is a many-to-one mapping of parameters → data

Since there is a many-to-one mapping, many objects are probable to have produced the observed data

- the object reconstruction (the *image*) having the highest such probability is the *maximum likelihood estimate* of the original object

Goal:

- estimate the model parameters using the observed data

Solution:

- EM will converge to a solution of maximum likelihood (but not necessarily the global maximum)

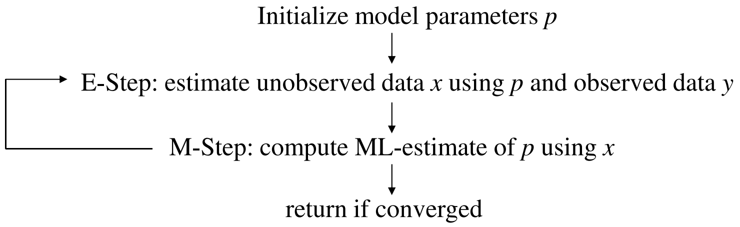
Overall Concept of ML-EM

Initialization step: choose an initial setting of the model parameters

Then proceed to EM, which has two steps, executed iteratively:

- E (expectation) step: estimate the unobserved data from the current estimate of the model parameters and the observed data
- M (maximization) step: compute the maximum-likelihood estimate of the model parameters using the estimated unobserved data

Stop when converged



Initialize model parameters p

E-Step: estimate unobserved data x using p and observed data y

M-Step: compute ML-estimate of p using x

return if converged

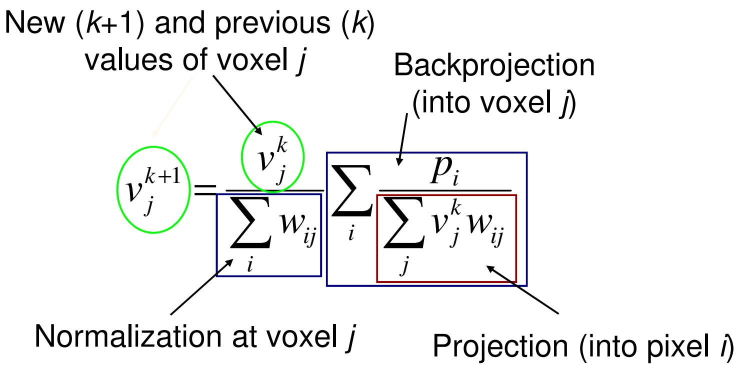
Maximum Likelihood Expectation Maximization (ML-EM)

After combining the E-step and the ML-step:

$$v_j^{k+1} = \frac{v_j^k}{\sum_i w_{ij}} \sum_i \frac{p_i}{\sum_j v_j^k w_{ij}}$$

Maximum Likelihood Expectation Maximization (ML-EM)

Maximizes the likelihood of the values of (object) voxels *j*, given values at (detector) pixels *i*



New (*k*+1) and previous (*k*) values of voxel *j*

Backprojection (into voxel *j*)

$$v_j^{k+1} = \frac{v_j^k}{\sum_i w_{ij}} \sum_i \frac{p_i}{\sum_j v_j^k w_{ij}}$$

Normalization at voxel *j*

Projection (into pixel *i*)

Algorithm Comparison

SART:

- projection ordering important
- ensure that consecutively selected projections are approximately orthogonal
- random selection works well in practice

CG:

- much depends on the condition number of the (system) matrix A
- various pre-conditioning methods exist in the literature
- also, line search can be expensive and inaccurate
- various methods and heuristics for line search have been described in the literature

EM:

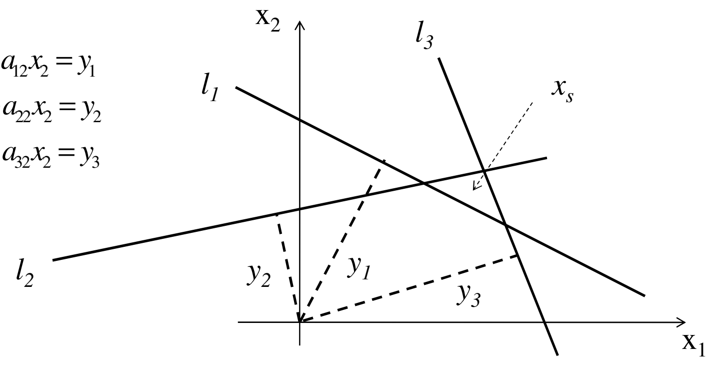
- convergence slow if all projections are applied before voxel update
- use OS-EM (Ordered Subsets EM): only a subset of projections are applied per iteration

Inconsistent Equations

Real life data (as mentioned earlier)

- typically equations (the data) are not consistent
- you may have more equations (data) than unknowns or not enough
- solution falls within a *convex* shape spanned by the intersection set
- need further criteria to determine the true solution (some *prior model*)

$$a_{11}x_1 + a_{12}x_2 = y_1$$
$$a_{21}x_1 + a_{22}x_2 = y_2$$
$$a_{31}x_1 + a_{32}x_2 = y_3$$



Determining the True Solution

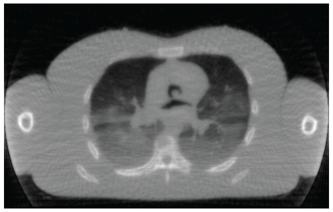
Need further criteria to determine the true solution

Use some *prior model*

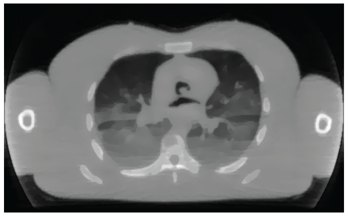
- smoothness, approximate shape, sharp edges, …
- incorporate this model into the reconstruction procedure

Example:

- enforce smoothness by intermittent blurring
- but at the same time preserve edges



streak artifacts, good edges



smooth, good edges