Chapter 4.
Improving Triangulations with Curvature Equalization

Both hierarchical triangulations described in chapter 3 can produce excellent surface approximations, given the right initial triangulation. Yet a poor initial triangulation can throw these methods off, introducing artifacts that persist through all levels of detail. For example, the triangulation may contain more triangles than are needed, which would increase storage space and time required to render the surface. It may also contain too many slivery triangles — characterized by at least one very acute angle — which cause artifacts in the display and anomalies in some analysis functions like finite element analysis. Refinement techniques often introduce artifacts at the coarser levels of detail because critical features are not always evident at those levels. Instead, several refinement iterations are required before these features are revealed. The work described here solves this problem of finding a good initial triangulation.

4.1. Foundations

Intuitively speaking, a good triangulation will approximate a surface with a few large triangles over smooth areas, and many small triangles over rough areas. I hypothesize that this may be achieved by equalizing the curvature under each triangle.

Consider, for example, the triangulation of a paraboloid which has equal curvature everywhere. The best triangulation is achieved with a mesh of triangles with equal area; the placement of the vertices is unimportant. This is demonstrated by showing that for an equilateral triangle approximating the surface of the paraboloid — with each vertex on the surface of the paraboloid — the point of greatest error is in the center of the triangle, regardless of where the triangle is. Furthermore, the error at that point is $h^2/3$, where h is the length of each side of the equilateral triangle.

To check the consistency of this idea with the strategy of my adaptive hierarchical triangulation, I re-ran adaptive hierarchical triangulation on the eight test cases described earlier and measured the curvature of triangles that required refinement versus those that didn't. On average, triangles requiring refinement had curvature values three times greater than those that didn't.

Then question then is how to equalize the curvatures in a triangulated model of a surface? This section provides some of the background for my search for an answer.

4.1.1. Polygonal curve approximations

An analogy to this problem may be found in the one-dimensional case where a curve is approximated with a series of straight line segments, as shown in figure 4.1. Starting with a line segment connecting the two endpoints of the curve (a), an approximation may be produced by successively splitting line segments at points of greatest error (b) until the line segments fit the curve within a given tolerance (c). Although the resulting approximation is error-free, it contains more points than necessary. In the one-dimensional case, this problem is solved by merging line segments (d). As shown, the two middle segments are merged so that the curvature of each interval represented by a line segment is roughly the same. Pavlidis' text [Pav77] describes this split-and-merge technique in detail.

The literature on polygonal approximations is extensive [Pav77, McC75, SK76, Mon70, T74, DH73]. Some of these methods are based on the result that in an optimal polygonal approximation vertices are placed so that an integral of the curvature takes the same value over all intervals [McC75]. In contrast, the literature on triangulated approximations is comparatively limited [MS88, Nad86, DLR90]. Extending polygonal approximation techniques to triangulations is difficult because there is no direct counterpart of merging triangles for the two-dimensional case.

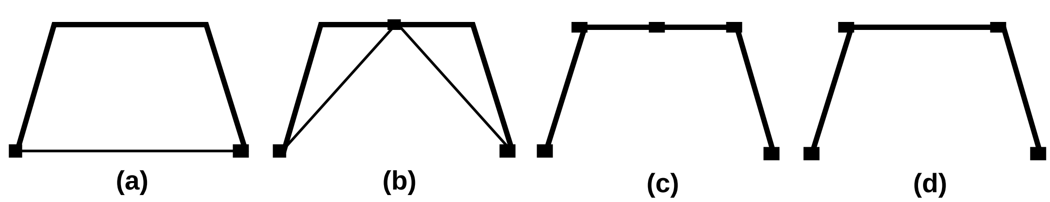
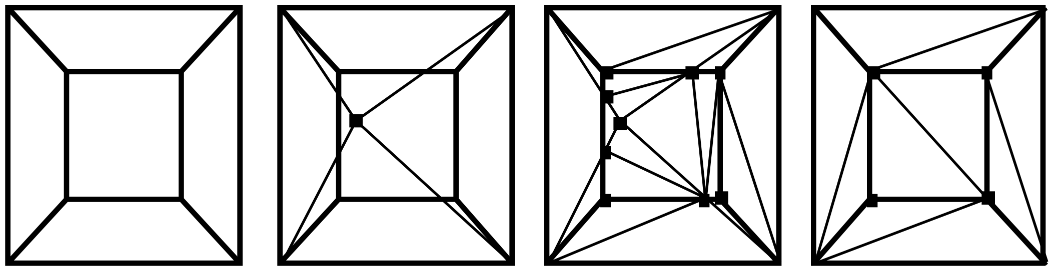


Figure 4.1. Improving linear approximations with a slit-and-merge technique



(a) (b) (c) (d)

Figure 4.2. Split-and-merge analogy for a surface

Consider, for example, the solid in Figure 4.2 which has a square base and a smaller square top. An initial triangulation may be created by connecting the four base points to a point of greatest error found anywhere on the top plane (a). Using such an initial approximation can produce more triangles than necessary (b). Instead, I would like to produce an optimal triangulation with a minimal number of triangles (c).

4.1.2. Estimating curvature

In a recent paper [MS88], McClure and Shwartz discuss methods of surface reconstruction based on triangulations. Although their focus is on image data compression, they view this as being analogous to the problem of defining concise and accurate approximations for surfaces. One of the questions that they explore is how fine a regular triangulation needs to be in order to provide a consistent representation of an underlying surface. To determine this level of detail, they develop an estimator for the surface which is based on the Hessian matrix of second partial derivatives $H(p_{i,j})$. On a discretely sampled surface where each point pi,j maps to a single elevation $f(x_i,y_j)$, the Hessian matrix is

$$H(p_{i,j}) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}$$

For each point $p_{i,j}$ the curvature estimator is expressed by

$$\begin{aligned} MS(p_{i,j}) &= 3(\text{trace } H(p_{i,j}))2 - 8(\det H(p_{i,j})) \\ &= 3(f_{xx}^2 + 2f_{xx}f_{yy} + f_{yy}^2) - 8(f_{xx}f_{yy} - f_{xy}^2) \\ &= 2f_{xx}^2. + 2f_{yy}^2 + (f_{xx} - f_{yy})^2 + 8f_{xy}^2 \end{aligned}$$

which is always a positive value.

There are several ways of generating this Hessian matrix. The second partial derivatives of $p_{i,j}$ may be approximated as follows:

$$f_{xx} = f(x_{i-1},y_j) + f(x_{i+1},y_j) - 2f(x_i,y_j)$$
$$f_{yy} = f(x_i,y_{j-1}) + f(x_i,y_{j+1}) - 2f(x_i,y_j)$$
$$f_{xy} = f_{yx} = f(x_{i-1},y_{j-1}) + f(x_{i+1},y_{j+1}) - f(x_{i-1},y_{j+1}) - f(x_{i+1},y_{j-1}).$$

For continuous or irregularly sampled data, these second partial derivatives can come directly from the quadratic equation for the curvature at that point, expressed as

$$ax^2 + bxy + cy^2 + dx + ey + f = z.$$

This equation may be found through Gaussian elimination (if five nearest neighbors are used) or least squares approximation (if more neighbors are used).

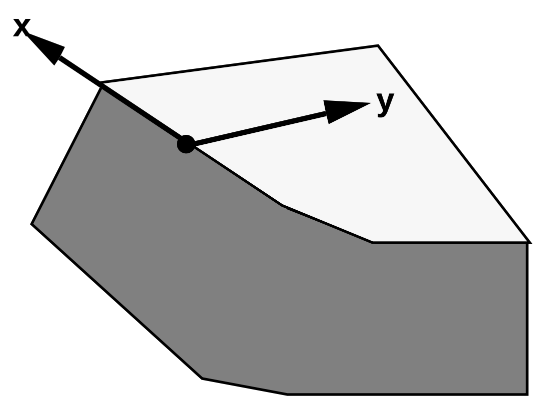


Figure 4.3. Gaussian curvature alone will not detect all critical features, such as this edge of a cliff

Although this curvature estimator is not curvature in the classical sense, I use the term loosely because this measure detects the salient features of a surface. In fact, the above measure incorporates both the mean curvature (approximated by the trace of the Hessian) and the Gaussian curvature (approximated by the determinant of the Hessian). This improves on simpler measures such as the determinant of the Hessian which fail to detect many critical features in real-world situations. Consider, for example, the edge of a cliff as shown in Figure 4.3. Although points along this edge are clearly critical to the model, the determinant of the Hessian at these points is zero.

With this curvature estimate for each point on the surface, the curvature of a triangle can be expressed as an integral of this measure over the entire triangle. When the surface is represented by a set of discrete sample points, the curvature of a triangle may be approximated by summing these measures for all points within it.

In their extensions section of [MS88], McClure and Shwartz discuss how this estimator relates to the selection of nonhomogeneous (irregular) triangles for approximating the surface. Although they provide an expression for the ideal density of distributed sample points and triangles, they do not give an actual algorithm for selecting these points and triangles.

4.2. Approach

My approach is to extend the ideas of polygonal approximation to a higher dimension. Since it is not feasible to extend the split-and-merge algorithm to triangulations, I follow the alternative strategy of moving vertices. I move these vertices — and collapse very thin pairs of triangles — so that the triangles approximate surface patches of similar curvature.

Curvature equalization starts with a triangulation that covers the surface with the required level of granularity, but is not optimal. It may, for example, contain more triangles than are needed and/or too many slivery triangles. I assume that the triangulation corresponds to a known underlying surface or bivariate function which is sampled at regular intervals, such as a digital terrain model. Error in the triangulation is measured by projecting each of these sample points to the appropriate triangle, then taking the difference between the actual and projected values. The triangulation meets error constraints if none of these errors exceeds a given tolerance value.

I attempt to improve the model by moving triangle vertices so that curvatures associated with the triangles are as nearly equivalent as possible. This must be done without introducing errors to the model. The curvature of each triangle is approximated by summing the curvature estimates for all sample points (as described above) within a triangle. Points coincident with triangle edges are not included in these sums. Indeed the goal of the algorithm is to place such boundaries over areas of high curvature, keeping triangle interiors relatively flat.

I use a three-step procedure for moving vertices. The first step shifts triangle vertices, attempting to equalize curvatures within the triangles. The second step collapses unnecessary triangles, further reducing the number of triangles in the model. The third step switches edge directions across pairs of triangles to reduce sliveriness without introducing errors. All three algorithms are described in greater detail below.

4.2.1. Initial Triangulation

Because the purpose of this algorithm is to improve existing triangulations, one must first provide the initial triangulation. For gridded surface models, the initial triangulation is simply generated for a subsampled set of points.

For sparse irregular surface models, such as those provided by Michigan State University [LS91], care must be taken that the initial triangulation only covers valid data areas. To accomplish this, I first find the convex hull of the data. This is used to generate the α-shape of the data set [Ede87][1] with an operator-specified α. A set of interior points that have high curvature are also selected for the triangulation. To ensure that these interior points are evenly distributed over the surface, the surface is partitioned into subregions. One point of highest curvature in then selected in each subregion. The MSU data also requires an adaptive grid for the surface as described in chapter 3. Once these points have been found, the results are triangulated with an inward spiral TIN generation algorithm [McK87].

For full three-dimensional data, the initial triangulation may be generated from a subsampling of the model produced using Hoppe et al.'s surface reconstruction algorithm [HDDMS91]. This algorithm finds neighborhoods and surface tangents for each data point, which are used to approximate curvature and generate a regular initial triangulation. Once point-triangle relationships have been established, any triangulation may be processed by curvature equalization.

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1. The α-shape captures the general shape of a set of points, separating "foreground" from "background". The parameter α determines the level of detail captured in this shape.

4.2.2. Equalizing curvature

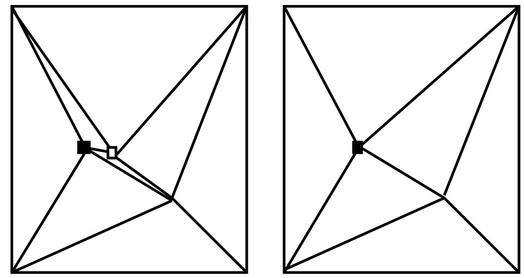

Figure 4.4. Collapsing very thin triangles

The algorithm of the first step attempts to equalize curvature by iteratively reducing the size of the triangles with greatest curvature. This size reduction is achieved by moving each vertex of the triangle inward, one at a time. A vertex may conceivably move anywhere within the triangle, although preference is given to points of highest curvature. This movement is further restricted to an area that is visible to all of the vertex's adjacencies. If moving a vertex makes the triangle very thin and slivery then I collapse the triangle and one of its neighbors, effectively removing them both. This is done by merging two of its vertices as shown in Figure 4.4. Points along the boundary of the domain of the triangulation may only move along that boundary, and that points defining the corners of the boundary may not move at all.

After determining where the vertex moves to, I recalculate the curvatures of the affected triangles to see if the overall curvature of the surface is more equalized. Because points on the triangle edges do not contribute to this overall curvature measure, positioning triangle edges over critical edges on the surface will reduce overall curvature for the surface. If the resulting triangulation is no worse than the previous one, then the triangulation is updated to reflect this move. Otherwise, I try moving the vertex to another point in the high-curvature triangle. Eventually, either the vertex will move, or all points will have been tried to no avail, causing the algorithm to skip to the next triangle vertex.

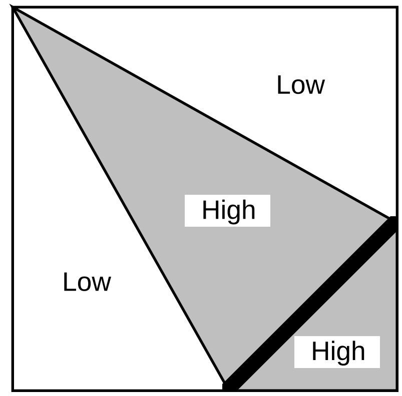

Figure 4.5. Curvature of a surface model cannot always be truly equalized

There are several reasons why triangle curvatures may never be completely equal. One factor is the discrete nature of the data. Vertices may only move to known points on the underlying surface. Another factor is the finite bounds on the triangulated area. Consider, for example, the ridge in Figure 4.5, indicated by the dark line. Because points of high curvature occur only along this ridge, curvature cannot be equalized in any triangulation in which this ridge corresponds to a triangle edge. To prevent cycling — moving a vertex back and forth between two points on the surface — I keep a record of all attempted moves. Any move that has been tried before is automatically rejected, and the algorithm proceeds to the next vertex. If no vertices may move in the triangle of highest curvature, the pointer is incremented to a new triangle with maximum curvature.

Curvature equalization continues to try moving vertices until the triangle with “maximum” curvature has curvature equal to the smallest curvature in the triangulation. If it is unable to move any vertices on the triangle of highest curvature, it proceeds to the triangle of next-highest curvature and considers that as the triangle of “maximum” curvature. Because the algorithm skips over triangles in this manner — and doesn’t allow repeated moves — either the curvature within the triangulation will become equal, or the “maximum” triangle will eventually be the same as the “minimum”, and the algorithm will halt.

This algorithm for this first step is summarized in the pseudo-code of figure 4.6.

```
procedure equalize_curvatures (Point, Triangle, Member, Point_Curvature,
Triangle_Curvature )
Input :   An initial triangulation Triangle that references a Point  set defining
          the surface, with a Point_Curvature  measure and 0..2 triangle
          Members  for each p      Point , and a summed Triangle_Curvature
          for each t     Triangle.
Output :  Updated Triangle, Triangle_Curvature , and Members .
begin
    sort triangles on Triangle_Curvature  in descending order;
    max_tri  = triangle at head of the sorted list;
    min_tri  = triangle at tail of the sorted list;
    while Triangle_Curvature[max_tri] >Triangle_Curvature[min_tri]  do begin
        for each current vertex v_i  of Triangle[max_tri]  do begin
            generate queue, in order of descending Point_Curvature,
                of all points p  such that Triangle[max_tri]      Members[p]
                and p  is visible from all points adjacent to v_i  ;
            repeat
                get candidate vertex point p  from queue;
                calculate new Triangle_Curvature  values if v_i  moves to p ;
            until  new triangulation is better or queue is empty;
            if new triangulation is better then begin (* update triangulation *)
                remove triangles that are too skinny;
                split neighbors of skinny triangles as necessary;
                update variables: Triangle, Triangle_Curvature , Members ;
                update sorted list of triangles;
                max_tri  = triangle at head of the sorted list;
            end;
            else if no vertices of Triangle[max_tri]  could move then
                max_tri  = next triangle on sorted list;
        end;
    end;
end;
```

Figure 4.6. Algorithm for equalizing curvature

4.2.3. Remove unnecessary triangles

The second step further improves the triangulation by removing unnecessary pairs of triangles. For example, a planar patch bounded by a rectangle may be accurately approximated by several triangles with equal curvature, as shown in Figure 4.7 (indicated by the lighter lines). However, only two triangles are actually needed (represented by the darker lines).

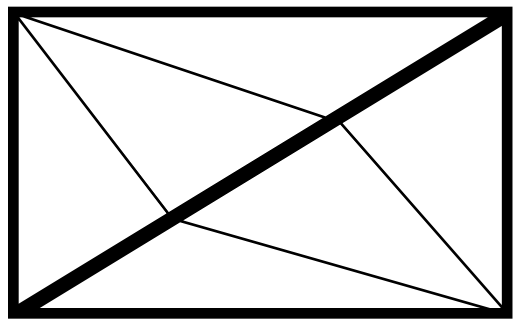

Figure 4.7. Triangulations with equalized curvature may still contain more surface patches than are necessary

The second step in curvature equalization remedies this by attempting to remove unnecessary pairs of triangles. Because triangle areas will increase, curvature may also increase. It is more important to examine the actual planar equations: it is only desirable to merge surface patches that approximate parts of the same plane.

Two triangles are removed from the triangulation by collapsing their common edge into a single point as shown in Figure 4.4. It tries to remove every pair of triangles within the triangulation, resulting in less than 3/2T iterations, where T is the number of triangles initially. Because an edge may be collapsed into either of its endpoints, the algorithm tries merging both ways and chooses the better way.

This algorithm for the second step is summarized in figure 4.8.

procedure *remove_unnecessary_triangles* (*Point, Triangle, Triangle_Curvature*)
Input : An initial triangulation *Triangle* that references a *Point* set defining
the surface, with a *Point_Curvature* measure and 0..2 triangle
Members for each *p* *Point* , and a summed *Triangle_Curvature*
for each *t* *Triangle*.
Output : Updated *Triangle*, *Triangle_Curvature* , and *Members* .
begin
for each pair of triangles t_i ,t_j *Triangle* that share edge *pq* **do**
if t_i and t_j are co-planar **then begin**
try moving point *p* to point *q* ;
try moving point *q* to point *p* ;
select the better move to make;
if this does not adversely affect neighboring triangles **then**
update triangulation;
end;
end;

Figure 4.8. Algorithm for eliminating unnecessary triangles

4.2.4. Switch edges

The third step improves the triangulation by switching the direction of an edge that splits the quadrilateral formed by two adjacent triangles. As shown in Figure 4.9, this switch from (a) to (b) can reduce sliveriness.

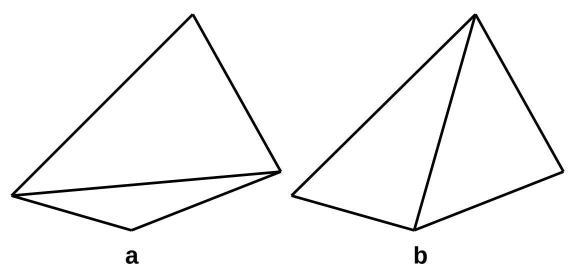

Figure 4.9. Motivation for switching edge directions: slivery triangles (a) can be made less slivery (b)

This step can also reduce error in the model. Although the previous steps allow vertices to move around and even collapse onto one another, they do not change the adjacency relationships among the vertices. As observed earlier, if the initial triangulation contains an edge that crosses the model surface the wrong way, the previous two steps will not be able to remedy this problem. For example, a triangulation of a ridge — where points and edges were selected without regard to topographic features — could contain edges that cut across the ridge rather than follow it. As shown in a recent paper by Southard [So91], this is a common problem which may be solved more directly by allowing the adjacencies to change. The triangle conditioning described in this paper inspired my algorithm for the third step.

My approach is to consider all pairs of triangles that have opposing obtuse angles. The algorithm switches the direction of the dividing edge if making such a switch produces less slivery triangles and introduces no additional error. This algorithm is summarized in figure 4.10.

---

procedure *switch_edges* (*Point, Triangle, Member, Point_Curvature, Triangle_Curvature*)

Input : An initial triangulation *Triangle* that references a *Point* set defining
    the surface, with a *Point_Curvature* measure and 0..2 triangle
    Members for each p *Point* , and a summed *Triangle_Curvature*
    for each t *Triangle*.

Output : Updated *Triangle*, *Triangle_Curvature* , and *Members* .

begin

  for each pair of triangles t_i ,t_j *Triangle* that share edge *pq* **do**

    if opposing angles φ_i and φ_j are both obtuse **then begin**

      calculate new *Triangle_Curvature* values for t_i ,t_j

        given that *pq* is replaced with a new edge splitting φ_i and φ_j ;

      if this triangulation is better **then**

        update triangulation;

    end;

end;

Figure 4.10. Algorithm for switching edge directions

4.3. Results

I first ran the three curvature equalization algorithms on simple artificial surfaces that were pre-triangulated. These surfaces — a ridge, a pyramid, and a paraboloid — are illustrated in Figure 4.11 along with the results of running my algorithms on them. In this figure, column A shows the original triangulations. Column B shows the results after equalizing curvature in the first step of my procedure. Column C shows the results after removing unnecessary triangles in the second step of my procedure. Column D shows the results after edge-switching. In these diagrams darkened lines represent ridges and highlighted points move in the next step. As shown, the three curvature equalization steps produced optimal triangulations in all cases, although some figures require more of the steps than others. This demonstrates that my methods work well for curved as well as polyhedral surfaces.

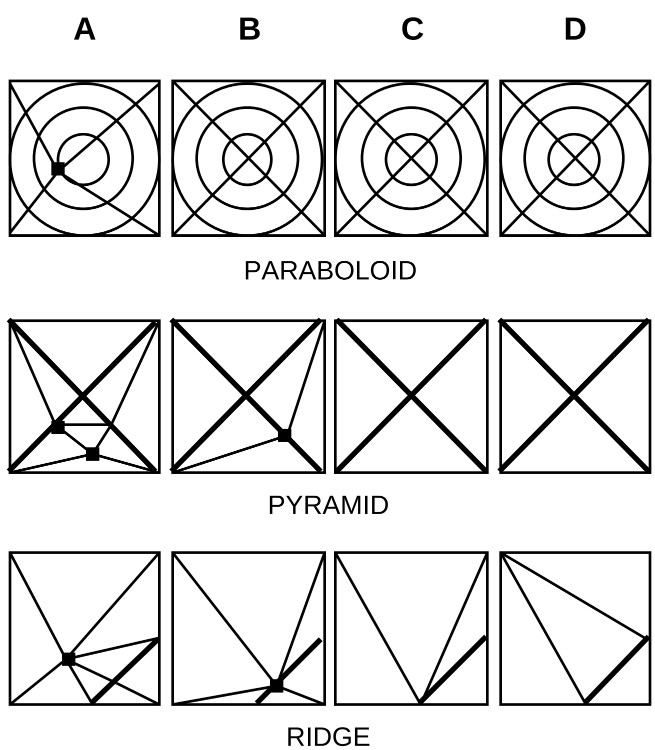

Figure 4.11. Curvature equalization applied to artificial test cases

4.3.1. Applying curvature equalization to terrain models

Having demonstrated the effectiveness of curvature equalization on artificial surfaces, I conducted a number of experiments with real terrain data as described below. In each experiment I consider three factors: numeric accuracy of the model, number of triangles in the model, and sliveriness of the triangles as described in previous chapters.

4.3.1.1. Improving Triangulations that Consider Surface Topology

My first experiment was to apply curvature equalization to the levels of detail produced by my adaptive hierarchical triangulation described in chapter 3.2. Overall, the results were disappointing. Curvature equalization made little or no improvement to these surface models. This is because the adaptive hierarchical triangulation accomplishes similar results by considering surface topology. As mentioned earlier, adaptive hierarchical triangulation typically splits triangles of high curvature anyway.

The next experiment was to use curvature equalization to simply improve the initial triangulation for the adaptive hierarchical triangulation. I hypothesized that such an initial triangulation would produce better, more consistent hierarchical triangulations.

To test my hypothesis I used the same eight test areas used to test adaptive hierarchical triangulation. These represent a variety of terrain types, from flat plains to rolling hills to jagged mountains. I generated three initial triangulations for each of the test areas. The first initial triangulation simply split the square area of interest (AOI) in half along the diagonal. The second initial triangulation found the point farthest from the surface of a bilinear patch defined by the four AOI corners, and split the area into four triangles meeting at that point. The third initial triangulation was created by applying curvature equalization to a regular triangulation with 16 triangles.

AOI	Split in Half	Bilinear Split	Curvature Equalized
1	5.086	5.087	**2.871**
2	9.645	9.543	**3.202**
3	5.934	4.646	**2.971**
4	11.315	5.497	**3.408**
5	14.843	10.576	**3.432**
6	5.305	7.691	**3.028**
7	4.352	4.921	**3.407**
8	5.949	6.958	**3.060**

Table 4.1. Effects of initial triangulation on sliveriness

I then ran adaptive hierarchical triangulation on the 24 initial triangulations, producing 5 levels of detail with an error tolerance of 10 meters at the highest level of detail. Tables 4.1 and 4.2 show my results with the best results shown in bold face for each test case. Table 4.1 shows the average sliveriness over all levels of detail in the

AOI	Split in Half	Bilinear Split	Curvature Equalized
1	1852	1852	**1834**
2	2330	**2260**	2282
3	1353	**1285**	1308
4	1123	1015	**953**
5	3072	3100	**2990**
6	1998	2096	**1933**
7	2745	2701	**2643**
8	4418	4489	**4390**

Table 4.2. Effects of initial triangulation on number of triangles in finest level of detail

model, where sliveriness has been normalized so that an equilateral triangle has a value of 1. In all cases the curvature equalized initial triangulation produced the least slivery results. Table 4.2 shows the number of triangles in the highest level of detail. In most cases the curvature equalized initial triangulation required fewer triangles to achieve the specified level of accuracy, although the differences are almost negligible in this case.

Figure 4.12 provides visual verification of the results for area of interest (AOI) 6, a region in Nevada. The original digital elevation model (a) contains

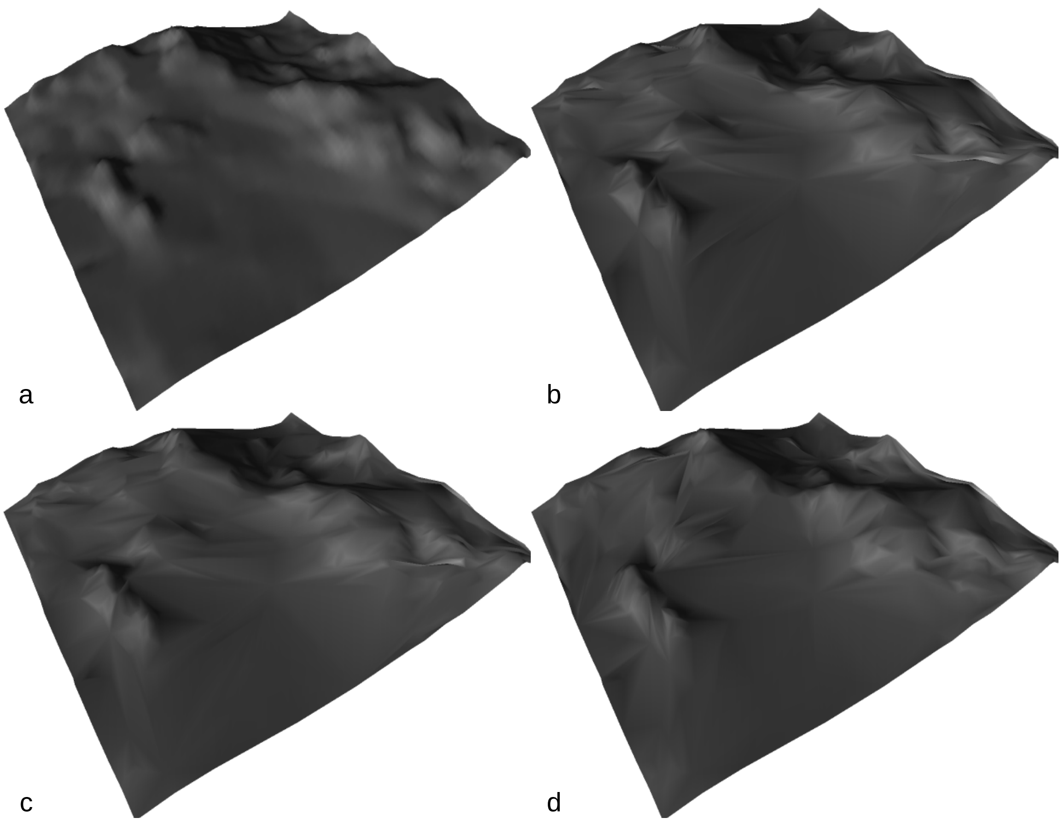

Figure 4.12. AOI 6 modeled with (a) original digital elevation model, (b) adaptive hierarchical triangulation developed from AOI split in half, (c) curvature equalization applied to (b), and (d) adaptive hierarchical triangulation developed from curvature equalized initial triangulation

10,952 triangles. The highest level of detail from an adaptive hierarchical triangulation that initially split the area in half (b) contains 1979 triangles. Here an artificial ridge imposed by the initial triangulation is clearly visible in the right corner. Applying curvature equalization to this final level of detail (c) smoothed over this ridge somewhat. Yet the best results are the 1933 triangles produced from an initial triangulation that was curvature equalized (d).

Figure 4.13 shows similar results when applied to AOI 1, a region in Alabama (a). Notice how the severe artifact introduced by the initial half-split (b)

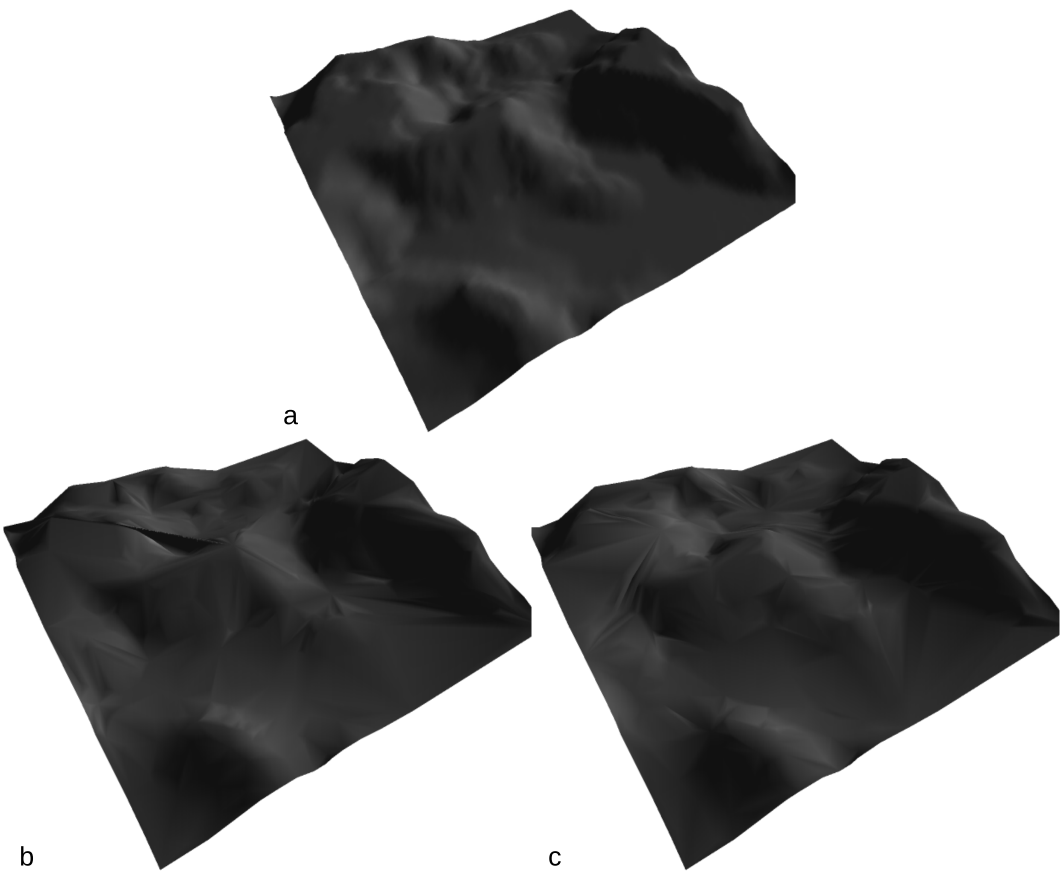

Figure 4.13. AOI 1 modeled with (a) original digital elevation model, (b) adaptive hierarchical triangulation developed from AOI split in half, and (c) adaptive hierarchical triangulation developed from curvature equalized initial triangulation

is nowhere evident in the model produced from a curvature-equalized initial triangulation (c).

4.3.1.2. Improving Triangulations that Disregard Surface Topology

In my next experiment I explored the results of applying curvature equalization to regular triangulations. Many real- and near-real-time applications such as flight simulators reduce the number of triangles by producing coarser triangulations from a subsampled grid. Although this has the advantage of simplicity, vertices selected this way are not necessarily important topologically. Likewise, discarded points are not necessarily unimportant. I hypothesized that curvature equalization could improve these regular triangulations.

	Subsampled Every 9th Point Curvature Equalization Reduces		**Subsampled Every 4th Point Curvature Equalization Reduces**	
AOI	**# of Triangles**	**Error**	**# of Triangles**	**Error**
1	32.0%	12.4%	21.9%	0.7%
2	32.0%	7.0%	2.8%	5.3%
3	63.3%	22.7%	7.4%	5.1%
4	44.5%	9.0%	38.1%	3.1%
5	17.2%	10.3%	15.1%	2.8%
6	43.0%	6.5%	4.2%	10.3%
7	44.5%	0.0%	0.3%	0.3%
8	41.4%	21.5%	19.9%	0.0%

Table 4.3. Effects of equalizing curvature of subsampled digital elevation models

I used the same eight AOIs mentioned previously to test my hypothesis. Table 4.3 shows the results of equalizing curvature of triangulations produced from two subsamplings. Taking every 4th point reduced each model from 10,952 to 648 triangles. Likewise, taking every 9th point reduced the model to 128 triangles. The columns in the table show, in both cases, how curvature equalization was able to further reduce both the number of triangles and numeric error in the model. Results are more significant for the larger subsampling rate because sparser samples are more likely to miss the critical features of the surface.

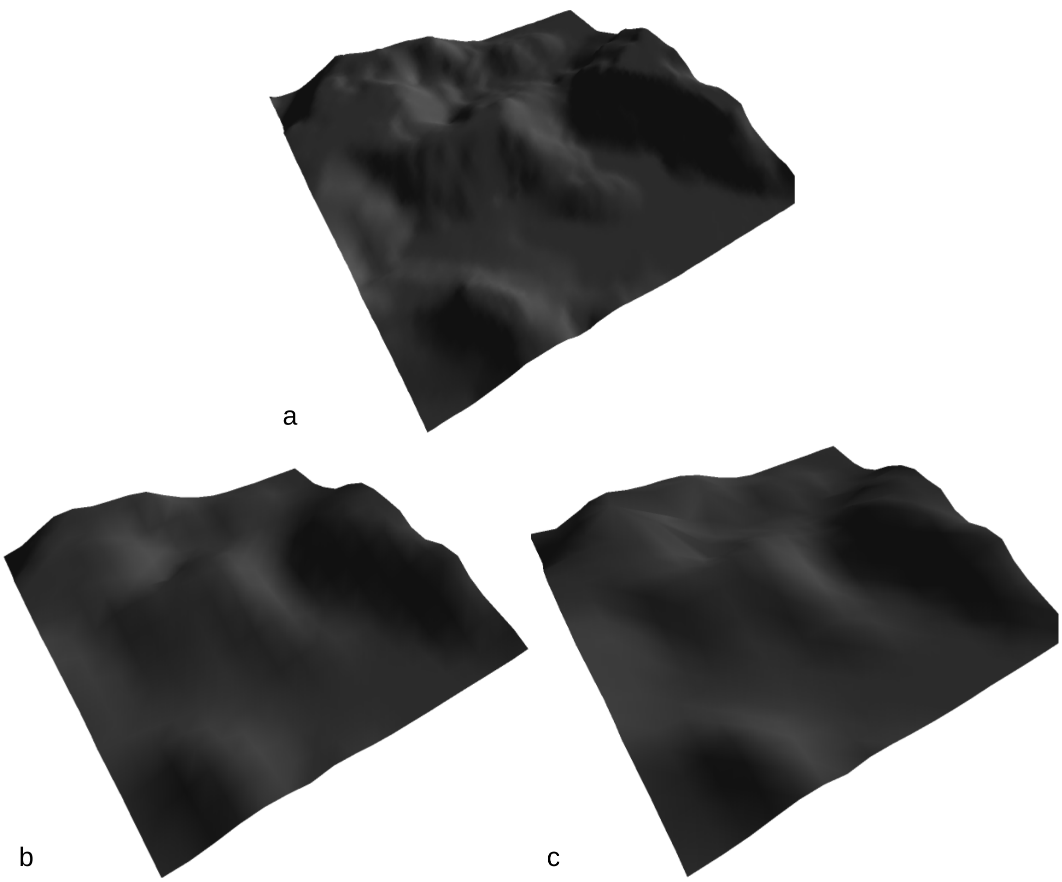

Figure 4.14. AOI 1 modeled with (a) original digital elevation model, (b) subsampled regular grid, and (c) curvature equalized subsampled grid

Figures 4.14 through 4.16 show the results of curvature equalization in three cases. Each figure shows a) the original data, b) the data subsampled at every fourth point, and c) the result of applying curvature equalization to the subsampled model. Figure 4.14 shows the effects of this on AOI 1 where curvature equalization reduced the number of triangles by 21.9% and the error by 0.7%. Figure 4.15 shows AOI 2 where curvature equalization reduced the number of triangles by 2.8% and the error by 5.3%. Figure 4.16 shows AOI 6 where curvature equalization reduced the number of triangles by 4.2% and the error by 10.3%.

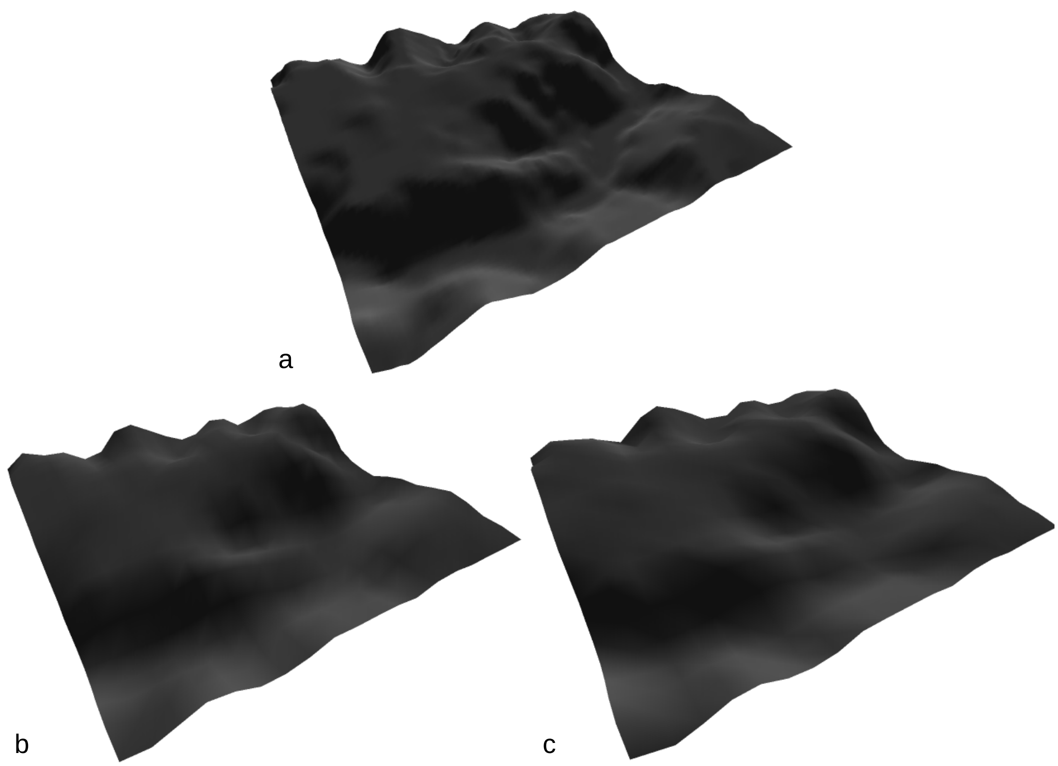

Figure 4.15. AOI 2 modeled with (a) original digital elevation model, (b) subsampled regular grid, and (c) curvature equalized subsampled grid

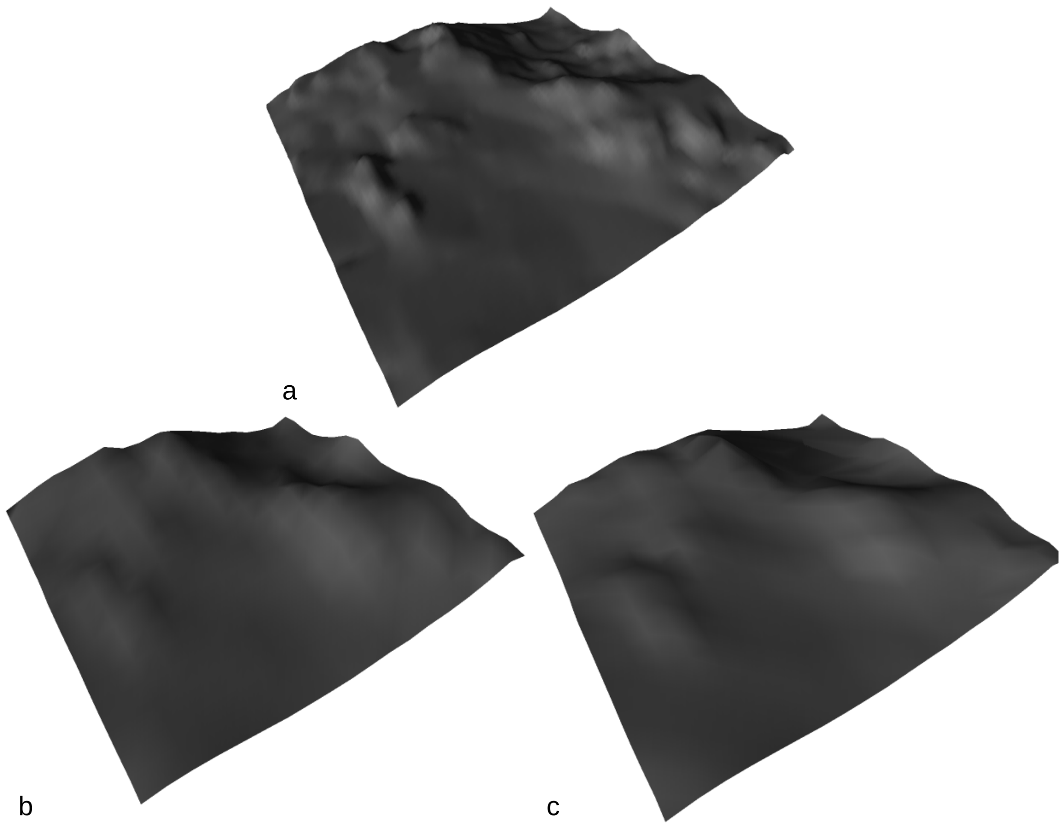

Figure 4.16. AOI 6 modeled with (a) original digital elevation model, (b) subsampled regular grid, and (c) curvature equalized subsampled grid

4.3.2. Applying curvature equalization to range data

With curvature equalization working well on terrain data, I ventured to try the method on range images of simple geometric objects. The purpose of this experiment was to demonstrate the applicability of curvature equalization to computer aided design (CAD). The source of this data is the Pattern Recognition and Image Processing Laboratory at Michigan State University where a set of test range images have been produced using the Technical Arts scanner [LS91].

Differences between this range data and the terrain data are numerous. First, the range data represent points scattered over the surface. Large areas where the data is invalid (i.e. not on the object surface) are blank. This necessitates the use of an artificial grid to organize the data, as described in section 3.2, and a triangulation of the α-shape of the data, as mentioned in 4.2. The second difference is the data type. The floating point values of the range data introduce precision errors that were not evident with the integer elevation values. To overcome this problem, I've re-introduced the idea of nearness used in the adaptive hierarchical triangulation: a point near an edge is treated as if it is on the edge. A point is considered near an edge if the edge passes through the grid cell containing that point.

Yet the greatest difference of all is between the surfaces represented. Terrain generally forms a continuous surface with few sharp discontinuities. Hence the approximations described earlier for the second partial derivatives produce reasonable values for calculating the Gaussian and mean curvature at each point. Geometric objects, on the other hand, are often characterized by sharp discontinuities. Because curvature is not defined on such edges, common estimation methods are prone to error.

I ran curvature equalization on three of these geometric objects with disappointing results. Although the algorithm did equalize curvature, it did not move the vertices to critical points and edges to critical lines on the surface as I had expected. The desired results were finally achieved by first running adaptive hierarchical triangulation on the initial triangulation to add vertices and edges at the critical junctures. Even then several iterations of the three curvature equalization steps were required before the optimal solution was achieved.

0	h	0		h	0	0
0	h	0		0	h	0
0	h	0		0	0	h

Figure 4.17. Window over a ridge of height *h* at two different orientations

Closer inspection revealed that unexpected curvature values are the culprit. On a square-based pyramid with ridges along the diagonals, points on the ridges have greater curvature values than the peak. Yet if the pyramid is rotated 45° about the z axis, the peak will have the greatest curvature value. This phenomenon is illustrated in figure 4.17 which shows two windows over a ridge of height *h*. Using the simple estimate of the second partial derivatives described earlier, the trace of the Hessian matrix is -2*h* for the window on the left and -4*h* for the window on the right. This indicates that large errors are creeping into the mean curvature which is supposed to be rotation invariant.

The trouble lies with the estimation of the second partial derivatives. Unlike terrain, the pyramid has sharp discontinuities that foul up this estimator. Using numerical differentiation — adding correction terms using data beyond the immediate locale of a point [DB74] — reduces the difference in the trace values, but does not eliminate it.

Naturally I was disappointed that the extension to a new data type did not work immediately. Yet I have not yet exhausted the realm of possibilities for curvature equalization. Chapter 7 lists future extensions to this work which, I believe, will render it applicable to the geometric data typical to computer aided design (CAD).