Introduction to Medical Imaging

Lecture 4: Fourier Theory

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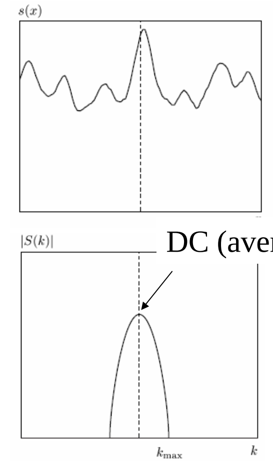
Introduction

Theory developed by Joseph Fourier (1768-1830)

The Fourier transform of a signal s(x) yields its *frequency spectrum S*(*k*)



$s(x)$

DC (average) term

$S(k)$

forward transform

$$S(k) = F\{s(x)\} = \int_{-\infty}^{+\infty} s(x) e^{-2\pi ikx} dx$$

inverse transform

$$s(x) = F^{-1}\{S(k)\} = \int_{-\infty}^{+\infty} S(k) e^{2\pi ikx} dk$$

Extension to Higher Dimensions

The Fourier transform generalizes to higher dimensions

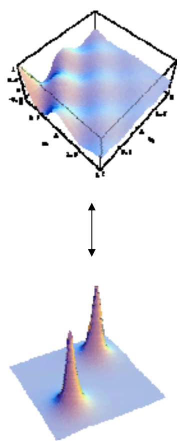
Consider the 2D case:

forward transform

$$S(k,l) = F\{s(x,y)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s(x,y) e^{-2\pi i(kx+ly)} dxdy$$

inverse transform

$$s(x,y) = F^{-1}\{S(k,l)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} S(k,l) e^{2\pi i(kx+ly)} dkdl$$

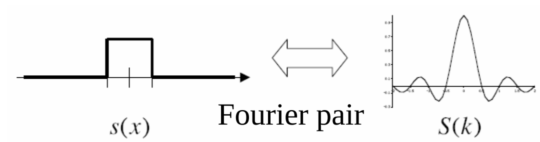


Calculation: Rect Function

$$S(k) = F\{A\Pi(\frac{x}{2L})\} = \int_{-\infty}^{+\infty} A\Pi(\frac{x}{2L}) e^{-i2\pi kx} dx = \int_{-L}^{+L} A e^{-i2\pi kx} dx$$

$$= -\frac{A}{2\pi ki}(e^{-i2\pi kL} - e^{i2\pi kL}) = \frac{A}{2\pi k} 2\sin(2\pi kL)$$

$$= 2AL \ \mathrm{sinc}(2\pi kL)$$

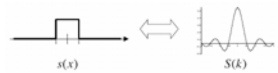


We see that a finite signal in the *x*-domain creates an infinite signal in the *k*-domain (the frequency domain)

- the same is true vice versa

Properties

Scaling:



- consider the rect (box): the greater L…
  - … the higher the spectrum (factor AL)
  - … the narrower the spectrum (factor L)

$$S(k) = 2AL\ \mathrm{sinc}(2\pi kL)$$

- the scaling rule is therefore:

$$F\{s(ax)\} = \frac{1}{|a|} S\left(\frac{k}{a}\right) \qquad \begin{array}{l} a>1 \text{ shrinks } s \\ a<1 \text{ stretches } s \end{array}$$

Symmetry: $F\{S(x)\} = s(-k)$

Linearity: $F\{as_1(x) + bs_2(x)\} = F\{as_1(x)\} + F\{bs_2(x)\}$

Translation: $F\{s(x - x_0)\} = S(k)e^{-2\pi i x_0 k}$ phase shift

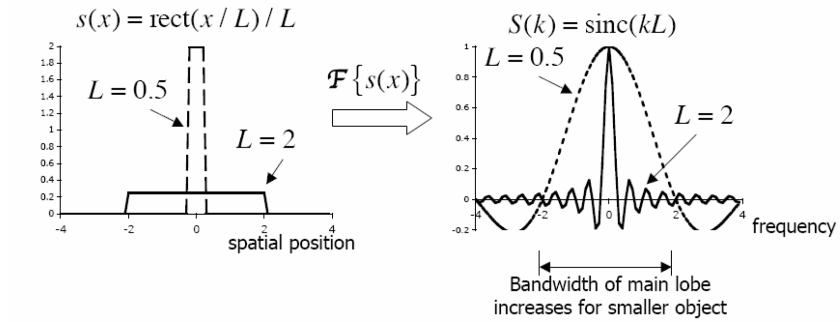
Convolution: $F\{s_1(x) * s_2(x)\} = S_1(k) \cdot S_2(k)$

$$F\{s_1(x) \cdot s_2(x)\} = S_1(k) * S_2(k)$$

Scaling Property

The rect function provides good insight into the relationship of fine detail and frequency bandwidth

- a thin rect can represent/resolve fine detail (think of a signal being represented as an array of thin rects
- a thin rect gives rise to a wide frequency lobe
- this illustrates that signals with more detail will have broader frequency spectra
- or, in turn, signals with thin frequency spectra will have low spatial resolution



Influence of Transfer Function *H*

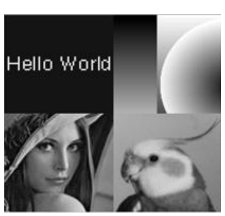
We know (from the last lecture) that:

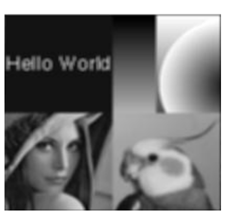
$$s_0(x) = \int_{-\infty}^{+\infty} S_i(k) e^{2\pi i k x} H(k) dk$$

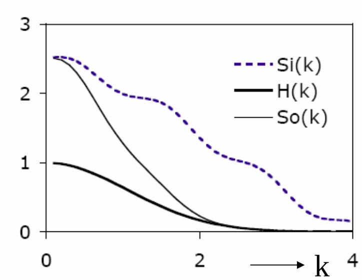
$$s_o(x) = s_i(x) * h(x) \leftrightarrow S_i(k) \cdot H(k) = S_o(k)$$

Let's look at a concrete example:

- *H* is a *lowpass (blurring) filter*: it reduces the higher frequencies of *S* more than the lower ones





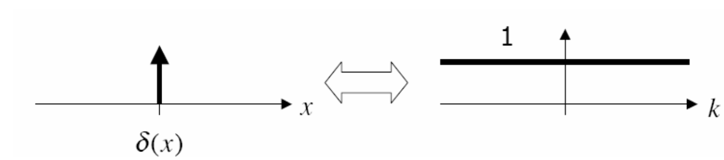


after application of *H*

Calculation: Dirac Impulse

For s(*x*)=δ(*x*):

$$S(k) = F\{\delta(x)\} = \int_{-\infty}^{+\infty} \delta(x) e^{-i2\pi kx} dx = e^{-i2\pi k0} = 1$$



Recall that the Dirac is an extremely thin rect function

- the frequency spectrum is therefore extremely broad (1 everywhere)

This illustrates a key feature of the Fourier Transform:

- the narrower the *s*(*x*), the wider the *S*(*k*)
- sharp objects need higher frequencies to represent that sharpness

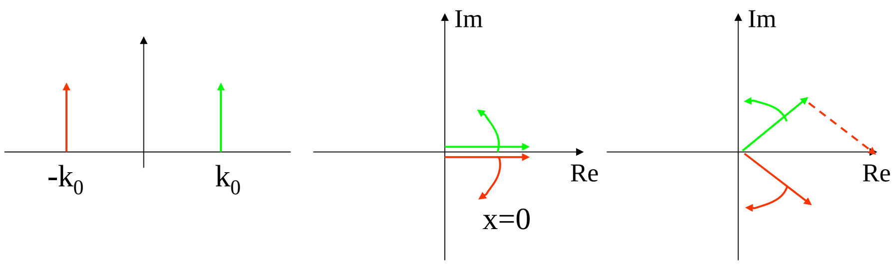
Important Fourier Pairs: Sinusoids

Sinusoids of frequency k_0 give rise to two spikes in the frequency domain at $\pm k_0$

$$\cos(2\pi k_0 x) \leftrightarrow (\delta(k+k_0)+\delta(k-k_0))/2$$
$$\sin(2\pi k_0 x) \leftrightarrow i(\delta(k+k_0)-\delta(k-k_0))/2$$

Recall the pointer analogon in the complex plane

for the cos(): the real signal is given by the addition of the two vectors (divided by 2), projected onto the real axis



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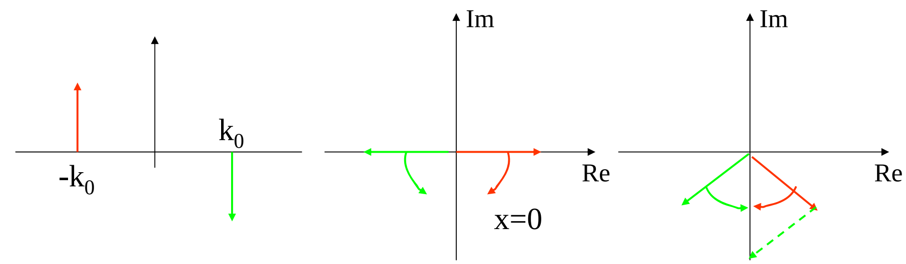
Recall the pointer analogon in the complex plane

for the sin(): the real signal is given by the addition of the two vectors (divided by 2), projected onto the imaginary axis (note the *i* in the equation)



More Important Fourier Pairs

$$\delta(x) \leftrightarrow 1$$
$$1 \leftrightarrow \delta(k)$$
$$\cos(2\pi k_0 x) \leftrightarrow (\delta(k+k_0)+\delta(k-k_0))/2$$
$$\sin(2\pi k_0 x) \leftrightarrow i(\delta(k+k_0)-\delta(k-k_0))/2$$
$$\Pi(\frac{x}{2L}) \leftrightarrow 2L\mathrm{sinc}(2\pi Lk)$$
$$\Lambda(\frac{x}{2L}) \leftrightarrow L\mathrm{sinc}^2(\pi Lk)$$
$$e^{\frac{-x^2}{2\sigma^2}} \leftrightarrow e^{-2\sigma^2 k^2}$$

the Gaussian width is inversely related

Some Notes

In the 2D transform, if *f*(*x*,*y*) is separable, that is, f(*x*,*y*)=*f*(*x*)*f*(*y*), one may write:

$$S(k,l) = F\{s(x,y)\} = \int_{-\infty}^{+\infty} s(y)e^{-2\pi ily}(\int_{-\infty}^{-\infty} s(x)e^{-2\pi ikx}dx)dy$$
$$s(x,y) = F^{-1}\{S(k,l)\} = \int_{-\infty}^{+\infty} S(l)e^{-2\pi ily}(\int_{-\infty}^{-\infty} s(k)e^{-2\pi ikx}dk)dl$$

- this comes in handy sometimes

Some Notes

Sometimes the factor $2\pi k$ is used as ω:

$$s_0(x) = \int_{-\infty}^{+\infty} S_i(\omega) e^{i\omega x} H(\omega) d\omega$$

So far, we have only discussed the continuous space with (potentially) infinite spectra and signals

- that is where it makes sense to use ω
- but in reality we deal with finite, discrete signals (here k matters)
- we shall discuss this next

Fourier Transform of Discrete Signals: DTFT

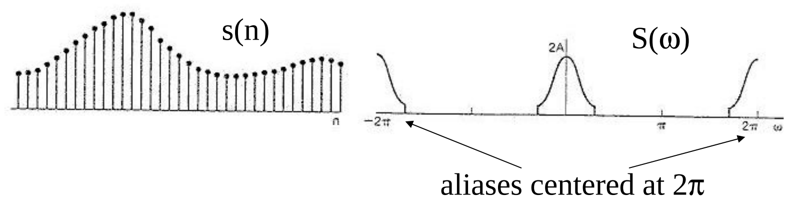
Discrete-Time Fourier Transform (DTFT)

- assumes that the signal is discrete, but infinite

$$S(\omega) = \sum_{n=-\infty}^{+\infty} s(n) e^{-i\omega n}$$

$$s(n) = \int_{-\pi}^{+\pi} S(\omega) e^{i\omega n}$$

- the frequency spectrum is <u>continuous</u>, but is <u>periodic</u> (has *aliases*)



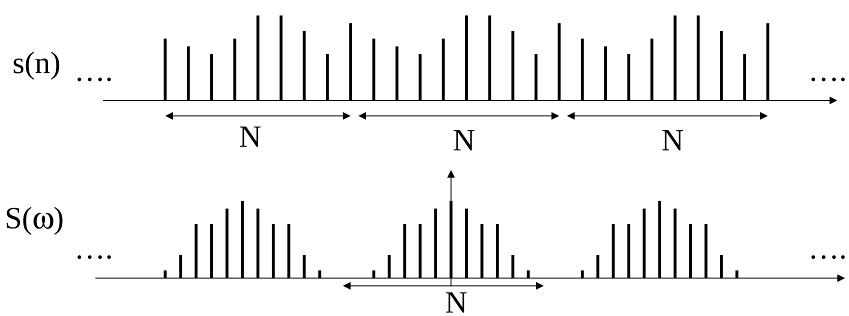
Fourier Transform of Discrete Signals: DFT

Discrete Fourier Transform (DFT)

- assumes that the signal is discrete and finite

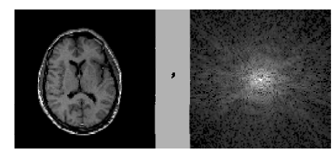
$$S(k) = \sum_{n=0}^{N-1} s(n) e^{\frac{-i2\pi kn}{N}} \qquad s(n) = \frac{1}{N} \sum_{n=0}^{N-1} S(k) e^{\frac{i2\pi kn}{N}}$$

- now we have only N samples, and we can calculate N frequencies
- the frequency spectrum is now <u>discrete</u>, and it is <u>periodic</u> in N



Fourier Transform in Higher Dimensions

The 2D transform:



$$S(k,l) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} s(n,m) e^{\frac{-i2\pi(kn+lm)}{NM}}$$

$$s(n,m) = \frac{1}{NM} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} S(k,l) e^{\frac{i2\pi(kn+lm)}{NM}}$$

Separability:

$$S(k,l) = \frac{1}{NM} \sum_{m=0}^{M-1} e^{\frac{-i2\pi lm}{M}} P(k,m) \quad \text{where } P(k,m) = \sum_{n=0}^{N-1} s(n,m) e^{\frac{-i2\pi kn}{N}}$$

$$s(n,m) = \frac{1}{NM} \sum_{l=0}^{M-1} e^{\frac{-i2\pi lm}{M}} p(n,l) \quad \text{where } \mathrm{p}(n,l) = \sum_{k=0}^{N-1} S(n,m) e^{\frac{-i2\pi kn}{N}}$$

- if M=N, complexity is $2 \cdot O(2N^3)$

Fast Fourier Transform (1)

Recursively breaks up the FT sum into odd and even terms:

$$S(k)=\sum_{n=0}^{N-1} s(n)e^{\frac{-i2\pi kn}{N}} = \sum_{n=0}^{N/2-1} s(2n)e^{\frac{-i2\pi k2n}{N}} + \sum_{n=0}^{N/2-1} s(2n+1)e^{\frac{-i2\pi k(2n+1)}{N}}$$

$$=\sum_{n=0}^{N/2-1} s_{even}(n)e^{\frac{-i2\pi kn}{N/2}} + e^{\frac{-i2\pi k}{N}} \sum_{n=0}^{N/2-1} s_{odd}(n)e^{\frac{-i2\pi kn}{N/2}}$$

Results in an O(n·log(n)) algorithm (in 1D)

- O(n^2·log(n)) for 2D (and so on)

Fast Fourier Transform (1)

Gives rise to the well-known butterfly architecture: