Introduction to Medical Imaging

Lecture 4: Fourier Theory

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Introduction

Theory developed by Joseph Fourier (1768-1830)
The Fourier transform of a signal \( s(x) \) yields its frequency spectrum \( S(k) \)

\[
S(k) = F\{s(x)\} = \int_{-\infty}^{+\infty} s(x)e^{-2\pi ikx} \, dx
\]

\[
s(x) = F^{-1}\{S(k)\} = \int_{-\infty}^{+\infty} S(k)e^{2\pi ikx} \, dk
\]

Forward transform

Inverse transform

DC (average) term

Extension to Higher Dimensions

The Fourier transform generalizes to higher dimensions
Consider the 2D case:

forward transform

\[
S(k,l) = F\{s(x,y)\} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} s(x,y)e^{-2\pi i(kx+ly)} \, dx \, dy
\]

inverse transform

\[
s(x,y) = F^{-1}\{S(k,l)\} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} S(k,l)e^{2\pi i(kx+ly)} \, dk \, dl
\]

Calculation: Rect Function

\[
S(k) = F\{A \Pi(\frac{x}{2L})\} = \int_{-\infty}^{+\infty} A \Pi(\frac{x}{2L})e^{-i2\pi kx} \, dx = \int_{-L}^{+L} A e^{-i2\pi kx} \, dx
\]

\[
= \frac{A}{2\pi ki} (e^{-i2\pi kL} - e^{i2\pi kL}) = \frac{A}{2\pi k} 2\sin(2\pi kL)
\]

\[= 2AL \text{sinc}(2\pi kL)\]

We see that a finite signal in the \( x \)-domain creates an infinite signal in the \( k \)-domain (the frequency domain)

* the same is true vice versa
Properties

- **Scaling:**
  - consider the rect (box): the greater \( L \)…
  - the higher the spectrum (factor \( AL \))
  - the narrower the spectrum (factor \( L \))
  - the scaling rule is therefore:
    \[
    F\{s(ax)\} = \left[\frac{1}{a}\right] S\left(\frac{k}{a}\right)
    \]
    \( a>1 \) shrinks \( s \)
    \( a<1 \) stretches \( s \)

- **Symmetry:**
  \( F\{S(-x)\} = S(-k) \)

- **Linearity:**
  \( F\{as_1(x) + bs_2(x)\} = F\{as_1(x)\} + F\{bs_2(x)\} \)

- **Translation:**
  \( F\{s(x - x_0)\} = S(k)e^{-2\pi i x_0 k} \)

- **Convolution:**
  \( F\{s_1(x) * s_2(x)\} = S_1(k) \cdot S_2(k) \)

  \( F\{s_1(x) \cdot s_2(x)\} = S_1(k) * S_2(k) \)

Influence of Transfer Function \( H \)

We know (from the last lecture) that:
\[
s_0(x) = \int_{-\infty}^{+\infty} S_1(k)e^{2\pi i x k}H(k)dk
\]
\[
s_0(x) = s_1(x) * h(x) \leftrightarrow S_1(k) \cdot H(k) = S_0(k)
\]

Let’s look at a concrete example:
- \( H \) is a **lowpass** (blurring) filter: it reduces the higher frequencies of \( S \) more than the lower ones

Calculation: Dirac Impulse

For \( s(x) = \delta(x) \):
\[
S(k) = F\{\delta(x)\} = \int_{-\infty}^{+\infty} \delta(x)e^{-2\pi i x k}dx = e^{-2\pi k 0} = 1
\]

Recall that the Dirac is an extremely thin rect function
- the frequency spectrum is therefore extremely broad (1 everywhere)

This illustrates a key feature of the Fourier Transform:
- the narrower the \( s(x) \), the wider the \( S(k) \)
- sharp objects need higher frequencies to represent that sharpness
Important Fourier Pairs: Sinusoids

Sinusoids of frequency $k_0$ give rise to two spikes in the frequency domain at $\pm k_0$

$$\cos(2\pi k_0 x) \leftrightarrow (\delta(k + k_0) + \delta(k - k_0))/2$$
$$\sin(2\pi k_0 x) \leftrightarrow i(\delta(k + k_0) - \delta(k - k_0))/2$$

Recall the pointer analogon in the complex plane for the $\cos()$: the real signal is given by the addition of the two vectors (divided by 2), projected onto the real axis.

Recall the pointer analogon in the complex plane for the $\sin()$: the real signal is given by the addition of the two vectors (divided by 2), projected onto the imaginary axis (note the $i$ in the equation).

More Important Fourier Pairs

$$\delta(x) \leftrightarrow 1$$
$$1 \leftrightarrow \delta(k)$$
$$\cos(2\pi k_0 x) \leftrightarrow (\delta(k + k_0) + \delta(k - k_0))/2$$
$$\sin(2\pi k_0 x) \leftrightarrow i(\delta(k + k_0) - \delta(k - k_0))/2$$

$$\Pi\left(\frac{x}{2L}\right) \leftrightarrow 2L \text{sinc}(2\pi L k)$$

$$\Lambda\left(\frac{x}{2L}\right) \leftrightarrow L \text{sinc}^2(\pi L k)$$

$$\frac{x^2}{2\sigma^2} \leftrightarrow e^{-\sigma^2 k^2}$$

the Gaussian width is inversely related

Some Notes

In the 2D transform, if $f(x,y)$ is separable, that is, $f(x,y) = f(x)f(y)$, one may write:

$$S(k, l) = F\{s(x, y)\} = \int s(y)e^{-2\pi i y l} \left( \int s(x)e^{-2\pi i k x} \, dx \right) \, dy$$

$$s(x, y) = F^{-1}\{S(k, l)\} = \int S(l)e^{-2\pi i l y} \left( \int s(k)e^{-2\pi i k x} \, dk \right) \, dl$$

• this comes in handy sometimes
**Some Notes**

Sometimes the factor $2\pi k$ is used as $\omega$:

$$s_0(x) = \int_{-\infty}^{+\infty} S_1(\omega) e^{i\omega x} H(\omega) d\omega$$

So far, we have only discussed the continuous space with (potentially) infinite spectra and signals
- that is where it makes sense to use $\omega$
- but in reality we deal with finite, discrete signals (here $k$ matters)
- we shall discuss this next

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**Fourier Transform of Discrete Signals: DTFT**

Discrete-Time Fourier Transform (DTFT)
- assumes that the signal is discrete, but infinite

$$S(\omega) = \sum_{n=-\infty}^{+\infty} s(n)e^{-i\omega n}$$

$$s(n) = \int S(\omega)e^{i\omega n}$$

- the frequency spectrum is continuous, but is periodic (has aliases)

![Discrete-Time Fourier Transform](image)

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**Fourier Transform of Discrete Signals: DFT**

Discrete Fourier Transform (DFT)
- assumes that the signal is discrete and finite

$$S(k) = \sum_{n=0}^{N-1} s(n)e^{-i\frac{2\pi kn}{N}}$$

$$s(n) = \frac{1}{N} \sum_{k=0}^{N-1} S(k)e^{i\frac{2\pi kn}{N}}$$

- now we have only $N$ samples, and we can calculate $N$ frequencies
- the frequency spectrum is now discrete, and it is periodic in $N$

![Discrete Fourier Transform](image)

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**Fourier Transform in Higher Dimensions**

The 2D transform:

$$S(k,l) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} s(n,m)e^{-i\frac{2\pi km+ln}{NM}}$$

$$s(n,m) = \frac{1}{NM} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} S(k,l)e^{i\frac{2\pi km+ln}{NM}}$$

Separability:

$$S(k,l) = \frac{1}{NM} \sum_{m=0}^{M-1} e^{-i\frac{2\pi km}{M}} P(k,m)$$

where $P(k,m) = \sum_{n=0}^{N-1} s(n,m)e^{-i\frac{2\pi km}{N}}$

$$s(n,m) = \frac{1}{NM} \sum_{l=0}^{N-1} e^{-i\frac{2\pi l}{M}} p(n,l)$$

where $p(n,l) = \sum_{k=0}^{M-1} S(n,m)e^{-i\frac{2\pi km}{M}}$

- if $M=N$, complexity is $2 \cdot O(2N^3)$
Recursively breaks up the FT sum into odd and even terms:

\[
S(k) = \sum_{n=0}^{N-1} s(n) e^{-\frac{i2\pi kn}{N}} = \sum_{n=0}^{N/2-1} s(2n) e^{-\frac{i2\pi kn}{N}} + \sum_{n=0}^{N/2-1} s(2n+1) e^{-\frac{i2\pi kn}{N}}
\]

\[
= \sum_{n=0}^{N/2-1} s_{\text{even}}(n) e^{-\frac{i2\pi kn}{N/2}} + e^{-\frac{i2\pi kn}{N}} \sum_{n=0}^{N/2-1} s_{\text{odd}}(n) e^{-\frac{i2\pi kn}{N/2}}
\]

Results in an O(n·log(n)) algorithm (in 1D)

- O(n²·log(n)) for 2D (and so on)