Inheritance in Rule-Based Frame Systems: Semantics and Inference

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Abstract. Knowledge representation languages that combine rules with object-oriented features akin to frame systems have recently attracted a lot of research interest, and F-logic is widely seen as a basis to achieve this integration. In this paper we extend the original F-logic formalism with an array of salient features that are essential for representing and reasoning with commonsense knowledge. In particular, we extend the syntax and semantics of F-logic to incorporate nonmonotonic multiple inheritance of class and instance methods in the presentence of class hierarchies defined via rules. The new semantics is completely model-theoretic and is free of the defects that caused the original F-logic to produce unintuitive results due to the unusual interaction between default inheritance and inference via rules. Moreover, we provide a computational framework for the new F-logic semantics which can be implemented by inference engines using either forward or backward chaining mechanisms.

1 Introduction

With computer systems getting more powerful and once esoteric information management problems becoming commonplace, attention is again shifting to knowledge representation languages that combine rules with object-oriented features akin to frame systems. Recently, W3C created a new working group, which is chartered with producing a recommendation for a standardized rule language that could serve as an interchange format for various rule-based systems [38]. According to the charter, the future language will support features inspired by object-oriented and frame-based languages.

As a prominent formalism in applications where both rules and frame-based representation are highly desired, F-logic has found its success in many areas, including Semantic Web [9,10,3,8], intelligent networking [24], software engineering [17,13], and industrial knowledge management [2,39]. F-logic based systems are available both commercially [33] and from the academia [14,46,40,30]. These systems were built for different purposes and offer different degrees of completeness with respect to the original specification.

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One major technical difficulty in this field is inheritance semantics, especially the issues related to overriding and conflict resolution [22,45]. A recent study [29] shows that inheritance — especially multiple inheritance — permeates RDF schemas developed by various communities over the past few years. Multiple inheritance is therefore likely to arise in Semantic Web applications as they grow in complexity and as rule engines start playing a more prominent role in such applications. However, current Semantic Web standards do not support multiple inheritance. Some of the F-logic based systems mentioned earlier do not support it either. Support provided by other systems is either incomplete or problematic in various ways.

The difficulty in defining a semantics for inheritance is due to the intricate interaction between inference by default inheritance and inference via rules. We illustrate this problem in Section 3. Although this problem was known at the time of the original publication on F-logic [22], no satisfactory solution was found then. Subsequent works either tried to rationalize the original solution or to impose unreasonable restrictions on the language [32,19,31]. We discuss these limitations of the related work in Section 11.

Our earlier work [45] proposed a solution to the above problem by developing a semantics that is both theoretically sound and computationally feasible. However, this semantics (like the one in [22] and most other related works) is restricted to the so called *class methods* [36] (or *static methods* in Java terminology) and to a particular type of inheritance, known as *value inheritance*, which is more common in Artificial Intelligence. The notion of *instance methods* — a much more important object-oriented modeling tool — was not supported in the language or its semantics. In this paper we extend F-logic to include instance methods and a new kind of inheritance, called *code inheritance*, which is analogous to inheritance used in programming languages like C^{++} and Java (and is different from inheritance typically found in AI systems).

Of course, neither instance method nor code inheritance is new by itself. Our contribution is in porting these notions to a logic-based language and the development of a complete model theory and inference procedure for this new class of methods and inheritance. Furthermore, these concepts are defined for vastly more general frameworks than what is found in programming languages or in the literature on logic-based inheritance. This includes systems with class hierarchies defined via rules (*intensional* class hierarchies), multiple inheritance with overriding, deductive systems with inheritance, both instance and class methods, and both value and code inheritance.

This paper is organized as follows. Section 2 introduces the basic F-logic syntax that is used throughout the paper. Section 3 motivates the research problems concerning inheritance and rules by presenting several motivating examples. The new three-valued semantics for F-logic is introduced in Section 4. Section 5 defines inheritance postulates, which bridge the formal semantics and its "real world" interpretation, and Section 6 formalizes the associated notion of object models. The computational framework is presented in Section 7. Section 8 introduces the notion of stable object models. Section 9 further develops the cautious object model semantics and discusses its properties. It is shown that every F-logic knowledge base has a unique cautious object model. Implementation of the cautious object model semantics and its computational complexity is described in Section 10. This implementation can be realized using any deductive engine that supports the well-founded semantics for negation [15] and therefore can be done using either forward or backward chaining mechanisms. Related work is discussed in Section 11 and Section 12 concludes the paper. Since some of the proofs are rather subtle and lengthy, we relegate them to the Appendix in the hope that this will help the reader focus on the main story line. Shorter proofs appear directly in the main text.

2 Preliminaries

F-logic provides frame-based syntax and semantics. It treats instances, classes, properties, and methods as objects in a uniform way. For instance, in one context, the object ostrich can be viewed as a class by itself (with members such as tweety and fred); in a different context, this object can be a member of another class (e.g., species). Whether an object functions as an instance or a class depends on its syntactic position in a logical statement. F-logic does not require instances and classes to be disjoint.[1]

To focus the discussion, we will use a subset of the F-logic syntax and include only three kinds of *atomic* formulas. A formula of the form $o : c$ says that object o is a member of class c; $s :: c$ says that class s is a (not necessarily immediate) subclass of class c; and $e[m \rightarrow v]$ says that object e has an *inheritable* method, m, whose result is a *set* that contains object v.[2] The symbols o, c, s, e, m, and v here are the usual *first-order* terms.[3]

Traditional object-oriented languages distinguish between two different kinds of methods: *instance methods* and *class methods* (also known as "static" methods in Java). The former apply to all instances of a class while the latter to classes themselves. In object-oriented data modeling, especially in the case of semistructured objects, it is useful to be able to define *object methods*, which are explicitly attached to individual objects. These explicitly attached methods override the methods inherited from superclasses. Object methods are similar to class methods except that they are not intended to be inherited. In F-logic both instance and class/object methods are specified using rules.

Let A be an atom. A literal of the form A is called a *positive* literal and $\neg$A is called a *negative* literal. An F-logic *knowledge base* (abbr. KB) is a finite

[1] The same idea is adopted in RDF and OWL-Full.

[2] The syntax for inheritable methods in [22] and in systems like $\mathcal{F}$LORA-2 is $e[m \star\!\!\rightarrow v]$, while atoms of the form $e[m \rightarrow v]$ are used for *noninheritable* methods. However, noninheritable methods are of no interest here, so we opted for a simpler notation.

[3] Recall that a first-order term is a constant, a variable, or a structure of the form $f(t_1, \ldots, t_n)$, where f is an n-ary function symbol and $t_1, \ldots, t_n$ are first-order terms.

set of rules where all variables are *universally* quantified at the front of a rule. There are two kinds of rules: *regular rules* and *template rules*. Regular rules were introduced in the original F-logic [22] while the concept of template rules is one of the new contributions of this paper. Generally, regular rules define class membership, subclass relationship, and class/object methods. Template rules represent pieces of *code* that define instance methods.

A regular rule has the form $\mathsf{H} :- \mathsf{L}_1, \ldots, \mathsf{L}_\mathsf{n}$, where $\mathsf{n} \geq 0$, H is a positive literal, called the rule *head*, and each L_i is either a positive or a negative literal. The conjunction of L_i's is called the rule *body*. A template rule for class c has the form $\mathsf{code(c)}$ @$\mathsf{this}[\mathsf{m} \rightarrow \mathsf{v}] :- \mathsf{L}_1, \ldots, \mathsf{L}_\mathsf{n}$. It is similar to a regular rule except that: (i) it is prefixed with the special notation $\mathsf{code(c)}$; (ii) its head must specify a method (i.e., it cannot be $\mathsf{o}:\mathsf{c}$ or $\mathsf{s}::\mathsf{c}$); and (iii) the object-position in the head literal is occupied by the *template term* @this (which can also appear in other parts of the rule). We will also assume that c is ground (i.e., variable-free) and will say that such a template rule defines instance method m for class c.

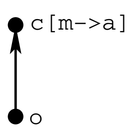
In the rest of this paper, we will use uppercase names to denote variables and lowercase names to denote constants. A rule with an empty body is called a *fact*. When a regular rule or a template rule has an empty body, we will call it a *regular fact* or a *template fact*, respectively. For facts, the symbol " :− " will be omitted.

3 Motivating Examples

We will now illustrate some of the problems that arise from unusual interaction among inference via rules, default inheritance, and intensional class hierarchies (i.e., class hierarchies that are defined using rules). In the following examples, a solid arrow from a node x to another node y indicates that x is either an instance or a subclass of y. All examples in this section are discussed informally. The formal treatment is given in Sections 4, 5, 7, and 9.

3.1 Interaction Between Default Inheritance and Rules

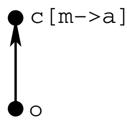
Inheritance Triggering Further Inheritance. Consider the KB in Figure 1. Without inheritance, this KB has a unique model, which consists of the first two facts. With inheritance, however, the common intuition tells us that o ought to inherit $\mathsf{m} \rightarrow \mathsf{a}$ from c. But if we only add $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{a}]$, the new set of facts would not be a model, since the last rule is no longer satisfied: with the inherited fact



o : c.
c[m → a].
c[m → b] :− o[m → a].

Fig. 1. Inheritance Leading to More Inheritance

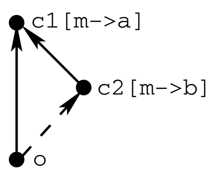
included, the least model must also contain c[m → b]. However, this begs the question as to whether o should inherit m → b from c as well. The intuition suggests that the intended model should be "stable" with respect to not only inference via rules but default inheritance as well. Therefore o[m → b] should also be in that model. This problem was recognized in [22], but the proposed solution was not stable in the above sense — it was based on plausible, ad hoc fixpoint computations rather than semantic principles.



o : c.
c[m → a].
o[m → b] :− o[m → a].

Fig. 2. Interaction between Derived and Inherited Facts

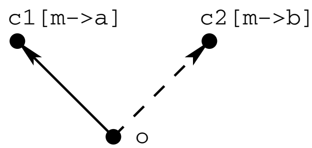
Derived vs. Inherited Information. Now consider Figure 2, which has the same KB as in Figure 1 except for the head of the last rule. Again, the intuition suggests that o[m → a] ought to be inherited, and o[m → b] be derived to make the resulting set of facts into a model in the conventional sense. This, however, leads to the following observation. The method m of o now has one value, a, which is inherited, and another, b, which is derived via a rule. Although the traditional frameworks for inheritance were developed without deduction in mind, it is clear that derived facts like o[m → b] in this example are akin to "explicit" method definitions and should be treated differently. Typically, explicit definitions should override inheritance. Thus our conclusion is that although derivation is done "after" inheritance, this derivation undermines the original reason for inheritance. Again, the framework presented in this paper, which is based on semantic principles, differs from the ad hoc computation in [22] (which keeps both derived and inherited facts).



o : c_1.
c_2 :: c_1.
c_1[m → a].
c_2[m → b].
o : c_2 :− o[m → a].

Fig. 3. Inheritance and Intensional Class Hierarchy

Intensionally Defined Class Hierarchy. Figure 3 is an example of an intensional class hierarchy. Initially, o is not known to be an instance of c_1. So, it seems that o can inherit m → a from c_1. However, this makes the fact o[m → a] true, which in turn causes o : c_2 to be derived by the last rule of the KB. Since this makes c_2 a more specific superclass of o than c_1 is, it appears that o ought to inherit $\mathsf{m} \rightarrow \mathsf{b}$ from c_2 rather than $\mathsf{m} \rightarrow \mathsf{a}$ from c_1. However, this would make the fact $\mathsf{o}:\mathsf{c}_2$ unsupported. Either way, the deductive inference enabled by the original inheritance undermines the support for the inheritance itself. Unlike [22], a logically correct solution in this case would be to leave both $\mathsf{o}:\mathsf{c}_2$ and $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{a}]$ *underdefined.* The dashed arrow from o to c_2 in Figure 3 represents the underdefinedness of $\mathsf{o}:\mathsf{c}_2$.



$\mathsf{c}_1[\mathsf{m} \rightarrow \mathsf{a}]$.
$\mathsf{c}_2[\mathsf{m} \rightarrow \mathsf{b}]$.
$\mathsf{o}:\mathsf{c}_1$.
$\mathsf{o}:\mathsf{c}_2$:− $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{a}]$.

Fig. 4. Derivation Causing Multiple Inheritance Conflicts

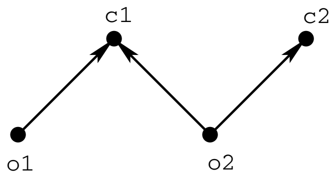
Derivation and Multiple Inheritance Conflicts. The example in Figure 4 illustrates a similar problem, but this time it occurs in the context of nonmonotonic multiple inheritance. Initially c_2 is not known to be a superclass of o. So there is no multiple inheritance conflict and the intuition suggests that o should inherit $\mathsf{m} \rightarrow \mathsf{a}$ from c_1. But then $\mathsf{o}:\mathsf{c}_2$ has to be added to the model in order to satisfy the last rule. This makes c_2 a superclass of o and introduces a multiple inheritance conflict. As in the previous example, although this conflict became apparent only after inheritance took place, it undermines the original reason for inheritance (which was based on the assumption that $\mathsf{c}_1[\mathsf{m} \rightarrow \mathsf{a}]$ is the only source of inheritance for o). Therefore, both $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{a}]$ and $\mathsf{o}:\mathsf{c}_2$ should be left underdefined. Again, this conclusion differs from [22].

3.2 Inheritance of Code

The inheritance shown in the previous examples is called *value inheritance.* It is called so because what gets inherited are the individual values that methods have in particular classes rather than the definitions of those methods.

We should note that value inheritance is *data-dependent.* Consider the example in Figure 5. At first glance, it appears that there is a multiple inheritance conflict for object o_2 with respect to method m from class c_1 and c_2. Indeed, in a traditional programming language like C^{++}, the first two rules in Figure 5 would be considered as part of the code that defines method m in class c_1 and c_2, respectively. Since o_2 is an instance of both classes, we have a multiple inheritance conflict. In contrast, value inheritance takes into account what holds in the model of the KB. Clearly, in the example of Figure 5, the premise of the first rule is true whereas the second is false. This means that the model makes $\mathsf{c}_1[\mathsf{m} \rightarrow \mathsf{a}]$ true but $\mathsf{c}_2[\mathsf{m} \rightarrow \mathsf{b}]$ false.[4] Therefore, if we only look at the *values* of method m that actually hold in the model of the KB, then no conflict exists and $\mathsf{m} \rightarrow \mathsf{a}$ can be readily inherited from c_1 by o_2 (and o_1) through value inheritance.

[4] Our claims here rely on the closed world assumption.



$c_1[m \rightarrow a] :- p[f \rightarrow d].$
$c_2[m \rightarrow b] :- p[f \rightarrow e].$
$o_1 : c_1.$
$o_2 : c_1.$
$o_2 : c_2.$
$p[f \rightarrow d].$

Fig. 5. Value Inheritance vs. Code Inheritance

Code inheritance, in contrast, behaves like in traditional programming languages and the above example would require conflict resolution. In this paper, we resolve multiple inheritance conflicts *cautiously* — whenever a conflict arises, nothing is inherited. To appreciate the difference between value and code inheritance, let us revisit the example of Figure 5 using code inheritance. Now suppose the first two regular rules in Figure 5 are replaced by the following two template rules (introduced in Section 2):

$$\mathsf{code}(c_1)\ @\mathsf{this}[m \rightarrow a] :- p[f \rightarrow d].$$
$$\mathsf{code}(c_2)\ @\mathsf{this}[m \rightarrow b] :- p[f \rightarrow e].$$

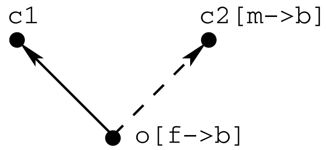
Note that template rules are prefixed with the notation $\mathsf{code}(c)$, for some class c, to indicate that they make up the code that defines the instance methods for a particular class.

We call the above rules template rules because they are not the actual rules that we require to hold true in the model of the KB. Instead, once inherited, they will be "instantiated" to the actual regular rules that are required to hold true. In the case of o_2, no code is inherited due to multiple inheritance conflict, as we just explained above. However, o_1 *can* inherit the first template rule from c_1, since there is no conflict. Inheritance of such a template rule is achieved by substituting the template term @this with o_1 in the rule. This results in a *regular rule* of the form $o_1[m \rightarrow a] :- p[f \rightarrow d]$. This rule and the fact $p[f \rightarrow d]$ together enable the derivation of a new fact, $o_1[m \rightarrow a]$.

The above example illustrates the intended use of template rules. The template term in a template rule acts as a *placeholder* for instances of that class. When the rule is inherited by an instance, the template term is replaced by that instance and the result is a regular rule. This is akin to *late binding* in traditional object-oriented languages.

The treatment of template rules should make it clear that the method m in our example above behaves like an instance method in a language like Java: the template rule does not define anything for class c_1 as an object; instead, it defines the method m for all instances of c_1. This is because template rules are not meant to be true in the model of the KB — but those regular rules resulting from code inheritance are.

The above example also alludes to the fact that value inheritance is a more "model-theoretic" notion than code inheritance, and that developing a model theory for code inheritance is not straightforward. We develop a suitable model theory in Section 6.



$$
\begin{array}{ll}
\mathsf{code}(\mathsf{c}_1) & @\mathsf{this}[\mathsf{m} \to \mathsf{a}] :- @\mathsf{this}[\mathsf{f} \to \mathsf{b}]. \\
 & \mathsf{o} : \mathsf{c}_1. \\
 & \mathsf{o}[\mathsf{f} \to \mathsf{b}]. \\
 & \mathsf{o} : \mathsf{c}_2 :- \mathsf{o}[\mathsf{m} \to \mathsf{a}]. \\
 & \mathsf{c}_2[\mathsf{m} \to \mathsf{b}].
\end{array}
$$

Fig. 6. Interaction between Template Rules and Regular Rules

Subtle interaction may arise between template rules and regular rules. To illustrate the issue, Figure 6 shows a template rule that defines instance method m for class c_1. On the surface, it seems that o should inherit this piece of code from c_1 and thus acquire the regular rule $\mathsf{o}[\mathsf{m} \to \mathsf{a}] :- \mathsf{o}[\mathsf{f} \to \mathsf{b}]$. (Recall that the template term in the rule is replaced with the inheriting instance). This inheritance seems to be possible because o is a member of c_1 and at this moment we cannot conclude that o also belongs to c_2.

A more careful look indicates, however, that there is a multiple inheritance conflict. If o inherits the above rule, then we can derive $\mathsf{o}[\mathsf{m} \to \mathsf{a}]$. But then we can also derive $\mathsf{o} : \mathsf{c}_2$ using the fourth rule in Figure 6 (which is a regular rule). Now, since $\mathsf{c}_2[\mathsf{m} \to \mathsf{b}]$ is also true, we have a multiple inheritance conflict analogous to the example of Figure 4. As in the example of Figure 4, the logically correct solution here is to leave both $\mathsf{o}[\mathsf{m} \to \mathsf{a}]$ and $\mathsf{o} : \mathsf{c}_2$ underdefined.

We thus see that template rules can interact with regular rules in subtle ways and cause inheritance to be canceled out. In other cases, such interaction might enable more inheritance. For instance, if instead of $\mathsf{c}_2[\mathsf{m} \to \mathsf{b}]$ we had $\mathsf{c}_2[\mathsf{n} \to \mathsf{b}]$, then inheritance of the template rule by o would *not* be blocked. Furthermore, o would inherit $\mathsf{n} \to \mathsf{b}$ from c_2 by value inheritance.

3.3 Observations

Nonmonotonic Inheritance. Overriding of inheritance leads to nonmonotonic reasoning, since more specific definitions take precedence over more general ones. However, overriding is not the only source of nonmonotonicity here. When an object belongs to multiple incomparable classes, inheritance conflicts can arise and their "canceling" effects can also lead to nonmonotonic inheritance.

Intensional Class Hierarchies. A class hierarchy becomes *intensional* when class membership and/or subclass relationship is defined using rules. In such cases, the inheritance hierarchy can be decided only at runtime, as complex interactions may come into play between inference via default inheritance and inference via rules. In this interaction, an earlier inference by inheritance may trigger a chain of deductions via rules which can result in violation of the assumptions that led to the original inheritance.

Value Inheritance vs. Code Inheritance. Inheritance of values is fundamentally different from inheritance of code. Value inheritance is *data-dependent* — it depends on the set of assertions in the current KB. Code inheritance is not dependent on data. Recall that in the example of Figure 5 we derived the fact $o_2[m \rightarrow a]$ via value inheritance because the premise of the second rule was false and therefore inheritance was conflict-free from the perspective of value inheritance. If we add the fact $p[f \rightarrow e]$, then the second rule will derive $c_2[m \rightarrow b]$ and create a multiple inheritance conflict. In this case, o_2 may inherit nothing.

In contrast, if we turn the rules in Figure 5 into template rules, then a multiple inheritance conflict would always exist regardless of whether the premise of either rule can be satisfied. As a result, o_2 would inherit nothing — whether $p[f \rightarrow d]$ and $p[f \rightarrow e]$ hold true or not.

4 Three-Valued Semantics

The examples in Section 3 illustrate the complex interactions between inference via default inheritance and inference via rules. These interactions cause inference to behave nonmonotonically and in many ways like default negation. This suggests that stable models [16] or well-founded models [15] could be adopted as a basis for our semantics. Since default negation is part of our language anyway, adoption of one of these two approaches is fairly natural. In this paper we base the semantics on well-founded models. Since well-founded models are three-valued and the original F-logic models were two-valued [22], we first need to define a suitable three-valued semantics for F-logic KBs. We also need to extend this semantics to accommodate template rules and to make it possible to distinguish facts derived by default inheritance from facts derived via rules.

Let P be an F-logic KB. The *Herbrand universe* of P, denoted $\mathcal{HU}_{P}$, consists of all the *ground* (i.e., variable-free) terms constructed using the function symbols and constants found in the KB. The *Herbrand instantiation* of P, denoted *ground*(P), is the set of rules obtained by consistently substituting all the terms in $\mathcal{HU}_{P}$ for all variables in every rule of P. The *Herbrand base* of P, denoted $\mathcal{HB}_{P}$, consists of the following sorts of atoms: $o : c$, $s :: c$, $s[m \rightarrow v]_{ex}$, $o[m \rightarrow v]^{c}_{val}$, and $o[m \rightarrow v]^{c}_{code}$, where o, c, s, m, and v are terms from $\mathcal{HU}_{P}$.

An atom of the form $o : c$ is intended to represent the fact that o is an instance of class c; $s :: c$ states that s is a subclass of c. An atom of the form $s[m \rightarrow v]_{ex}$ states that $m \rightarrow v$ is *explicitly defined* at s via a regular rule. Atoms of the forms $o[m \rightarrow v]^{c}_{val}$ and $o[m \rightarrow v]^{c}_{code}$, where $o \neq c$, imply that object o inherits $m \rightarrow v$ from class c by value and code inheritance, respectively.

A *three-valued interpretation* $\mathcal{I}$ of an F-logic KB P is a pair $\langle T; U \rangle$, where T and U are *disjoint* subsets of $\mathcal{HB}_{P}$. The set T contains all atoms that are *true* whereas U contains all atoms that are *underdefined.* Underdefined atoms are called this way because there is insufficient evidence to establish their truth or falsehood. The set F of the *false* atoms in $\mathcal{I}$ is defined as $F = \mathcal{HB}_{P} - (T \cup U)$. It is easy to see that the usual two-valued interpretations are a special case of three-valued interpretations of the form $\langle T; \emptyset \rangle$.

Following [34], we will define the truth valuation functions for atoms, literals, and regular rules. The atoms in $\mathcal{HB}_\mathrm{P}$ can have one of the following three truth values: $\mathbf{t}$, $\mathbf{f}$, and $\mathbf{u}$. Intuitively, $\mathbf{u}$ (underdefined) means possibly true or possible false. Underdefined atoms are viewed as being "more true" than false atoms, but "less true" than true atoms. This is captured by the following *truth ordering* among the truth values: $\mathbf{f} < \mathbf{u} < \mathbf{t}$. Given an interpretation $\mathcal{I} = \langle \mathrm{T}; \mathrm{U} \rangle$ of an F-logic KB P, for any atom A from $\mathcal{HB}_\mathrm{P}$ we can define the corresponding truth valuation function $\mathcal{I}$ as follows:

$$\mathcal{I}(\mathsf{A}) = \begin{cases} \mathbf{t}, \text{ if } \mathsf{A} \in \mathrm{T}; \\ \mathbf{u}, \text{ if } \mathsf{A} \in \mathrm{U}; \\ \mathbf{f}, \text{ otherwise.} \end{cases}$$

Truth valuations are extended to conjunctions of atoms in $\mathcal{HB}_\mathrm{P}$ as follows:

$$\mathcal{I}(\mathsf{A}_1 \wedge \ldots \wedge \mathsf{A}_\mathsf{n}) = min\{\mathcal{I}(\mathsf{A}_\mathsf{i}) | 1 \leq \mathsf{i} \leq \mathsf{n}\}$$

The intuitive reading of a regular rule is as follows: its rule head acts as an *explicit definition* while its rule body as a *query*. In particular, if $\mathsf{s}[\mathsf{m} \rightarrow \mathsf{v}]$ is in the *head* of a regular rule and the body of this rule is satisfied, then $\mathsf{m} \rightarrow \mathsf{v}$ is *explicitly defined* for s. In the body of a regular rule, the literal $\mathsf{s}[\mathsf{m} \rightarrow \mathsf{v}]$ is true if s has either an explicit definition of $\mathsf{m} \rightarrow \mathsf{v}$, or s inherits $\mathsf{m} \rightarrow \mathsf{v}$ from one of its superclasses by value or code inheritance. Therefore, the truth valuation of a ground F-logic literal depends on whether it appears in a rule head or in a rule body. This is formally defined as follows.

Definition 1 (Truth Valuation of Literals). *Let $\mathcal{I}$ be an interpretation of an F-logic KB* P*. The truth valuation functions, $\mathcal{V}_\mathcal{I}^\mathbf{h}$ and $\mathcal{V}_\mathcal{I}^\mathbf{b}$ (*$\mathbf{h}$ *and* $\mathbf{b}$ *stand for head and body, respectively), on ground F-logic literals are defined as follows:*

$$\mathcal{V}_\mathcal{I}^\mathbf{h}(\mathsf{o}:\mathsf{c}) = \mathcal{V}_\mathcal{I}^\mathbf{b}(\mathsf{o}:\mathsf{c}) = \mathcal{I}(\mathsf{o}:\mathsf{c}) \qquad \mathcal{V}_\mathcal{I}^\mathbf{h}(\mathsf{s}::\mathsf{c}) = \mathcal{V}_\mathcal{I}^\mathbf{b}(\mathsf{s}::\mathsf{c}) = \mathcal{I}(\mathsf{s}::\mathsf{c})$$

$$\mathcal{V}_\mathcal{I}^\mathbf{h}(\mathsf{s}[\mathsf{m} \rightarrow \mathsf{v}]) = \mathcal{I}(\mathsf{s}[\mathsf{m} \rightarrow \mathsf{v}]_\mathrm{ex})$$

$$\mathcal{V}_\mathcal{I}^\mathbf{b}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]) = max_{\mathsf{c} \in \mathcal{HU}_\mathrm{P}}\{\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]_\mathrm{ex}),\ \mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\,\mathsf{c}}_{\,\mathrm{val}}),\ \mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\,\mathsf{c}}_{\,\mathrm{code}})\}$$

Let L *and* L_i *(*$1 \leq \mathsf{i} \leq \mathsf{n}$*) be ground literals. Then:*

$$\mathcal{V}_\mathcal{I}^\mathbf{b}(\neg\,\mathsf{L}) = \neg\,\mathcal{V}_\mathcal{I}^\mathbf{b}(\mathsf{L}) \qquad \mathcal{V}_\mathcal{I}^\mathbf{b}(\mathsf{L}_1 \wedge \ldots \wedge \mathsf{L}_\mathsf{n}) = min\{\mathcal{V}_\mathcal{I}^\mathbf{b}(\mathsf{L}_\mathsf{i}) \,|\, 1 \leq \mathsf{i} \leq \mathsf{n}\}$$

For completeness, we define the negation of a truth value as follows: $\neg\,\mathbf{f} = \mathbf{t}$, $\neg\,\mathbf{t} = \mathbf{f}$*, and* $\neg\,\mathbf{u} = \mathbf{u}$*.*

The following two lemmas follow directly from the above definitions.

Lemma 1. *Let* $\mathcal{I} = \langle \mathrm{T}; \mathrm{U} \rangle$ *be an interpretation of an F-logic KB* P*,* L *a ground literal in ground*(P)*,* $\mathcal{J} = \langle \mathrm{T}; \emptyset \rangle$*, and* $\mathcal{K} = \langle \mathrm{T} \cup \mathrm{U}; \emptyset \rangle$*. Then:*

(1) If L *is a positive literal, then* $\mathcal{V}_\mathcal{I}^\mathbf{b}(\mathsf{L}) = \mathbf{t}$ *iff* $\mathcal{V}_\mathcal{J}^\mathbf{b}(\mathsf{L}) = \mathbf{t}$.

(2) If L *is a negative literal, then* $\mathcal{V}_\mathcal{I}^\mathbf{b}(\mathsf{L}) = \mathbf{t}$ *iff* $\mathcal{V}_\mathcal{K}^\mathbf{b}(\mathsf{L}) = \mathbf{t}$.

(3) If L *is a positive literal, then* $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L}) \geq \mathbf{u}$ *iff* $\mathcal{V}^{\mathbf{b}}_{\mathcal{K}}(\mathsf{L}) = \mathbf{t}$.
(4) If L *is a negative literal, then* $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L}) \geq \mathbf{u}$ *iff* $\mathcal{V}^{\mathbf{b}}_{\mathcal{J}}(\mathsf{L}) = \mathbf{t}$.

Lemma 2. *Let* $\mathcal{I} = \langle \mathrm{A}; \emptyset \rangle$ *and* $\mathcal{J} = \langle \mathrm{B}; \emptyset \rangle$ *be two-valued interpretations of an F-logic KB* P *such that* $\mathrm{A} \subseteq \mathrm{B}$*, and let* L *be a ground literal in ground*(P)*. Then:*

(1) If L *is a positive literal and* $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L}) = \mathbf{t}$*, then* $\mathcal{V}^{\mathbf{b}}_{\mathcal{J}}(\mathsf{L}) = \mathbf{t}$.
(2) If L *is a negative literal and* $\mathcal{V}^{\mathbf{b}}_{\mathcal{J}}(\mathsf{L}) = \mathbf{t}$*, then* $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L}) = \mathbf{t}$.

Having defined the truth valuation functions $\mathcal{V}^{\mathbf{h}}_{\mathcal{I}}$ and $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}$ for ground literals, we now extend the truth valuation function $\mathcal{I}$ to ground regular rules. Intuitively, a ground regular rule is true if and only if the truth value of its head is at least as high as truth value of the rule body (according to the truth ordering). Note that the truth valuation of either the head or the body is three-valued, but the truth valuation of a rule is always two-valued.

Definition 2 (Truth Valuation of Regular Rules). *Given an interpretation* $\mathcal{I}$ *of an F-logic KB* P*, the truth valuation function* $\mathcal{I}$ *on a ground regular rule,* $\mathsf{H} :- \; \mathsf{B} \in$ *ground*(P)*, is defined as follows:*

$$\mathcal{I}(\mathsf{H} :- \; \mathsf{B}) = \begin{cases} \mathbf{t}, \textit{ if } \mathcal{V}^{\mathbf{h}}_{\mathcal{I}}(\mathsf{H}) \geq \mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{B}); \\ \mathbf{f}, \textit{ otherwise.} \end{cases}$$

Given a ground regular fact, $\mathsf{H} \in$ *ground*(P)*:*

$$\mathcal{I}(\mathsf{H}) = \begin{cases} \mathbf{t}, \textit{ if } \mathcal{V}^{\mathbf{h}}_{\mathcal{I}}(\mathsf{H}) = \mathbf{t}; \\ \mathbf{f}, \textit{ otherwise.} \end{cases}$$

Satisfaction of nonground regular rules in an interpretation is defined via instantiation, as usual.

Definition 3 (Regular Rule Satisfaction). *A three-valued interpretation* $\mathcal{I}$ *satisfies the regular rules of an F-logic KB* P *if* $\mathcal{I}(\mathsf{R}) = \mathbf{t}$ *for every regular rule* R *in ground*(P)*.*

5 Inheritance Postulates

Even if an interpretation $\mathcal{I}$ satisfies all the regular rules of an F-logic KB P, it does not necessarily mean that $\mathcal{I}$ is an *intended* model of P. An intended model must also include facts that are derived via inheritance and must not include unsupported facts. As we saw in Section 3, defining what should be inherited exactly is a subtle issue. The main purpose of this section is to formalize the common intuition behind default inheritance using what we call *inheritance postulates*.

5.1 Basic Concepts

Intuitively, c[m] is an *inheritance context* for object o, if o is an instance of class c, and either c[m → v] is defined as a regular fact or is derived via a regular rule (in this case we say that m → v is *explicitly defined* at c); or if there is a template rule which specifies the instance method m for class c. Inheritance context is necessary for inheritance to take place, but it is not sufficient: inheritance of m from c might be overridden by a more specific inheritance context that sits below c along the inheritance path. If an inheritance context is not overridden by any other inheritance context, then we call it an *inheritance candidate*. Inheritance candidates represent potential sources for inheritance. But there must be exactly one inheritance candidate for inheritance to take place — having more than one leads to a multiple inheritance conflict, which blocks inheritance.

The concepts to be defined in this section come in two flavors: *strong* or *weak*. The "strong" flavor of a concept requires that all relevant facts be positively established while the "weak" flavor allows some or all facts to be underdefined.

Definition 4 (Explicit Definition). *Let* P *be an F-logic KB and* $\mathcal{I}$ *an interpretation of* P. *We say that* s[m] *has a strong explicit definition in* $\mathcal{I}$, *if* $max\{\mathcal{I}(\mathsf{s}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}}) | \mathsf{v} \in \mathcal{HU}_{\mathrm{P}}\} = \mathbf{t}$. *We say that* s[m] *has a weak explicit definition in* $\mathcal{I}$ *if* $max\{\mathcal{I}(\mathsf{s}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}}) | \mathsf{v} \in \mathcal{HU}_{\mathrm{P}}\} = \mathbf{u}$.

Definition 5 (Value Inheritance Context). *Given an interpretation* $\mathcal{I}$ *of an F-logic KB* P, c[m] *is a strong value inheritance context for* o *in* $\mathcal{I}$, *if* $\mathsf{c} \neq \mathsf{o}$ *(i.e.,* c *and* o *are distinct terms) and* $min\{\mathcal{I}(\mathsf{o}:\mathsf{c}), max\{\mathsf{c}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}} | \mathsf{v} \in \mathcal{HU}_{\mathrm{P}}\}\} = \mathbf{t}$. *We say that* c[m] *is a weak value inheritance context for* o *in* $\mathcal{I}$, *if* $\mathsf{c} \neq \mathsf{o}$ *and* $min\{\mathcal{I}(\mathsf{o}:\mathsf{c}), max\{\mathsf{c}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}} | \mathsf{v} \in \mathcal{HU}_{\mathrm{P}}\}\} = \mathbf{u}$.

Definition 6 (Code Inheritance Context). *Given an interpretation* $\mathcal{I}$ *of an F-logic KB* P, c[m] *is a strong (respectively, weak) code inheritance context for* o *in* $\mathcal{I}$, *if* $\mathsf{c} \neq \mathsf{o}$, $\mathcal{I}(\mathsf{o}:\mathsf{c}) = \mathbf{t}$ *(respectively,* $\mathcal{I}(\mathsf{o}:\mathsf{c}) = \mathbf{u}$*), and there is a template rule in* P *of the form* code(c) @this[m → ...] :− ..., *i.e., there is a template rule that defines instance method* m *for class* c.

When the specific type of an inheritance context is immaterial as, for example, in the following definitions, we will use the term *inheritance context* without indicating whether a value or a code inheritance context is meant.

Definition 7 (Overriding). *Let* $\mathcal{I}$ *be an interpretation of an F-logic KB* P. *We will say that class* s *strongly overrides inheritance context* c[m] *for* o, *if* $\mathsf{s} \neq \mathsf{c}$, $\mathcal{I}(\mathsf{s} :: \mathsf{c}) = \mathbf{t}$, *and* s[m] *is a strong (value or code) inheritance context for* o.

We will say class s *weakly overrides* c[m] *for* o, *if either*

(1) $\mathcal{I}(\mathsf{s} :: \mathsf{c}) = \mathbf{t}$ *and* s[m] *is a weak inheritance context for* o; *or*
(2) $\mathcal{I}(\mathsf{s} :: \mathsf{c}) = \mathbf{u}$ *and* s[m] *is a weak or a strong inheritance context for* o.

Definition 8 (Value Inheritance Candidate). *Given an interpretation* $\mathcal{I}$ *of an F-logic KB* P, c[m] *is a strong value inheritance candidate for* o, *denoted*

$\mathsf{c}[\mathsf{m}] \overset{s.val}{\leadsto}_{\mathcal{I}} \mathsf{o}$, *if* $\mathsf{c}[\mathsf{m}]$ *is a strong value inheritance context for* o *and there is no class* s *that strongly or weakly overrides* $\mathsf{c}[\mathsf{m}]$ *for* o.

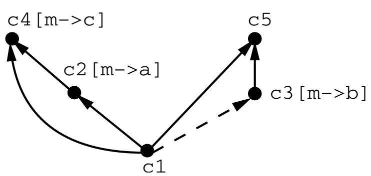
$\mathsf{c}[\mathsf{m}]$ *is a weak value inheritance candidate for* o*, denoted* $\mathsf{c}[\mathsf{m}] \overset{w.val}{\leadsto}_{\mathcal{I}} \mathsf{o}$*, if the above conditions are relaxed by allowing* $\mathsf{c}[\mathsf{m}]$ *to be a weak value inheritance context and/or allowing weak overriding. Formally, this means that there is no class* s *that strongly overrides* $\mathsf{c}[\mathsf{m}]$ *for* o *and either*

(1) $\mathsf{c}[\mathsf{m}]$ *is a weak value inheritance context for* o*; or*
(2) $\mathsf{c}[\mathsf{m}]$ *is a strong value inheritance context for* o *and there is some class* s *that weakly overrides* $\mathsf{c}[\mathsf{m}]$ *for* o.

Definition 9 (Code Inheritance Candidate). *Let* $\mathcal{I}$ *be an interpretation for an F-logic KB* P*.* $\mathsf{c}[\mathsf{m}]$ *is called a strong code inheritance candidate for* o*, denoted* $\mathsf{c}[\mathsf{m}] \overset{s.code}{\leadsto}_{\mathcal{I}} \mathsf{o}$*, if* $\mathsf{c}[\mathsf{m}]$ *is a strong code inheritance context for* o *and there is no* s *that strongly or weakly overrides* $\mathsf{c}[\mathsf{m}]$ *for* o.

$\mathsf{c}[\mathsf{m}]$ *is a weak code inheritance candidate for* o*, denoted* $\mathsf{c}[\mathsf{m}] \overset{w.code}{\leadsto}_{\mathcal{I}} \mathsf{o}$*, if the above conditions are relaxed by allowing* $\mathsf{c}[\mathsf{m}]$ *to be a weak code inheritance context and/or allowing weak overriding. Formally, this means that there is no class* s *that strongly overrides* $\mathsf{c}[\mathsf{m}]$ *for* o *and either*

(1) $\mathsf{c}[\mathsf{m}]$ *is a weak code inheritance context for* o*; or*
(2) $\mathsf{c}[\mathsf{m}]$ *is a strong code inheritance context for* o *and there is some class* s *that weakly overrides* $\mathsf{c}[\mathsf{m}]$ *for* o.



$\mathsf{c}_1 : \mathsf{c}_2.$ $\mathsf{c}_1 : \mathsf{c}_5.$
$\mathsf{c}_2 :: \mathsf{c}_4.$ $\mathsf{c}_3 :: \mathsf{c}_5.$
$\mathsf{c}_2[\mathsf{m} \rightarrow \mathsf{a}].$ $\mathsf{c}_3[\mathsf{m} \rightarrow \mathsf{b}].$ $\mathsf{c}_4[\mathsf{m} \rightarrow \mathsf{c}].$
$\mathsf{code}(\mathsf{c}_5)$ $@\mathsf{this}[\mathsf{m} \rightarrow \mathsf{v}]$ $:-$ $@\mathsf{this}[\mathsf{f} \rightarrow \mathsf{x}].$

Fig. 7. Inheritance Context, Overriding, and Inheritance Candidate

Example 1. Consider an interpretation $\mathcal{I} = \langle \mathrm{T}; \mathrm{U} \rangle$ of an F-logic KB P, where

$$\begin{aligned} \mathrm{T} = &\{\mathsf{c}_1 : \mathsf{c}_2, \mathsf{c}_1 : \mathsf{c}_4, \mathsf{c}_1 : \mathsf{c}_5, \mathsf{c}_2 :: \mathsf{c}_4, \mathsf{c}_3 :: \mathsf{c}_5\} \cup \\ &\{\mathsf{c}_2[\mathsf{m} \rightarrow \mathsf{a}]_{\mathrm{ex}}, \mathsf{c}_3[\mathsf{m} \rightarrow \mathsf{b}]_{\mathrm{ex}}, \mathsf{c}_4[\mathsf{m} \rightarrow \mathsf{c}]_{\mathrm{ex}}\} \\ \mathrm{U} = &\{\mathsf{c}_1 : \mathsf{c}_3\} \end{aligned}$$

$\mathcal{I}$ and P are shown in Figure 7, where solid and dashed arrows represent true and underdefined values, respectively.

In the interpretation $\mathcal{I}$, $\mathsf{c}_2[\mathsf{m}]$ and $\mathsf{c}_4[\mathsf{m}]$ are strong value inheritance contexts for c_1. $\mathsf{c}_5[\mathsf{m}]$ is a strong code inheritance context for c_1. $\mathsf{c}_3[\mathsf{m}]$ is a weak value inheritance context for c_1. The class c_2 strongly overrides $\mathsf{c}_4[\mathsf{m}]$ for c_1, while c_3 weakly overrides $\mathsf{c}_5[\mathsf{m}]$ for c_1. $\mathsf{c}_2[\mathsf{m}]$ is a strong value inheritance candidate for c_1. $\mathsf{c}_3[\mathsf{m}]$ is a weak value inheritance candidate for c_1. $\mathsf{c}_5[\mathsf{m}]$ is a weak code inheritance candidate for c_1. Finally, $\mathsf{c}_4[\mathsf{m}]$ is neither a strong nor a weak value inheritance candidate for c_1.

For convenience, we will simply write $\mathsf{c}[\mathsf{m}] \rightsquigarrow_{\mathcal{I}} \mathsf{o}$ when it does not matter whether $\mathsf{c}[\mathsf{m}]$ is a strong or a weak value/code inheritance candidate. Now we are ready to introduce the postulates for nonmonotonic multiple value and code inheritance. The inheritance postulates consist of two parts: core inheritance postulates and cautious inheritance postulates. We formalize the core inheritance postulates first.

5.2 Core Inheritance Postulates

The following definition says that class membership and subclass relationship must satisfy the usual transitive closure property.

Definition 10 (Positive ISA Transitivity). *An interpretation $\mathcal{I}$ of an F-logic KB* P *satisfies the positive ISA transitivity constraint if the set of true class membership and subclass relationship atoms is transitively closed. Formally this means that the following two conditions hold:*

(1) *for all* s *and* c*: if there is* x *such that* $\mathcal{I}(\mathsf{s} :: \mathsf{x}) = \mathbf{t}$ *and* $\mathcal{I}(\mathsf{x} :: \mathsf{c}) = \mathbf{t}$*, then* $\mathcal{I}(\mathsf{s} :: \mathsf{c}) = \mathbf{t}$*;*
(2) *for all* o *and* c*: if there is* x *such that* $\mathcal{I}(\mathsf{o} : \mathsf{x}) = \mathbf{t}$ *and* $\mathcal{I}(\mathsf{x} :: \mathsf{c}) = \mathbf{t}$*, then* $\mathcal{I}(\mathsf{o} : \mathsf{c}) = \mathbf{t}$*.*

The context consistency constraint below captures the idea that only explicit definitions are inherited and that explicit definitions override inheritance.

Definition 11 (Context Consistency). *An interpretation $\mathcal{I}$ of an F-logic KB* P *satisfies the context consistency constraint, if the following conditions hold:*

(1) *for all* $\mathsf{o}, \mathsf{m}, \mathsf{v}$*:* $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{o}}_{\text{val}}) = \mathbf{f}$ *and* $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{o}}_{\text{code}}) = \mathbf{f}$*;*
(2) *for all* $\mathsf{c}, \mathsf{m}, \mathsf{v}$*: if* $\mathcal{I}(\mathsf{c}[\mathsf{m} \rightarrow \mathsf{v}]_{\text{ex}}) = \mathbf{f}$*, then* $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\text{val}}) = \mathbf{f}$ *for all* o*;*
(3) *for all* c, m*: if ground*(P) *has no template rule that defines instance method* m *for class* c*, then* $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\text{code}}) = \mathbf{f}$ *for all* o, v*;*
(4) *for all* o, m*: if* $\mathsf{o}[\mathsf{m}]$ *has a strong explicit definition, then for all* v*,* c*,* $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\text{val}}) = \mathbf{f}$ *and* $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\text{code}}) = \mathbf{f}$*.*

The first condition in the above definition rules out self inheritance. The second condition states that if $\mathsf{m} \rightarrow \mathsf{v}$ is not explicitly defined at c, then no one can inherit $\mathsf{m} \rightarrow \mathsf{v}$ from c by value inheritance. The third condition says that if a class c does not explicitly specify an instance method m, then no object should inherit $\mathsf{m} \rightarrow \mathsf{v}$ from c by code inheritance, for any v. The fourth condition states that if o has an explicit definition for method m, then this definition should prevent o from inheriting $\mathsf{m} \rightarrow \mathsf{v}$ from any other class for any v (either by value or by code inheritance).

Intuitively, we want our semantics to have the property that if inheritance is allowed, then it should take place from a *unique* source. This is captured by the following definition.

Definition 12 (Unique Source Inheritance). *An interpretation $\mathcal{I}$ of an F-logic KB* P *satisfies the unique source inheritance constraint, if the following three conditions hold:*

(1) for all $\mathsf{o}, \mathsf{m}, \mathsf{v}, \mathsf{c}$*: if* $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}}) = \mathbf{t}$ *or* $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\mathrm{code}}) = \mathbf{t}$*, then* $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{z}]^{\mathsf{x}}_{\mathrm{val}}) = \mathbf{f}$ *and* $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{z}]^{\mathsf{x}}_{\mathrm{code}}) = \mathbf{f}$ *for all* z, x *such that* $\mathsf{x} \neq \mathsf{c}$.

(2) for all $\mathsf{c}, \mathsf{m}, \mathsf{o}$*: if* $\mathsf{c}[\mathsf{m}] \overset{s.val}{\leadsto}_{\mathcal{I}} \mathsf{o}$ *or* $\mathsf{c}[\mathsf{m}] \overset{s.code}{\leadsto}_{\mathcal{I}} \mathsf{o}$*, then* $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{x}}_{\mathrm{val}}) = \mathbf{f}$ *and* $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{x}}_{\mathrm{code}}) = \mathbf{f}$ *for all* v*,* x *such that* $\mathsf{x} \neq \mathsf{c}$.

(3) for all $\mathsf{o}, \mathsf{m}, \mathsf{v}, \mathsf{c}$*:* $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}}) = \mathbf{t}$ *iff*

- *(i)* $\mathsf{o}[\mathsf{m}]$ *has neither strong nor weak explicit definitions; and*
- *(ii)* $\mathsf{c}[\mathsf{m}] \overset{s.val}{\leadsto}_{\mathcal{I}} \mathsf{o}$*; and*
- *(iii)* $\mathcal{I}(\mathsf{c}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}}) = \mathbf{t}$*; and*
- *(iv) there is no* x *such that* $\mathsf{x} \neq \mathsf{c}$ *and* $\mathsf{x}[\mathsf{m}] \leadsto_{\mathcal{I}} \mathsf{o}$.

Uniqueness of an inheritance source is captured via three conditions. The first condition above says that an object can inherit from a class only if it does not already inherit from another class. The second condition states that if a strong inheritance candidate, $\mathsf{c}[\mathsf{m}]$, exists, then inheritance of method m cannot take place from any other sources (because there would then be a multiple inheritance conflict). The third condition specifies when value inheritance takes place. An object o inherits $\mathsf{m} \rightarrow \mathsf{v}$ from class c by value inheritance iff: (i) o has *no* explicit definition for method m; (ii) $\mathsf{c}[\mathsf{m}]$ is a strong value inheritance candidate for o; (iii) $\mathsf{c}[\mathsf{m} \rightarrow \mathsf{v}]$ is explicitly defined; and (iv) there are no other inheritance candidates — weak or strong — from which o could inherit method m.

5.3 Cautious Inheritance Postulates

The core postulates introduced so far impose restrictions only on the part of an interpretation that contains the facts known to be true. For three-valued interpretations, we still need to describe the underdefined part more tightly. Since "underdefined" means possibly true or possibly false, it is natural to expect that the conclusions drawn from underdefined facts remain underdefined. As is typical for three-valued semantics, such as the well-founded semantics, we do not jump to negative conclusions from underdefined facts. This is why we call our semantics "cautious".

Definition 13 (Cautious ISA Transitivity). *We will say that an interpretation* $\mathcal{I}$ *of an F-logic KB* P *satisfies the cautious ISA transitivity constraint if the underdefined part of the class hierarchy is transitively closed; i.e.,*

(1) *for all* s*,* c*: if there is* x *such that* $\mathcal{I}(\mathsf{s} :: \mathsf{x} \wedge \mathsf{x} :: \mathsf{c}) = \mathbf{u}$ *and* $\mathcal{I}(\mathsf{s} :: \mathsf{c}) \neq \mathbf{t}$*, then* $\mathcal{I}(\mathsf{s} :: \mathsf{c}) = \mathbf{u}$*;*

(2) *for all* o*,* c*: if there is* x *such that* $\mathcal{I}(\mathsf{o} : \mathsf{x} \wedge \mathsf{x} :: \mathsf{c}) = \mathbf{u}$ *and* $\mathcal{I}(\mathsf{o} : \mathsf{c}) \neq \mathbf{t}$*, then* $\mathcal{I}(\mathsf{o} : \mathsf{c}) = \mathbf{u}$.

Definition 14 (Cautious Inheritance). *We will say that an interpretation* $\mathcal{I}$ *of an F-logic KB* P *satisfies the cautious inheritance constraint, if for all* o*,* m*,* v*,* c*:* $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}}) = \mathbf{u}$ *iff*

(1) $\mathsf{o}[\mathsf{m}]$ *does not have a strong explicit definition; and*

(2) $\mathsf{c}[\mathsf{m}] \overset{s.val}{\leadsto}_{\mathcal{I}} \mathsf{o}$ *or* $\mathsf{c}[\mathsf{m}] \overset{w.val}{\leadsto}_{\mathcal{I}} \mathsf{o}$*; and*

(3) $\mathcal{I}(\mathsf{c}[\mathsf{m} \to \mathsf{v}]_{\text{ex}}) \geq \mathbf{u}$; *and*
(4) *there is no* $\mathsf{x} \neq \mathsf{c}$ *such that* $\mathsf{x}[\mathsf{m}] \overset{s.val}{\leadsto}_{\mathcal{I}} \mathsf{o}$ *or* $\mathsf{x}[\mathsf{m}] \overset{s.code}{\leadsto}_{\mathcal{I}} \mathsf{o}$; *and*
(5) $\mathcal{I}(\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\text{val}}) \neq \mathsf{t}$.

The cautious inheritance constraint captures the intuition behind multiple inheritance based on underdefined knowledge. The conditions above state when cautious value inheritance takes place. An object o *cautiously* inherits $\mathsf{m} \to \mathsf{v}$ from class c by value inheritance if and only if: (i) there is no strong evidence that method m has an explicitly defined value at o; (ii) c[m] is either a strong or a weak value inheritance candidate for o; (iii) $\mathsf{m} \to \mathsf{v}$ is explicitly defined at c; (iv) there are no other strong inheritance candidates that can block value inheritance from c (by the unique source inheritance constraint); and (v) o does not already inherit $\mathsf{m} \to \mathsf{v}$ from c by value inheritance.

6 Object Models

A model of an F-logic KB should satisfy all the rules in it. In Section 4 we formalized the notion of regular rule satisfaction. Here we will extend this notion to template rules. Recall that when an object inherits a template rule, the rule is evaluated in the context of that object.

Definition 15 (Binding). *Let* R *be the following template rule which defines instance method* m *for class* c*:* code(c) @this[m → v] :− B*. The binding of* R *with respect to object* o*, denoted* $\mathsf{R}_{\|\mathsf{o}}$*, is obtained from* R *by substituting* o *for every occurrence of* @this *in* R*. In general, we will use* $\mathsf{X}_{\|\mathsf{o}}$ *to represent the term that is obtained from* X *by substituting* o *for every occurrence of* @this *in* X*.*

We call the above process "binding" because it is akin to late binding in traditional programming languages like C^{++}. Recall from Section 3 that template rules are just templates for the regular rules that are obtained via binding when template rules are inherited. Therefore, satisfaction of template rules in a model will have to be defined via satisfaction of their bindings. When an object inherits template rules from a class, the bindings of these template rules with respect to this object should be satisfied similarly to regular rules. However, because only those template rules that are actually inherited need to be satisfied, satisfaction of template rules depends on how they are inherited: strongly or weakly.

Definition 16 (Strong Code Inheritance). *Let* $\mathcal{I}$ *be an interpretation of an F-logic KB* P *and* R ≡ code(c) @this[m → v] :− B *a template rule in ground*(P)*. An object* o *strongly inherits* R*, if the following conditions hold:*

(1) $\mathsf{c}[\mathsf{m}] \overset{s.code}{\leadsto}_{\mathcal{I}} \mathsf{o}$;
(2) o[m] *has neither strong nor weak explicit definitions;*
(3) *there is no* $\mathsf{x} \neq \mathsf{c}$ *such that* $\mathsf{x}[\mathsf{m}] \leadsto_{\mathcal{I}} \mathsf{o}$.

In other words, strong code inheritance happens when there is a strong code inheritance candidate, which is not overwritten and which does not have a rival inheritance candidate of any kind.

Definition 17 (Weak Code Inheritance). *Let $\mathcal{I}$ be an interpretation of an F-logic KB* P *and* $\mathsf{R} \equiv \mathsf{code(c)}\ @\mathsf{this[m} \rightarrow \mathsf{v]}\ :-\ \mathsf{B}$ *a template rule in ground*(P). *An object* o *weakly inherits* R*, if all of the following holds:*

(1) $\mathsf{c[m]} \overset{s.code}{\leadsto}_{\mathcal{I}} \mathsf{o}$ *or* $\mathsf{c[m]} \overset{w.code}{\leadsto}_{\mathcal{I}} \mathsf{o}$*;*
(2) o[m] *has no strong explicit definitions;*
(3) *there is no* $\mathsf{x} \neq \mathsf{c}$ *such that* $\mathsf{x[m]} \overset{s.val}{\leadsto}_{\mathcal{I}} \mathsf{o}$ *or* $\mathsf{x[m]} \overset{s.code}{\leadsto}_{\mathcal{I}} \mathsf{o}$*;*
(4) o *does not strongly inherit* R*.*

In other words, o weakly inherits R, if: c[m] is a code inheritance candidate for o (strong or weak); o[m] has no strong explicit definitions; there are no other strong conflicting inheritance candidates; and, of course, o does not strongly inherit R.

For convenience, we define a function, $imode_{\mathcal{I}}$, on the bindings of ground template rules, which returns the "inheritance mode" of a binding:

$$imode_{\mathcal{I}}(\mathsf{R}_{\|\mathsf{o}}) = \begin{cases} \mathbf{t}, \text{ if o strongly inherits R;} \\ \mathbf{u}, \text{ if o weakly inherits R;} \\ \mathbf{f}, \text{ otherwise.} \end{cases}$$

When $imode_{\mathcal{I}}(\mathsf{R}_{\|\mathsf{o}}) = \mathbf{t}$, we will say that $\mathsf{R}_{\|\mathsf{o}}$ is in strong code inheritance mode. Similarly, we will say $\mathsf{R}_{\|\mathsf{o}}$ is in weak code inheritance mode if $imode_{\mathcal{I}}(\mathsf{R}_{\|\mathsf{o}}) = \mathbf{u}$. Now we can extend the truth valuation function to template rules as follows.

Definition 18 (Truth Valuation of Template Rules). *Let $\mathcal{I}$ be an interpretation and* $\mathsf{R} \equiv \mathsf{code(c)}\ @\mathsf{this[m} \rightarrow \mathsf{v]}\ :-\ \mathsf{B}$ *a ground template rule. The truth valuation function $\mathcal{I}$ on* $\mathsf{R}_{\|\mathsf{o}}$ *is defined as follows:*

$$\mathcal{I}(\mathsf{R}_{\|\mathsf{o}}) = \begin{cases} \mathbf{t}, \textit{ if } imode_{\mathcal{I}}(\mathsf{R}_{\|\mathsf{o}}) \geq \mathbf{u} \textit{ and} \\ \quad \mathcal{I}(\mathsf{o[m} \rightarrow \mathsf{v]}^{\mathsf{c}}_{\text{code}}) \geq min\{\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{B}_{\|\mathsf{o}}), imode_{\mathcal{I}}(\mathsf{R}_{\|\mathsf{o}})\}; \\ \mathbf{t}, \textit{ if } imode_{\mathcal{I}}(\mathsf{R}_{\|\mathsf{o}}) = \mathbf{f} \textit{ and } \mathcal{I}(\mathsf{o[m} \rightarrow \mathsf{v]}^{\mathsf{c}}_{\text{code}}) = \mathbf{f}; \\ \mathbf{f}, \textit{ otherwise.} \end{cases}$$

For ground template facts of the form $\mathsf{F} \equiv \mathsf{code(c)}\ @\mathsf{this[m} \rightarrow \mathsf{v]}$*, their truth valuation is defined similarly:*

$$\mathcal{I}(\mathsf{F}_{\|\mathsf{o}}) = \begin{cases} \mathbf{t}, \textit{ if } imode_{\mathcal{I}}(\mathsf{F}_{\|\mathsf{o}}) \geq \mathbf{u} \textit{ and } \mathcal{I}(\mathsf{o[m} \rightarrow \mathsf{v]}^{\mathsf{c}}_{\text{code}}) \geq imode_{\mathcal{I}}(\mathsf{F}_{\|\mathsf{o}}); \\ \mathbf{t}, \textit{ if } imode_{\mathcal{I}}(\mathsf{F}_{\|\mathsf{o}}) = \mathbf{f} \textit{ and } \mathcal{I}(\mathsf{o[m} \rightarrow \mathsf{v]}^{\mathsf{c}}_{\text{code}}) = \mathbf{f}; \\ \mathbf{f}, \textit{ otherwise.} \end{cases}$$

Recall that atoms of the form $\mathsf{o[m} \rightarrow \mathsf{v]}^{\mathsf{c}}_{\text{code}}$ represent those facts that are derived via code inheritance. Note that when $imode_{\mathcal{I}}(\mathsf{R}_{\|\mathsf{o}}) = \mathbf{f}$, i.e., o does not inherit R, it is required that $\mathcal{I}(\mathsf{o[m} \rightarrow \mathsf{v]}^{\mathsf{c}}_{\text{code}}) = \mathbf{f}$ in order for $\mathsf{R}_{\|\mathsf{o}}$ to be satisfied. This means that if an object, o, does not inherit a template rule, then the binding of that rule with respect to o should not be used to make inference.

Now the idea of template rule satisfaction and the notion of an *object model* can be formalized as follows.

Definition 19 (Template Rule Satisfaction). *An interpretation* $\mathcal{I}$ *satisfies the template rules of an F-logic KB* P*, if* $\mathcal{I}(\mathsf{R}_{\|\mathsf{o}}) = \mathbf{t}$ *for all template rule* $\mathsf{R} \in ground(\mathrm{P})$ *and all* $\mathsf{o} \in \mathcal{HU}_{\mathrm{P}}$.

Observe that in the event of strong code inheritance, $imode_{\mathcal{I}}(\mathsf{R}_{\|\mathsf{o}}) = \mathbf{t}$ and so the truth valuation function on template rules reduces to that on regular rules. Indeed, for template rules, we have from Definition 18 that $\mathcal{I}(\mathsf{R}_{\|\mathsf{o}}) = \mathbf{t}$ iff $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\text{code}}) \geq min\{\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{B}_{\|\mathsf{o}}), imode_{\mathcal{I}}(\mathsf{R}_{\|\mathsf{o}})\} = \mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{B}_{\|\mathsf{o}})$. A similar conclusion can be drawn for template facts.

Definition 20 (Object Model). *An interpretation* $\mathcal{I}$ *is called an object model of an F-logic KB* P *if* $\mathcal{I}$ *satisfies:*

- *all the regular rules in* P*,*
- *all the template rules in* P*, and*
- *all the core inheritance postulates (including the positive ISA transitivity constraint, the context consistency constraint, and the unique source inheritance constraint).*

7 Computation

In this section we will define a series of operators, which will form the basis for a bottom-up procedure for computing object models of F-logic KBs.

First we need to extend the definition of an interpretation in Section 4 to include book-keeping information used by the computation. This book-keeping information is cast out at the last stage when the final object model is produced. The *extended Herbrand base* of an F-logic KB P, denoted $\widehat{\mathcal{HB}}_{\mathrm{P}}$, consists of atoms from $\mathcal{HB}_{\mathrm{P}}$ and *auxiliary* atoms of the forms $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o}$ and $\mathsf{c}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o}$, where c, m, and o are terms from $\mathcal{HU}_{\mathrm{P}}$. During the computation, these auxiliary atoms will be used to approximate value and code inheritance candidates (with which they should not be confused). An *extended atom set* is a subset of $\widehat{\mathcal{HB}}_{\mathrm{P}}$. In the sequel, we will use symbols with a hat (e.g., $\widehat{\mathrm{I}}$) to denote extended atom sets. The *projection* of an extended atom set $\widehat{\mathrm{I}}$, denoted $\pi(\widehat{\mathrm{I}})$, is $\widehat{\mathrm{I}}$ with the auxiliary atoms removed.

We will often need to compare a normal atom set with the projection of an extended atom set. In such cases, when confusion does not arise, we will omit the projection operator π.

It is easy to generalize the definitions of the truth valuation functions in Section 4 to extended atom sets, since the auxiliary atoms do not occur in F-logic KBs. Formally, given an extended atom set $\widehat{\mathrm{I}}$, let $\mathcal{I} = \langle \pi(\widehat{\mathrm{I}}); \emptyset \rangle$. We define: (i) $val^{\mathbf{h}}_{\widehat{\mathrm{I}}}(\mathsf{H}) \overset{\text{def}}{=} \mathcal{V}^{\mathbf{h}}_{\mathcal{I}}(\mathsf{H})$, for a ground rule head H; (ii) $val^{\mathbf{b}}_{\widehat{\mathrm{I}}}(\mathsf{B}) \overset{\text{def}}{=} \mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{B})$, for a ground rule body B; (iii) $val_{\widehat{\mathrm{I}}}(\mathsf{R}) \overset{\text{def}}{=} \mathcal{I}(\mathsf{R})$, for a ground regular rule R; and (iv) $val_{\widehat{\mathrm{I}}}(\mathsf{R}_{\|\mathsf{o}}) \overset{\text{def}}{=} \mathcal{I}(\mathsf{R}_{\|\mathsf{o}})$, for a binding of a ground template rule R.

The computation model for F-logic KBs with regular and template rules was inspired by the alternating fixpoint operator [42] and extends it. The new element here is the book-keeping mechanism, which is necessary for recording inheritance information.

Definition 21. *Given a ground literal* L *of an F-logic KB* P *and an atom* $\mathsf{A} \in \mathcal{HB}_{\mathrm{P}}$*, we say that* L *matches* A*, if one of the following conditions is true: (i)* $\mathsf{L} = \mathsf{o}:\mathsf{c}$ *and* $\mathsf{A} = \mathsf{o}:\mathsf{c}$*; or (ii)* $\mathsf{L} = \mathsf{s}::\mathsf{c}$ *and* $\mathsf{A} = \mathsf{s}::\mathsf{c}$*; or (iii)* $\mathsf{L} = \mathsf{s}[\mathsf{m} \rightarrow \mathsf{v}]$ *and* $\mathsf{A} = \mathsf{s}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}}$.

Definition 22 (Regular Rule Consequence). *The regular rule consequence operator,* $\mathbf{RC}_{\mathrm{P},\widehat{\mathrm{I}}}$*, is defined for an F-logic KB* P *and an extended atom set* $\widehat{\mathrm{I}}$*. It takes as input an extended atom set,* $\widehat{\mathrm{J}}$*, and generates a new extended atom set,* $\mathbf{RC}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{J}})$*, as follows:*

$$\left\{ \mathsf{A} \;\middle|\; \begin{array}{l} \textit{ground}(\mathrm{P}) \textit{ has a regular rule, } \mathsf{H} :- \mathsf{L}_1, \ldots, \mathsf{L}_n\textit{, such that } \mathsf{H} \textit{ matches} \\ \mathsf{A} \textit{ and for every literal } \mathsf{L}_i \;\; (1 \le i \le n)\textit{: (i) if } \mathsf{L}_i \textit{ is positive, then} \\ \textit{val}^{\mathbf{b}}_{\widehat{\mathrm{J}}}(\mathsf{L}_i) = \mathbf{t}\textit{; and (ii) if } \mathsf{L}_i \textit{ is negative, then } \textit{val}^{\mathbf{b}}_{\widehat{\mathrm{I}}}(\mathsf{L}_i) = \mathbf{t}. \end{array} \right\}$$

The regular rule consequence operator is adopted from the usual alternating fixpoint computation. It derives new facts, including class membership, subclass relationship, and explicit method definitions for classes and objects, from the regular rules in an F-logic KB.

Definition 23 (Inheritance Blocking). *The inheritance blocking operator,* $\mathbf{IB}_{\mathrm{P}}$*, is defined for an F-logic KB* P*. It takes as input an extended atom set,* $\widehat{\mathrm{I}}$*, and generates the set,* $\mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{I}})$*, which is the union of the following sets of atoms.*

Explicit inheritance conflicts:

$$\left\{ ec(\mathsf{o},\mathsf{m}) \mid \exists \mathsf{v} \textit{ such that } \mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}} \in \widehat{\mathrm{I}} \right\}$$

Multiple inheritance conflicts:

$$\left\{ mc(\mathsf{c},\mathsf{m},\mathsf{o}) \;\middle|\; \exists \mathsf{x} \neq \mathsf{c} \textit{ such that } \mathsf{x}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{I}} \textit{ or } \mathsf{x}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o} \in \widehat{\mathrm{I}} \right\}$$

Overriding inheritance conflicts:

$$\left\{ ov(\mathsf{c},\mathsf{m},\mathsf{o}) \;\middle|\; \begin{array}{l} \exists \mathsf{x} \textit{ such that: (i) } \mathsf{x} \neq \mathsf{c},\; \mathsf{x} \neq \mathsf{o},\; \mathsf{x}::\mathsf{c} \in \widehat{\mathrm{I}}, \\ \mathsf{o}:\mathsf{x} \in \widehat{\mathrm{I}}\textit{; and (ii) } \exists \mathsf{v} \textit{ such that } \mathsf{x}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}} \in \widehat{\mathrm{I}} \\ \textit{or there is a template rule in ground}(\mathrm{P}) \textit{ of the} \\ \textit{form } \mathsf{code}(\mathsf{x}) \;\; @\mathsf{this}[\mathsf{m} \rightarrow \ldots] \;\; :- \;\ldots\textit{, which} \\ \textit{specifies the instance method } \mathsf{m} \textit{ for class } \mathsf{x}. \end{array} \right\}$$

The inheritance blocking operator is an auxiliary operator used in defining the template rule consequence operator and the inheritance consequence operator below. It returns book-keeping information that is needed to determine inheritance candidates.

Intuitively, $ec(\mathsf{o},\mathsf{m})$ means method m is explicitly defined at o; $mc(\mathsf{c},\mathsf{m},\mathsf{o})$ means inheritance of method m from c to o is not possible due to a multiple inheritance conflict (because there is a value or a code inheritance candidate other than c); $ov(\mathsf{c},\mathsf{m},\mathsf{o})$ means inheritance of method m from c by o would be overridden by another class that stands between o and c in the class hierarchy. From Definition 23 we can see that a class must have a explicitly defined value for a method or have an instance method definition to be able to override inheritance from its superclasses. The following lemmas follow directly from the above definitions.

Lemma 3. *Given an interpretation* $\mathcal{I} = \langle \mathrm{T}; \mathrm{U} \rangle$ *of an F-logic KB* P*:*

(1) *for all* $\mathsf{c},\mathsf{m},\mathsf{o}$*: there is* x *such that* x *strongly overrides* $\mathsf{c}[\mathsf{m}]$ *for* o *iff* $ov(\mathsf{c},\mathsf{m},\mathsf{o}) \in \mathbf{IB}_{\mathrm{P}}(\mathrm{T})$.

(2) *for all* $\mathsf{c},\mathsf{m},\mathsf{o}$*: there is* x *such that* x *strongly or weakly overrides* $\mathsf{c}[\mathsf{m}]$ *for* o *iff* $ov(\mathsf{c},\mathsf{m},\mathsf{o}) \in \mathbf{IB}_{\mathrm{P}}(\mathrm{T} \cup \mathrm{U})$.

Lemma 4. *Given an interpretation* $\mathcal{I} = \langle \mathrm{T}; \mathrm{U} \rangle$ *of an F-logic KB* P*:*

(1) *for all* $\mathsf{c},\mathsf{m},\mathsf{o}$*:* $\mathsf{c}[\mathsf{m}] \overset{s.val}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$ *iff (i)* $\mathsf{c} \neq \mathsf{o}$, $\mathsf{o}:\mathsf{c} \in \mathrm{T}$*; (ii)* $\mathsf{c}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}} \in \mathrm{T}$ *for some* v*; and (iii)* $ov(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\mathrm{T} \cup \mathrm{U})$.

(2) *for all* $\mathsf{c},\mathsf{m},\mathsf{o}$*:* $\mathsf{c}[\mathsf{m}] \overset{s.code}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$ *iff (i)* $\mathsf{c} \neq \mathsf{o}$, $\mathsf{o}:\mathsf{c} \in \mathrm{T}$*; (ii) there is a template rule in ground*(P) *which specifies the instance method* m *for class* c*; and (iii)* $ov(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\mathrm{T} \cup \mathrm{U})$.

(3) *for all* $\mathsf{c},\mathsf{m},\mathsf{o}$*:* $\mathsf{c}[\mathsf{m}] \overset{s.val}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$ *or* $\mathsf{c}[\mathsf{m}] \overset{w.val}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$ *iff (i)* $\mathsf{c} \neq \mathsf{o}$, $\mathsf{o}:\mathsf{c} \in \mathrm{T} \cup \mathrm{U}$*; (ii)* $\mathsf{c}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}} \in \mathrm{T} \cup \mathrm{U}$ *for some* v*; and (iii)* $ov(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\mathrm{T})$.

(4) *for all* $\mathsf{c},\mathsf{m},\mathsf{o}$*:* $\mathsf{c}[\mathsf{m}] \overset{s.code}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$ *or* $\mathsf{c}[\mathsf{m}] \overset{w.code}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$ *iff (i)* $\mathsf{c} \neq \mathsf{o}$, $\mathsf{o}:\mathsf{c} \in \mathrm{T} \cup \mathrm{U}$*; (ii) there is a template rule in ground*(P) *which specifies the instance method* m *for class* c*; and (iii)* $ov(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\mathrm{T})$.

(5) *for all* $\mathsf{c},\mathsf{m},\mathsf{o}$*:* $\mathsf{c}[\mathsf{m}] \rightsquigarrow_{\mathcal{I}} \mathsf{o}$ *iff (i)* $\mathsf{c} \neq \mathsf{o}$, $\mathsf{o}:\mathsf{c} \in \mathrm{T} \cup \mathrm{U}$*; (ii) there is a template rule in ground*(P) *which specifies the instance method* m *for class* c *or* $\mathsf{c}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}} \in \mathrm{T} \cup \mathrm{U}$ *for some* v*; and (iii)* $ov(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\mathrm{T})$.

Definition 24 (Template Rule Consequence). *The template rule consequence operator,* $\mathbf{TC}_{\mathrm{P},\widehat{\mathrm{I}}}$*, is defined for an F-logic KB* P *and an extended atom set* $\widehat{\mathrm{I}}$*. It takes as input an extended atom set,* $\widehat{\mathrm{J}}$*, and generates a new extended atom set,* $\mathbf{TC}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{J}})$*, as follows:*

$$\left\{ \mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\mathrm{code}} \;\middle|\; \begin{array}{l} \mathsf{c}[\mathsf{m}] \overset{code}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{J}},\ ec(\mathsf{o},\mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{I}}),\ mc(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{I}}); \\ ground(\mathrm{P}) \text{ has a template rule } \mathsf{code(c)}\ @\mathsf{this}[\mathsf{m} \rightarrow \mathsf{v}] :- \mathsf{B} \\ \text{and for every literal } \mathsf{L} \in \mathsf{B}_{\|\mathsf{o}}: \\ \qquad (i) \text{ if } \mathsf{L} \text{ is positive, then } val^{\mathbf{b}}_{\widehat{\mathrm{J}}}(\mathsf{L}) = \mathbf{t}, \text{ and} \\ \qquad (ii) \text{ if } \mathsf{L} \text{ is negative, then } val^{\mathbf{b}}_{\widehat{\mathrm{I}}}(\mathsf{L}) = \mathbf{t}. \end{array} \right\}$$

The template rule consequence operator is used to derive new facts as a result of code inheritance. It is similar to the regular rule consequence operator except that the regular rule consequence operator is applied to all regular rules whereas the template rule consequence operator is applied only to those selected template rules that could be inherited according to our inheritance semantics.

Given an object o and a template rule, $\mathsf{code(c)}$ @this$[\mathsf{m} \to \mathsf{v}]$:− B, which defines instance method m for class c, we first need to decide whether o can inherit this instance method definition from c. If so, then we will bind this instance method definition for o and evaluate it (note that $\mathsf{B}_{\|\mathsf{o}}$ is obtained from B by substituting o for every occurrence of @this in B). If the rule body is satisfied in the context of o, we will derive $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{code}}$ to represent the fact that $\mathsf{m} \to \mathsf{v}$ is established for o by inheritance of an instance method definition from c.

We can decide whether object o can inherit the definitions of instance method m from class c by looking up the two sets $\widehat{\mathrm{J}}$ and $\mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{I}})$. In particular, such code inheritance can happen only if the following conditions are true: (i) $\mathsf{c}[\mathsf{m}]$ is a code inheritance candidate for o ($\mathsf{c}[\mathsf{m}] \overset{code}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{J}}$); (ii) method m is not explicitly defined at o ($ec(\mathsf{o},\mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{I}})$); and (iii) there is no multiple inheritance conflict ($mc(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{I}})$).

Definition 25 (Inheritance Consequence). *The inheritance consequence operator,* $\mathbf{IC}_{\mathrm{P},\widehat{\mathrm{I}}}$*, where* P *is an F-logic KB and* $\widehat{\mathrm{I}}$ *is an extended atom set, takes as input an extended atom set,* $\widehat{\mathrm{J}}$*, and generates a new extended atom set as follows:*

$$\mathbf{IC}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{J}}) \overset{\mathrm{def}}{=} \mathbf{IC}^t(\widehat{\mathrm{J}}) \cup \mathbf{IC}^c_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{J}}) \cup \mathbf{IC}^i_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{J}}), \quad \textit{where}$$

$$\mathbf{IC}^t(\widehat{\mathrm{J}}) = \left\{ \mathsf{o}:\mathsf{c} \;\middle|\; \exists \mathsf{x} \textit{ such that } \mathsf{o}:\mathsf{x} \in \widehat{\mathrm{J}}, \mathsf{x}::\mathsf{c} \in \widehat{\mathrm{J}} \right\} \cup \left\{ \mathsf{s}::\mathsf{c} \;\middle|\; \exists \mathsf{x} \textit{ such that } \mathsf{s}::\mathsf{x} \in \widehat{\mathrm{J}}, \mathsf{x}::\mathsf{c} \in \widehat{\mathrm{J}} \right\}$$

$$\mathbf{IC}^c_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{J}}) = \left\{ \mathsf{c}[\mathsf{m}] \overset{val}{\rightsquigarrow} \mathsf{o} \;\middle|\; \begin{array}{l} \mathsf{o}:\mathsf{c} \in \widehat{\mathrm{J}},\ \mathsf{c} \neq \mathsf{o},\ \mathsf{c}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}} \in \widehat{\mathrm{J}}, \textit{ and} \\ ov(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{I}}) \end{array} \right\} \cup \left\{ \mathsf{c}[\mathsf{m}] \overset{code}{\rightsquigarrow} \mathsf{o} \;\middle|\; \begin{array}{l} \mathsf{o}:\mathsf{c} \in \widehat{\mathrm{J}},\ \mathsf{c} \neq \mathsf{o}, \textit{ there is a template rule in} \\ \textit{ground}(\mathrm{P}) \textit{ which specifies the instance method } \mathsf{m} \\ \textit{for class } \mathsf{c}, \textit{ and } ov(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{I}}) \end{array} \right\}$$

$$\mathbf{IC}^i_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{J}}) = \left\{ \mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}} \;\middle|\; \begin{array}{l} \mathsf{c}[\mathsf{m}] \overset{val}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{J}},\ \mathsf{c}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}} \in \widehat{\mathrm{J}}, \\ ec(\mathsf{o},\mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{I}}), \textit{ and } mc(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{I}}) \end{array} \right\}$$

The inheritance consequence operator, $\mathbf{IC}_{\mathrm{P},\widehat{\mathrm{I}}}$, is the union of three operators: $\mathbf{IC}^t$, $\mathbf{IC}^c_{\mathrm{P},\widehat{\mathrm{I}}}$, and $\mathbf{IC}^i_{\mathrm{P},\widehat{\mathrm{I}}}$. The operator $\mathbf{IC}^t$ is used to perform transitive closure of the class hierarchy, including class membership and subclass relationship. Value and code inheritance candidates are computed by the operator $\mathbf{IC}^c_{\mathrm{P},\widehat{\mathrm{I}}}$, which relies on the overriding information provided by $\mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{I}})$. Finally, the operator $\mathbf{IC}^i_{\mathrm{P},\widehat{\mathrm{I}}}$ derives new facts by value inheritance. This operator also relies on the information provided by $\mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{I}})$.

Definition 26 (KB Completion). *The KB completion operator,* $\mathbf{T}_{\mathrm{P},\widehat{\mathrm{I}}}$*, where* P *is an F-logic KB and* $\widehat{\mathrm{I}}$ *an extended atom set, takes as input an extended atom set,* $\widehat{\mathrm{J}}$*, and generates a new extended atom set as follows:*

$$\mathbf{T}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{J}}) \stackrel{\text{def}}{=} \mathbf{RC}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{J}}) \cup \mathbf{TC}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{J}}) \cup \mathbf{IC}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{J}})$$

The KB completion operator is the union of the regular rule consequence operator, the template rule consequence operator, and the inheritance consequence operator. It derives new "explicit" method definitions (via regular rules in the KB), new inherited facts (by value and code inheritance), plus inheritance candidacy information that is used to decide which facts to inherit in the future.

We have the following lemma regarding the *monotonicity* property of the operators that we have defined so far.

Lemma 5. *Suppose* P *and* $\widehat{\mathrm{I}}$ *are fixed. Then the following operators are monotonic:* $\mathbf{RC}_{\mathrm{P},\widehat{\mathrm{I}}}$, $\mathbf{IB}_{\mathrm{P}}$, $\mathbf{TC}_{\mathrm{P},\widehat{\mathrm{I}}}$, $\mathbf{IC}^t$, $\mathbf{IC}^c_{\mathrm{P},\widehat{\mathrm{I}}}$, $\mathbf{IC}^i_{\mathrm{P},\widehat{\mathrm{I}}}$, $\mathbf{IC}_{\mathrm{P},\widehat{\mathrm{I}}}$, $\mathbf{T}_{\mathrm{P},\widehat{\mathrm{I}}}$.

Given an F-logic KB P, the set of all subsets of the extended Herbrand base $\widehat{\mathcal{HB}}_{\mathrm{P}}$ constitutes a complete lattice where the partial ordering is defined by set inclusion. Therefore, any monotonic operator, Φ, defined on this lattice has a unique least fixpoint lfp(Φ) [28].

Definition 27 (Alternating Fixpoint). *The alternating fixpoint operator,* Ψ_{P}*, for an F-logic KB* P *takes as input an extended atom set,* $\widehat{\mathrm{I}}$*, and generates a new extended atom set as follows:* $\Psi_{\mathrm{P}}(\widehat{\mathrm{I}}) \stackrel{\text{def}}{=} lfp(\mathbf{T}_{\mathrm{P},\widehat{\mathrm{I}}})$.

Definition 28 (F-logic Fixpoint). *The F-logic fixpoint operator,* $\mathbf{F}_{\mathrm{P}}$*, where* P *is an F-logic KB, takes as input an extended atom set,* $\widehat{\mathrm{I}}$*, and generates a new extended atom set as follows:* $\mathbf{F}_{\mathrm{P}}(\widehat{\mathrm{I}}) \stackrel{\text{def}}{=} \Psi_{\mathrm{P}}(\Psi_{\mathrm{P}}(\widehat{\mathrm{I}}))$.

Lemma 6. *Let* $\widehat{\mathrm{I}}$ *be an extended atom set of an F-logic KB* P*,* $\widehat{\mathrm{J}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{I}})$*. Then:*

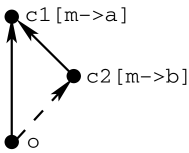
(1) *for all* c, m, o*: if* $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{J}}$ *then* $\mathsf{c} \neq \mathsf{o}$.
(2) *for all* c, m, o*: if* $\mathsf{c}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o} \in \widehat{\mathrm{J}}$ *then* $\mathsf{c} \neq \mathsf{o}$.
(3) *for all* o, m, v, c*:* $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}} \in \widehat{\mathrm{J}}$ *iff* $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}} \in \mathbf{IC}^i_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{J}})$.
(4) *for all* o, m, v, c*:* $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{code}} \in \widehat{\mathrm{J}}$ *iff* $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{code}} \in \mathbf{TC}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{J}})$.
(5) *for all* o, m, v, c*: if* $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}} \in \widehat{\mathrm{J}}$ *then* $\mathsf{c} \neq \mathsf{o}$.
(6) *for all* o, m, v, c*: if* $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{code}} \in \widehat{\mathrm{J}}$ *then* $\mathsf{c} \neq \mathsf{o}$.

Lemma 7. Ψ_{P} *is antimonotonic when* P *is fixed.*

Lemma 8. $\mathbf{F}_{\mathrm{P}}$ *is monotonic when* P *is fixed.*

8 Stable Object Models

Although the inheritance postulates rule out a large number of unintended interpretations of F-logic KBs, they still do not restrict object models tightly enough. There can be *unfounded* object models that do not match the common intuition behind inference. This problem is illustrated with the following example.



$\mathsf{o} : \mathsf{c}_1.$
$\mathsf{c}_2 :: \mathsf{c}_1.$
$\mathsf{c}_1[\mathsf{m} \rightarrow \mathsf{a}].$
$\mathsf{c}_2[\mathsf{m} \rightarrow \mathsf{b}].$
$\mathsf{o} : \mathsf{c}_2 \; :- \; \mathsf{o}[\mathsf{m} \rightarrow \mathsf{b}].$

Fig. 8. Unfounded Inference

Example 2. Consider the KB in Figure 8 and the following *two-valued* object model $\mathcal{I} = \langle\, \mathrm{T}; \emptyset \,\rangle$, where

$$\mathrm{T} = \{\mathsf{o} : \mathsf{c}_1, \mathsf{c}_2 :: \mathsf{c}_1, \mathsf{o} : \mathsf{c}_2, \mathsf{c}_1[\mathsf{m} \rightarrow \mathsf{a}]_{\mathrm{ex}}, \mathsf{c}_2[\mathsf{m} \rightarrow \mathsf{b}]_{\mathrm{ex}}, \mathsf{o}[\mathsf{m} \rightarrow \mathsf{b}]^{\mathsf{c}_2}_{\mathrm{val}}\}.$$

Clearly, $\mathcal{I}$ satisfies the regular rules of the KB in Figure 8 and all the inheritance postulates introduced in Section 5. However, we should note that in $\mathcal{I}$ the truth of $\mathsf{o} : \mathsf{c}_2$ and $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{b}]^{\mathsf{c}_2}_{\mathrm{val}}$ is not *well-founded*. Indeed, the truth of $\mathsf{o} : \mathsf{c}_2$ depends on $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{b}]$ being satisfied in the body of the last rule. Since $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{b}]$ does not appear in the head of any rule, there is no way for $\mathsf{m} \rightarrow \mathsf{b}$ to be explicitly defined for o. So the satisfaction of $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{b}]$ depends on o inheriting $\mathsf{m} \rightarrow \mathsf{b}$ from c_2, since c_2 is the only class that has an explicit definition for $\mathsf{m} \rightarrow \mathsf{b}$. However, o can inherit $\mathsf{m} \rightarrow \mathsf{b}$ from c_2 only if the truth of $\mathsf{o} : \mathsf{c}_2$ can be established first. We see that the inferences of $\mathsf{o} : \mathsf{c}_2$ and $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{b}]^{\mathsf{c}_2}_{\mathrm{val}}$ depend on each other like chicken and egg. Therefore, we should not conclude that both $\mathsf{o} : \mathsf{c}_2$ and $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{b}]^{\mathsf{c}_2}_{\mathrm{val}}$ are true as implied by the KB and our semantics for inheritance.

To overcome the problem, we will introduce a special class of *stable* object models, which do not exhibit the aforementioned anomaly.

Definition 29. *Given an interpretation $\mathcal{I} = \langle\, \mathrm{T}; \mathrm{U} \,\rangle$ of an F-logic KB* P*, let $\widehat{\mathrm{T}}_{\mathcal{I}}$ be the extended atom set constructed by augmenting* T *with the set of auxiliary atoms corresponding to the strong inheritance candidates in $\mathcal{I}$. Let $\widehat{\mathrm{U}}_{\mathcal{I}}$ be the extended atom set constructed by augmenting $\mathrm{T} \cup \mathrm{U}$ with the set of auxiliary atoms corresponding to the strong and weak inheritance candidates in $\mathcal{I}$. More precisely, we define $\widehat{\mathrm{T}}_{\mathcal{I}} \overset{\mathrm{def}}{=} \mathrm{T} \cup \mathrm{A}$, $\widehat{\mathrm{U}}_{\mathcal{I}} \overset{\mathrm{def}}{=} \mathrm{T} \cup \mathrm{U} \cup \mathrm{B}$, where*

$$\mathrm{A} = \{\mathsf{c}[\mathsf{m}] \overset{val}{\rightsquigarrow} \mathsf{o} \mid \mathsf{c}[\mathsf{m}] \overset{s.val}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}\} \cup \{\mathsf{c}[\mathsf{m}] \overset{code}{\rightsquigarrow} \mathsf{o} \mid \mathsf{c}[\mathsf{m}] \overset{s.code}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}\}$$

$$\mathrm{B} = \{\mathsf{c}[\mathsf{m}] \overset{val}{\rightsquigarrow} \mathsf{o} \mid \mathsf{c}[\mathsf{m}] \overset{s.val}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o} \textit{ or } \mathsf{c}[\mathsf{m}] \overset{w.val}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}\} \cup$$
$$\{\mathsf{c}[\mathsf{m}] \overset{code}{\rightsquigarrow} \mathsf{o} \mid \mathsf{c}[\mathsf{m}] \overset{s.code}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o} \textit{ or } \mathsf{c}[\mathsf{m}] \overset{w.code}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}\}$$

Definition 30 (Stable Interpretation). *Let* $\mathcal{I} = \langle\, \mathrm{T}; \mathrm{U} \,\rangle$ *be an interpretation of an F-logic KB* P*.* $\mathcal{I}$ *is called a stable interpretation of* P*, if* $\widehat{\mathrm{T}}_{\mathcal{I}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{U}}_{\mathcal{I}})$ *and* $\widehat{\mathrm{U}}_{\mathcal{I}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$*.*

Our definition of stable interpretations is closely related to that of stable models introduced in [16,35]. The idea is that given an interpretation $\mathcal{I}$ of an F-logic KB P, we first resolve all negative premises using the information in $\mathcal{I}$. The result is a residual positive KB without negation. Then $\mathcal{I}$ is said to be stable if and only if $\mathcal{I}$ can reproduce itself via the least fixpoint computation over the residual KB. This is how stable interpretations can prevent the kind of unfounded inference illustrated in Example 2.

We should note that Definition 30 only requires that a stable interpretation $\mathcal{I} = \langle\, \mathrm{T}; \mathrm{U} \,\rangle$ satisfy a certain computational property with respect to Ψ_{P}, i.e., $\widehat{\mathrm{T}}_{\mathcal{I}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{U}}_{\mathcal{I}})$ and $\widehat{\mathrm{U}}_{\mathcal{I}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$. In fact, it turns out that a stable interpretation of an F-logic KB P satisfies all the regular rules and template rules in P as well as all the core and cautious inheritance postulates.

Theorem 1. *Let* $\mathcal{I} = \langle\, \mathrm{T}; \mathrm{U} \,\rangle$ *be a stable interpretation of an F-logic KB* P*. Then* $\mathcal{I}$ *is an object model of* P*. Moreover,* $\mathcal{I}$ *satisfies the cautious ISA transitivity constraint and the cautious inheritance constraint.*

Proof. By Definition 20, and by Propositions 1, 2, 3, 4, 5, 6, and 7.

Since, by Theorem 1, stable interpretations satisfy all the requirements for object models, we will start referring to stable interpretations as *stable object models*.

There is an interesting correspondence between stable object models and fixpoints of $\mathbf{F}_{\mathrm{P}}$. On one hand, it can be easily seen that stable object models are essentially fixpoints of $\mathbf{F}_{\mathrm{P}}$. Let $\mathcal{I} = \langle\, \mathrm{T}; \mathrm{U} \,\rangle$ be a stable object model of an F-logic KB P. Then $\widehat{\mathrm{T}}_{\mathcal{I}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{U}}_{\mathcal{I}})$ and $\widehat{\mathrm{U}}_{\mathcal{I}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$, by Definition 30. It follows that $\widehat{\mathrm{T}}_{\mathcal{I}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{U}}_{\mathcal{I}}) = \Psi_{\mathrm{P}}(\Psi_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})) = \mathbf{F}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$ and so $\widehat{\mathrm{T}}_{\mathcal{I}}$ is a fixpoint of $\mathbf{F}_{\mathrm{P}}$. Similarly, $\widehat{\mathrm{U}}_{\mathcal{I}}$ is also a fixpoint of $\mathbf{F}_{\mathrm{P}}$. Moreover, $\widehat{\mathrm{T}}_{\mathcal{I}} \subseteq \widehat{\mathrm{U}}_{\mathcal{I}}$ by Definition 29.

The following theorem shows that stable object models can be constructed using *certain* fixpoints of $\mathbf{F}_{\mathrm{P}}$.

Theorem 2. *Let* P *be an F-logic KB,* $\widehat{\mathrm{J}}$ *a fixpoint of* $\mathbf{F}_{\mathrm{P}}$*,* $\widehat{\mathrm{K}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{J}})$*, and* $\widehat{\mathrm{J}} \subseteq \widehat{\mathrm{K}}$*. Then* $\mathcal{I} = \langle\, \pi(\widehat{\mathrm{J}}); \pi(\widehat{\mathrm{K}}) - \pi(\widehat{\mathrm{J}}) \,\rangle$*, where* π *is the projection function defined in Section 7, is a stable object model of* P*.*

Proof. Let $\mathrm{T} = \pi(\widehat{\mathrm{J}})$ and $\mathrm{U} = \pi(\widehat{\mathrm{K}}) - \pi(\widehat{\mathrm{J}})$. Thus $\mathcal{I} = \langle\, \mathrm{T}; \mathrm{U} \,\rangle$. Since $\widehat{\mathrm{J}} \subseteq \widehat{\mathrm{K}}$, it follows that $\pi(\widehat{\mathrm{J}}) \subseteq \pi(\widehat{\mathrm{K}})$, and so $\mathrm{T} \cup \mathrm{U} = \pi(\widehat{\mathrm{K}})$. To show that $\mathcal{I}$ is a stable object model of P, we need to establish that $\widehat{\mathrm{T}}_{\mathcal{I}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{U}}_{\mathcal{I}})$ and $\widehat{\mathrm{U}}_{\mathcal{I}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$. Since $\widehat{\mathrm{J}}$ is a fixpoint of $\mathbf{F}_{\mathrm{P}}$ and $\widehat{\mathrm{K}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{J}})$, it follows that $\widehat{\mathrm{J}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{K}})$, by Definition 28. Therefore, if we can show that $\widehat{\mathrm{T}}_{\mathcal{I}} = \widehat{\mathrm{J}}$ and $\widehat{\mathrm{U}}_{\mathcal{I}} = \widehat{\mathrm{K}}$, then it follows that $\mathcal{I}$ is a stable object model of P.

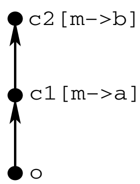
Since $\widehat{\mathrm{J}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{K}}) = \mathrm{lfp}(\mathbf{T}_{\mathrm{P},\widehat{\mathrm{K}}})$ and $\widehat{\mathrm{K}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{J}}) = \mathrm{lfp}(\mathbf{T}_{\mathrm{P},\widehat{\mathrm{J}}})$, we can derive the following equations, by Definitions 26 and 25:

$$\widehat{\mathrm{J}} = \mathbf{RC}_{\mathrm{P},\widehat{\mathrm{K}}}(\widehat{\mathrm{J}}) \cup \mathbf{TC}_{\mathrm{P},\widehat{\mathrm{K}}}(\widehat{\mathrm{J}}) \cup \mathbf{IC}^{t}(\widehat{\mathrm{J}}) \cup \mathbf{IC}^{c}_{\mathrm{P},\widehat{\mathrm{K}}}(\widehat{\mathrm{J}}) \cup \mathbf{IC}^{i}_{\mathrm{P},\widehat{\mathrm{K}}}(\widehat{\mathrm{J}})$$

$$\widehat{\mathrm{K}} = \mathbf{RC}_{\mathrm{P},\widehat{\mathrm{J}}}(\widehat{\mathrm{K}}) \cup \mathbf{TC}_{\mathrm{P},\widehat{\mathrm{J}}}(\widehat{\mathrm{K}}) \cup \mathbf{IC}^{t}(\widehat{\mathrm{K}}) \cup \mathbf{IC}^{c}_{\mathrm{P},\widehat{\mathrm{J}}}(\widehat{\mathrm{K}}) \cup \mathbf{IC}^{i}_{\mathrm{P},\widehat{\mathrm{J}}}(\widehat{\mathrm{K}})$$

First we will show that for all $\mathsf{c},\mathsf{m},\mathsf{o}$: $\mathsf{c}[\mathsf{m}]\overset{val}{\rightsquigarrow}\mathsf{o} \in \widehat{\mathrm{J}}$ iff $\mathsf{c}[\mathsf{m}]\overset{s.val}{\rightsquigarrow}_{\mathcal{I}}\mathsf{o}$. Indeed, $\mathsf{c}[\mathsf{m}]\overset{val}{\rightsquigarrow}\mathsf{o} \in \widehat{\mathrm{J}}$, iff $\mathsf{c}[\mathsf{m}]\overset{val}{\rightsquigarrow}\mathsf{o} \in \mathbf{IC}^{c}_{\mathrm{P},\widehat{\mathrm{K}}}(\widehat{\mathrm{J}})$, iff $\mathsf{c} \neq \mathsf{o}$, $\mathsf{o}:\mathsf{c} \in \widehat{\mathrm{J}}$, $\mathsf{c}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}} \in \widehat{\mathrm{J}}$ for some v, and $ov(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{K}})$, by Definition 25, iff $\mathsf{c} \neq \mathsf{o}$, $\mathsf{o}:\mathsf{c} \in \pi(\widehat{\mathrm{J}})$, $\mathsf{c}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}} \in \pi(\widehat{\mathrm{J}})$ for some v, and $ov(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\pi(\widehat{\mathrm{K}}))$, iff $\mathsf{c} \neq \mathsf{o}$, $\mathsf{o}:\mathsf{c} \in \mathrm{T}$, $\mathsf{c}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}} \in \mathrm{T}$ for some v, and $ov(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\mathrm{T} \cup \mathrm{U})$, iff $\mathsf{c}[\mathsf{m}]\overset{s.val}{\rightsquigarrow}_{\mathcal{I}}\mathsf{o}$, by Lemma 4. Similarly, we can also show that (i) for all $\mathsf{c},\mathsf{m},\mathsf{o}$: $\mathsf{c}[\mathsf{m}]\overset{code}{\rightsquigarrow}\mathsf{o} \in \widehat{\mathrm{J}}$ iff $\mathsf{c}[\mathsf{m}]\overset{s.code}{\rightsquigarrow}_{\mathcal{I}}\mathsf{o}$; and (ii) for all $\mathsf{c},\mathsf{m},\mathsf{o}$: $\mathsf{c}[\mathsf{m}]\overset{val}{\rightsquigarrow}\mathsf{o} \in \widehat{\mathrm{K}}$ or $\mathsf{c}[\mathsf{m}]\overset{code}{\rightsquigarrow}\mathsf{o} \in \widehat{\mathrm{K}}$ iff $\mathsf{c}[\mathsf{m}]\rightsquigarrow_{\mathcal{I}}\mathsf{o}$. Therefore, it follows that $\widehat{\mathrm{T}}_{\mathcal{I}} = \widehat{\mathrm{J}}$ and $\widehat{\mathrm{U}}_{\mathcal{I}} = \widehat{\mathrm{K}}$ by Definition 29. This completes the proof.

It is worth pointing out that the condition $\widehat{\mathrm{J}} \subseteq \widehat{\mathrm{K}}$ in Theorem 2 is *not* necessary for constructing a stable object model out of the extended sets $\widehat{\mathrm{J}}$ and $\widehat{\mathrm{K}}$. In fact, the following example shows that there is an F-logic KB P such that $\widehat{\mathrm{J}}$ is a fixpoint of $\mathbf{F}_{\mathrm{P}}$, $\widehat{\mathrm{K}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{J}})$, and $\widehat{\mathrm{J}} \nsubseteq \widehat{\mathrm{K}}$, but $\mathcal{I} = \langle\, \pi(\widehat{\mathrm{J}}); \pi(\widehat{\mathrm{K}}) - \pi(\widehat{\mathrm{J}})\,\rangle$ is a stable object model of P.



$\mathsf{o}:\mathsf{c}_1.$
$\mathsf{c}_1[\mathsf{m} \rightarrow \mathsf{a}].$
$\mathsf{c}_1::\mathsf{c}_2 \;:-\; \mathsf{c}_2[\mathsf{m} \rightarrow \mathsf{b}].$
$\mathsf{c}_2[\mathsf{m} \rightarrow \mathsf{b}] \;:-\; \neg\, \mathsf{c}_1::\mathsf{c}_2.$
$\mathsf{c}_1::\mathsf{c}_2 \;:-\; \mathsf{o}[\mathsf{m} \rightarrow \mathsf{a}].$
$\mathsf{c}_2[\mathsf{m} \rightarrow \mathsf{b}] \;:-\; \mathsf{o}[\mathsf{m} \rightarrow \mathsf{a}].$

Fig. 9. Constructive Fixpoints

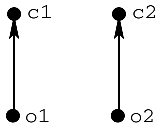
Example 3. Consider the F-logic KB P in Figure 9 and the following two extended sets $\widehat{\mathrm{J}}$ and $\widehat{\mathrm{K}}$:

$$\widehat{\mathrm{J}} = \{\mathsf{o}:\mathsf{c}_1, \mathsf{c}_1[\mathsf{m} \rightarrow \mathsf{a}]_{\mathrm{ex}}, \mathsf{o}[\mathsf{m} \rightarrow \mathsf{a}]^{\mathsf{c}_1}_{\mathrm{val}}, \mathsf{c}_1::\mathsf{c}_2, \mathsf{c}_2[\mathsf{m} \rightarrow \mathsf{b}]_{\mathrm{ex}}\} \cup \{\mathsf{c}_1[\mathsf{m}]\overset{val}{\rightsquigarrow}\mathsf{o}, \mathsf{c}_2[\mathsf{m}]\overset{val}{\rightsquigarrow}\mathsf{o}\}$$

$$\widehat{\mathrm{K}} = \{\mathsf{o}:\mathsf{c}_1, \mathsf{c}_1[\mathsf{m} \rightarrow \mathsf{a}]_{\mathrm{ex}}\} \cup \{\mathsf{c}_1[\mathsf{m}]\overset{val}{\rightsquigarrow}\mathsf{o}\}$$

One can verify that $\widehat{\mathrm{J}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{K}})$, $\widehat{\mathrm{K}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{J}})$, and so $\widehat{\mathrm{J}}$ is a fixpoint of Ψ_{P}. Moreover, $\pi(\widehat{\mathrm{J}}) = \{\mathsf{o}:\mathsf{c}_1, \mathsf{c}_1[\mathsf{m} \rightarrow \mathsf{a}]_{\mathrm{ex}}, \mathsf{o}[\mathsf{m} \rightarrow \mathsf{a}]^{\mathsf{c}_1}_{\mathrm{val}}, \mathsf{c}_1::\mathsf{c}_2, \mathsf{c}_2[\mathsf{m} \rightarrow \mathsf{b}]_{\mathrm{ex}}\}$, $\pi(\widehat{\mathrm{K}}) - \pi(\widehat{\mathrm{J}}) = \emptyset$. We can also verify that $\mathcal{I} = \langle\, \pi(\widehat{\mathrm{J}}); \pi(\widehat{\mathrm{K}}) - \pi(\widehat{\mathrm{J}})\,\rangle$ is a stable object model of P. But clearly $\widehat{\mathrm{J}} - \widehat{\mathrm{K}} \neq \emptyset$. Thus $\widehat{\mathrm{J}} \nsubseteq \widehat{\mathrm{K}}$.

Another interesting question is whether we can *always* construct stable object models of an F-logic KB P from fixpoints of Ψ_{P}. The answer turns out to be *no*. The following example shows that some F-logic KBs may have fixpoints from which we cannot even construct an object model for that KB.



$o_1 : c_1.$
$o_2 : c_2.$
$c_1[m \rightarrow a] \ :- \ \neg\, o_1[m \rightarrow a].$
$c_2[m \rightarrow b] \ :- \ \neg\, o_2[m \rightarrow b].$
$c_2[m \rightarrow c] \ :- \ o_1[m \rightarrow a], o_2[m \rightarrow b].$

Fig. 10. Nonconstructive Fixpoints

Example 4. Consider the F-logic KB P in Figure 10 and the following two extended sets $\widehat{J}$ and $\widehat{K}$:

$$\widehat{J} = \{o_1 : c_1, o_2 : c_2, c_1[m \rightarrow a]_{ex}, o_1[m \rightarrow a]^{c_1}_{val}\} \cup \{c_1[m] \overset{val}{\rightsquigarrow} o_1\}$$
$$\widehat{K} = \{o_1 : c_1, o_2 : c_2, c_2[m \rightarrow b]_{ex}, o_2[m \rightarrow b]^{c_2}_{val}\} \cup \{c_2[m] \overset{val}{\rightsquigarrow} o_2\}$$

One can verify that $\widehat{J} = \Psi_P(\widehat{K})$, $\widehat{K} = \Psi_P(\widehat{J})$, and so $\widehat{J}$ is a fixpoint of Ψ_P. However,

$$\pi(\widehat{J}) = \{o_1 : c_1, o_2 : c_2, c_1[m \rightarrow a]_{ex}, o_1[m \rightarrow a]^{c_1}_{val}\}$$
$$\pi(\widehat{K}) - \pi(\widehat{J}) = \{c_2[m \rightarrow b]_{ex}, o_2[m \rightarrow b]^{c_2}_{val}\}$$

It is easy to check that the interpretation $\mathcal{I} = \langle\, \pi(\widehat{J}); \pi(\widehat{K}) - \pi(\widehat{J})\,\rangle$ is not even an object model of P, because $\mathcal{I}$ does not satisfy the KB in Figure 10, namely, the last rule of the KB in Figure 10. But if we remove the last rule from the KB in Figure 10 then $\mathcal{I}$ would be an object model of this new KB, but *not* a stable object model.

9 Cautious Object Models

Here we introduce a special class of stable object models, called *cautious object model*. These models have an important property that every F-logic KB has a unique cautious object model. This notion relates stable object models and the fixpoint computation of $\mathbf{F}_P$. Recall that $\mathbf{F}_P \overset{\text{def}}{=} \Psi_P \cdot \Psi_P$ is monotonic and hence has a unique least fixpoint, denoted $\text{lfp}(\mathbf{F}_P)$.

Definition 31 (Cautious Object Model). *The cautious object model, $\mathcal{M}$, of an F-logic KB* P *is defined as follows: $\mathcal{M} = \langle\, T; U\,\rangle$, where*

$$T = \pi(\mathit{lfp}(\mathbf{F}_P))$$
$$U = \pi(\Psi_P(\mathit{lfp}(\mathbf{F}_P))) - \pi(\mathit{lfp}(\mathbf{F}_P))$$

and π is the projection function defined in Section 7.

Next we will list several important properties of cautious object models. First we need to introduce the notations used for representing the intermediate results of the least fixpoint computation of Ψ_P and $\mathbf{F}_P$.

Definition 32. *Let α range over all countable ordinals. We define the following extended atom sets for an F-logic KB* P*:*

$$\begin{array}{lll}
\widehat{\mathrm{T}}_0 = \emptyset & \widehat{\mathrm{U}}_0 = \Psi_{\mathrm{P}}(\widehat{\mathrm{T}}_0) & \textit{for limit ordinal } 0 \\
\widehat{\mathrm{T}}_\alpha = \Psi_{\mathrm{P}}(\widehat{\mathrm{U}}_{\alpha-1}) & \widehat{\mathrm{U}}_\alpha = \Psi_{\mathrm{P}}(\widehat{\mathrm{T}}_\alpha) & \textit{for successor ordinal } \alpha \\
\widehat{\mathrm{T}}_\alpha = \bigcup_{\beta<\alpha} \widehat{\mathrm{T}}_\beta & \widehat{\mathrm{U}}_\alpha = \Psi_{\mathrm{P}}(\widehat{\mathrm{T}}_\alpha) & \textit{for limit ordinal } \alpha \neq 0 \\
\widehat{\mathrm{T}}_\infty = \bigcup_{\alpha} \widehat{\mathrm{T}}_\alpha & \widehat{\mathrm{U}}_\infty = \Psi_{\mathrm{P}}(\widehat{\mathrm{T}}_\infty) &
\end{array}$$

Lemma 9. *Let α and β range over all countable ordinals. Then:*

(1) *for all α, β: if $\alpha < \beta$ then* $\widehat{\mathrm{T}}_\alpha \subseteq \widehat{\mathrm{T}}_\beta$
(2) $\widehat{\mathrm{T}}_\infty = \mathit{lfp}(\mathbf{F}_{\mathrm{P}})$
(3) *for all α:* $\widehat{\mathrm{T}}_\alpha \subseteq \widehat{\mathrm{T}}_\infty$
(4) $\widehat{\mathrm{U}}_\infty = \mathit{gfp}(\mathbf{F}_{\mathrm{P}})$
(5) *for all α:* $\widehat{\mathrm{U}}_\alpha \supseteq \widehat{\mathrm{U}}_\infty$
(6) *for all α, β: if $\alpha < \beta$ then* $\widehat{\mathrm{U}}_\alpha \supseteq \widehat{\mathrm{U}}_\beta$
(7) *for all α:* $\widehat{\mathrm{T}}_\alpha \subseteq \widehat{\mathrm{U}}_\alpha$
(8) *for all α, β:* $\widehat{\mathrm{T}}_\alpha \subseteq \widehat{\mathrm{U}}_\beta$

From Definition 31, Lemma 9, and from the definition of the projection function π in Section 7, we obtain a new characterization of the cautious object model.

Lemma 10. *If $\mathcal{M}$ is the cautious object model of an F-logic KB* P *then*

$$\mathcal{M} = \langle\, \pi(\widehat{\mathrm{T}}_\infty); \pi(\widehat{\mathrm{U}}_\infty) - \pi(\widehat{\mathrm{T}}_\infty) \,\rangle = \langle\, \pi(\widehat{\mathrm{T}}_\infty); \pi(\widehat{\mathrm{U}}_\infty - \widehat{\mathrm{T}}_\infty) \,\rangle$$

Let α be a countable ordinal. Given a pair of extended atom sets $\widehat{\mathrm{T}}_\alpha$ and $\widehat{\mathrm{U}}_\alpha$, we know that $\widehat{\mathrm{T}}_\alpha \subseteq \widehat{\mathrm{U}}_\alpha$ and so $\pi(\widehat{\mathrm{T}}_\alpha) \subseteq \pi(\widehat{\mathrm{U}}_\alpha)$ by Lemma 9. We can construct an interpretation $\mathcal{I}_\alpha$ as follows: $\mathcal{I}_\alpha = \langle\, \pi(\widehat{\mathrm{T}}_\alpha); \pi(\widehat{\mathrm{U}}_\alpha) - \pi(\widehat{\mathrm{T}}_\alpha) \,\rangle$. Then the set of atoms $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o}$ ($\mathsf{c}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o}$) in $\widehat{\mathrm{T}}_\alpha$ constitutes a *subset* of the set of strong value (code) inheritance candidates in $\mathcal{I}_\alpha$, whereas the set of atoms $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o}$ ($\mathsf{c}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o}$) in $\widehat{\mathrm{U}}_\alpha$ constitutes a *superset* of the set of strong and weak value (code) inheritance candidates in $\mathcal{I}_\alpha$. In other words, $\widehat{\mathrm{T}}_\alpha$ *underestimates* inheritance information whereas $\widehat{\mathrm{U}}_\alpha$ *overestimates* inheritance information. The following lemma illustrates this book-keeping mechanism of the alternating fixpoint computation.

Lemma 11. *Let $\mathcal{I}_\alpha = \langle\, \pi(\widehat{\mathrm{T}}_\alpha); \pi(\widehat{\mathrm{U}}_\alpha) - \pi(\widehat{\mathrm{T}}_\alpha) \,\rangle$ where α ranges over all countable ordinals. Then the following statements are true:*

(1) *for all* $\mathsf{c}, \mathsf{m}, \mathsf{o}$*: if* $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{T}}_\alpha$ *then* $\mathsf{c}[\mathsf{m}] \overset{s.val}{\leadsto}_{\mathcal{I}_\alpha} \mathsf{o}$
(2) *for all* $\mathsf{c}, \mathsf{m}, \mathsf{o}$*: if* $\mathsf{c}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o} \in \widehat{\mathrm{T}}_\alpha$ *then* $\mathsf{c}[\mathsf{m}] \overset{s.code}{\leadsto}_{\mathcal{I}_\alpha} \mathsf{o}$
(3) *for all* $\mathsf{c}, \mathsf{m}, \mathsf{o}$*: if* $\mathsf{c}[\mathsf{m}] \overset{s.val}{\leadsto}_{\mathcal{I}_\alpha} \mathsf{o}$ *or* $\mathsf{c}[\mathsf{m}] \overset{w.val}{\leadsto}_{\mathcal{I}_\alpha} \mathsf{o}$ *then* $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{U}}_\alpha$
(4) *for all* $\mathsf{c}, \mathsf{m}, \mathsf{o}$*: if* $\mathsf{c}[\mathsf{m}] \overset{s.code}{\leadsto}_{\mathcal{I}_\alpha} \mathsf{o}$ *or* $\mathsf{c}[\mathsf{m}] \overset{w.code}{\leadsto}_{\mathcal{I}_\alpha} \mathsf{o}$ *then* $\mathsf{c}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o} \in \widehat{\mathrm{U}}_\alpha$

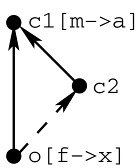
Lemma 12. *Let $\mathcal{M}$ be the cautious object model of an F-logic KB* P. *Then the following statements are true:*

(1) *for all* $\mathsf{c}, \mathsf{m}, \mathsf{o}$*:* $\mathsf{c}[\mathsf{m}] \overset{s.val}{\leadsto}_{\mathcal{M}} \mathsf{o}$ *iff* $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{T}}_{\infty}$
(2) *for all* $\mathsf{c}, \mathsf{m}, \mathsf{o}$*:* $\mathsf{c}[\mathsf{m}] \overset{s.code}{\leadsto}_{\mathcal{M}} \mathsf{o}$ *iff* $\mathsf{c}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o} \in \widehat{\mathrm{T}}_{\infty}$
(3) *for all* $\mathsf{c}, \mathsf{m}, \mathsf{o}$*:* $\mathsf{c}[\mathsf{m}] \overset{s.val}{\leadsto}_{\mathcal{M}} \mathsf{o}$ *or* $\mathsf{c}[\mathsf{m}] \overset{w.val}{\leadsto}_{\mathcal{M}} \mathsf{o}$ *iff* $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{U}}_{\infty}$
(4) *for all* $\mathsf{c}, \mathsf{m}, \mathsf{o}$*:* $\mathsf{c}[\mathsf{m}] \overset{s.code}{\leadsto}_{\mathcal{M}} \mathsf{o}$ *or* $\mathsf{c}[\mathsf{m}] \overset{w.code}{\leadsto}_{\mathcal{M}} \mathsf{o}$ *iff* $\mathsf{c}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o} \in \widehat{\mathrm{U}}_{\infty}$

The lemma above says that $\widehat{\mathrm{T}}_{\infty}$ includes exactly all the strong inheritance candidates while $\widehat{\mathrm{U}}_{\infty}$ includes exactly all the strong and weak inheritance candidates in $\mathcal{M}$. This essentially implies that the cautious object model is indeed a stable object model.

Theorem 3. *The cautious object model $\mathcal{M}$ of an F-logic KB* P *is a stable object model of* P*.*

Proof. Let $\mathcal{M} = \langle\, \mathrm{T}; \mathrm{U} \,\rangle$ be the cautious object model of P. Then $\mathrm{T} = \pi(\widehat{\mathrm{T}}_{\infty})$ and $\mathrm{U} = \pi(\widehat{\mathrm{U}}_{\infty}) - \pi(\widehat{\mathrm{T}}_{\infty})$. So by Definition 29 and Lemma 12, $\widehat{\mathrm{T}}_{\mathcal{M}} = \widehat{\mathrm{T}}_{\infty}$ and $\widehat{\mathrm{U}}_{\mathcal{M}} = \widehat{\mathrm{U}}_{\infty}$. Moreover, $\widehat{\mathrm{U}}_{\infty} = \Psi_{\mathrm{P}}(\widehat{\mathrm{T}}_{\infty})$ and $\widehat{\mathrm{T}}_{\infty} = \Psi_{\mathrm{P}}(\widehat{\mathrm{U}}_{\infty})$ by Definition 32 and Lemma 9. It follows that $\widehat{\mathrm{T}}_{\mathcal{M}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{U}}_{\mathcal{M}})$ and $\widehat{\mathrm{U}}_{\mathcal{M}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{M}})$. Therefore, $\mathcal{M}$ is a stable interpretation and thus a stable object model of P.



	$\mathsf{o} : \mathsf{c}_1.$
	$\mathsf{c}_2 :: \mathsf{c}_1.$
	$\mathsf{o} : \mathsf{c}_2 \;\; :- \;\; \mathsf{o}[\mathsf{m} \rightarrow \mathsf{a}].$
	$\mathsf{o}[\mathsf{f} \rightarrow \mathsf{x}].$
	$\mathsf{c}_1[\mathsf{m} \rightarrow \mathsf{a}].$
$\mathsf{code}(\mathsf{c}_2)$	$\mathsf{@this}[\mathsf{m} \rightarrow \mathsf{b}] \;\; :- \;\; \mathsf{@this}[\mathsf{f} \rightarrow \mathsf{x}].$

Fig. 11. Computation of Cautious Object Models

Example 5. We illustrate the computation of cautious object models using the F-logic KB P in Figure 11. First let T and U denote the following sets of atoms:

$$\mathrm{T} = \{\mathsf{o} : \mathsf{c}_1, \mathsf{c}_2 :: \mathsf{c}_1, \mathsf{o}[\mathsf{f} \rightarrow \mathsf{x}]_{\mathrm{ex}}, \mathsf{c}_1[\mathsf{m} \rightarrow \mathsf{a}]_{\mathrm{ex}}\}$$
$$\mathrm{U} = \{\mathsf{o} : \mathsf{c}_2, \mathsf{o}[\mathsf{m} \rightarrow \mathsf{a}]^{\mathsf{c}_1}_{\mathrm{val}}, \mathsf{o}[\mathsf{m} \rightarrow \mathsf{b}]^{\mathsf{c}_2}_{\mathrm{code}}\}$$

Then the computation process of Ψ_{P} is as follows:

$$\widehat{\mathrm{T}}_0 = \emptyset$$
$$\widehat{\mathrm{T}}_1 = \Psi_{\mathrm{P}}(\widehat{\mathrm{T}}_0) = \mathrm{T} \cup \mathrm{U} \cup \{\mathsf{c}_1[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o}, \mathsf{c}_2[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o}\}$$
$$\widehat{\mathrm{T}}_2 = \Psi_{\mathrm{P}}(\widehat{\mathrm{T}}_1) = \mathrm{T}$$
$$\widehat{\mathrm{T}}_3 = \Psi_{\mathrm{P}}(\widehat{\mathrm{T}}_2) = \widehat{\mathrm{T}}_1$$
$$\widehat{\mathrm{T}}_4 = \Psi_{\mathrm{P}}(\widehat{\mathrm{T}}_3) = \widehat{\mathrm{T}}_2$$

Therefore, $\mathrm{lfp}(\mathbf{F}_{\mathrm{P}}) = \widehat{\mathrm{T}}_2$ and $\Psi_{\mathrm{P}}(\mathrm{lfp}(\mathbf{F}_{\mathrm{P}})) = \widehat{\mathrm{T}}_1$, and so the cautious object model of the KB in Figure 11 is $\langle\, \mathrm{T}; \mathrm{U} \,\rangle$.

Theorem 3 gives a procedural characterization of the cautious object model, i.e., it is essentially defined as the least fixpoint of the extended alternating fixpoint computation. Next we will present two additional characterizations of the cautious object model semantics.

First, by comparing the amount of "definite" information, i.e., truth *and* falsehood, that is contained in different stable object models of an F-logic KB P, we can define a partial order, called *information ordering*, among stable object models.

Definition 33 (Information Ordering). *Let* $\mathcal{I}_1 = \langle \mathrm{P}_1; \mathrm{Q}_1 \rangle$, $\mathcal{I}_2 = \langle \mathrm{P}_2; \mathrm{Q}_2 \rangle$ *be two stable object models of an F-logic KB* P, $\mathrm{R}_1 = \mathcal{HB}_\mathrm{P} - (\mathrm{P}_1 \cup \mathrm{Q}_1)$, $\mathrm{R}_2 = \mathcal{HB}_\mathrm{P} - (\mathrm{P}_2 \cup \mathrm{Q}_2)$. *The information ordering on object models is defined as follows:* $\mathcal{I}_1 \preceq \mathcal{I}_2$ *iff* $\mathrm{P}_1 \subseteq \mathrm{P}_2$ *and* $\mathrm{R}_1 \subseteq \mathrm{R}_2$.

Intuitively, a stable object model is "smaller" in the information ordering, if it contains fewer true facts and fewer false facts. Therefore, the least stable object model contains the smallest set of true atoms and the smallest set of false atoms among all stable object models.

Definition 34 (Least Stable Object Model). *Let* $\mathcal{I}$ *be a stable object model of an F-logic KB* P. $\mathcal{I}$ *is the least stable object model of* P, *if* $\mathcal{I} \preceq \mathcal{J}$ *for any stable object model* $\mathcal{J}$ *of* P.

Theorem 4. *The cautious object model* $\mathcal{M}$ *of an F-logic KB* P *is the least stable object model of* P.

Proof. Let $\mathcal{I} = \langle \mathrm{T}; \mathrm{U} \rangle$ be any stable object model of P. We need to show that $\mathcal{M} \preceq \mathcal{I}$. Recall that $\mathcal{M} = \langle \pi(\widehat{\mathrm{T}}_\infty); \pi(\widehat{\mathrm{U}}_\infty) - \pi(\widehat{\mathrm{T}}_\infty) \rangle$. Therefore, to show that $\mathcal{M} \preceq \mathcal{I}$, it suffices to show that $\pi(\widehat{\mathrm{T}}_\infty) \subseteq \mathrm{T}$ and $\pi(\widehat{\mathrm{U}}_\infty) \supseteq \mathrm{T} \cup \mathrm{U}$. Since $\mathcal{I}$ is a stable object model of P, it follows that $\widehat{\mathrm{T}}_\mathcal{I} = \Psi_\mathrm{P}(\widehat{\mathrm{U}}_\mathcal{I})$ and $\widehat{\mathrm{U}}_\mathcal{I} = \Psi_\mathrm{P}(\widehat{\mathrm{T}}_\mathcal{I})$. Therefore, $\widehat{\mathrm{T}}_\mathcal{I} = \Psi_\mathrm{P}(\widehat{\mathrm{U}}_\mathcal{I}) = \Psi_\mathrm{P}(\Psi_\mathrm{P}(\widehat{\mathrm{T}}_\mathcal{I})) = \mathbf{F}_\mathrm{P}(\widehat{\mathrm{T}}_\mathcal{I})$ and so $\widehat{\mathrm{T}}_\mathcal{I}$ is a fixpoint of $\mathbf{F}_\mathrm{P}$. Similarly, $\widehat{\mathrm{U}}_\mathcal{I}$ is also a fixpoint of $\mathbf{F}_\mathrm{P}$. But $\widehat{\mathrm{T}}_\infty = \mathrm{lfp}(\mathbf{F}_\mathrm{P})$ and $\widehat{\mathrm{U}}_\infty = \mathrm{gfp}(\mathbf{F}_\mathrm{P})$, by Lemma 9. It follows that $\widehat{\mathrm{T}}_\infty \subseteq \widehat{\mathrm{T}}_\mathcal{I}$ and $\widehat{\mathrm{U}}_\infty \supseteq \widehat{\mathrm{U}}_\mathcal{I}$. Thus $\pi(\widehat{\mathrm{T}}_\infty) \subseteq \pi(\widehat{\mathrm{T}}_\mathcal{I})$ and $\pi(\widehat{\mathrm{U}}_\infty) \supseteq \pi(\widehat{\mathrm{U}}_\mathcal{I})$. Moreover, $\pi(\widehat{\mathrm{T}}_\mathcal{I}) = \mathrm{T}$ and $\pi(\widehat{\mathrm{U}}_\mathcal{I}) = \mathrm{T} \cup \mathrm{U}$, by Definition 29. So $\pi(\widehat{\mathrm{T}}_\infty) \subseteq \mathrm{T}$ and $\pi(\widehat{\mathrm{U}}_\infty) \supseteq \mathrm{T} \cup \mathrm{U}$.

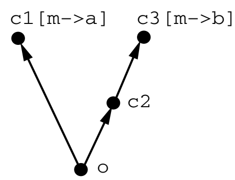
Besides comparing different models of a KB with respect to information ordering, it is also common to compare different models based on the amount of "truth" contained in the models. Typically, the true component of a model is minimized and the false component maximized. However, in F-logic we also need to deal with inheritance, which complicates the matters a bit, because some facts may be derived via inheritance. As a consequence, there are object models that look similar but are actually incomparable. This leads to the following definition of truth ordering among object models, which minimizes not only the set of true atoms of an object model, but also the amount of positive inheritance information implied by the object model.

Definition 35 (Truth Ordering). *Let* $\mathcal{I}_1 = \langle \mathrm{P}_1; \mathrm{Q}_1 \rangle$ *and* $\mathcal{I}_2 = \langle \mathrm{P}_2; \mathrm{Q}_2 \rangle$ *be two object models of an F-logic KB* P. *We write* $\mathcal{I}_1 \leq \mathcal{I}_2$ *iff*

(1) $\mathrm{P}_1 \subseteq \mathrm{P}_2$*; and*
(2) $\mathrm{P}_1 \cup \mathrm{Q}_1 \subseteq \mathrm{P}_2 \cup \mathrm{Q}_2$*; and*
(3) *for all* $\mathsf{c}, \mathsf{m}, \mathsf{o}$*:* $\mathsf{c}[\mathsf{m}] \overset{s.val}{\leadsto}_{\mathcal{I}_1} \mathsf{o}$ *implies* $\mathsf{c}[\mathsf{m}] \overset{s.val}{\leadsto}_{\mathcal{I}_2} \mathsf{o}$*; and*
(4) *for all* $\mathsf{c}, \mathsf{m}, \mathsf{o}$*:* $\mathsf{c}[\mathsf{m}] \overset{s.code}{\leadsto}_{\mathcal{I}_1} \mathsf{o}$ *implies* $\mathsf{c}[\mathsf{m}] \overset{s.code}{\leadsto}_{\mathcal{I}_2} \mathsf{o}$.

Definition 36 (Minimal Object Model). *An object model* $\mathcal{I}$ *is minimal iff there exists no object model* $\mathcal{J}$ *such that* $\mathcal{J} \leq \mathcal{I}$ *and* $\mathcal{J} \neq \mathcal{I}$.

The above definitions minimize the number of strong inheritance candidates implied by an object model *in addition to* the usual minimization of truth and maximization of falsehood. This is needed because increasing the number of false facts might inflate the number of strong inheritance candidates, which in turn might unjustifiably inflate the number of facts that are derived by inheritance.



$\mathsf{o} : \mathsf{c}_1.$
$\mathsf{o} : \mathsf{c}_2.$
$\mathsf{c}_2 :: \mathsf{c}_3.$
$\mathsf{c}_1[\mathsf{m} \rightarrow \mathsf{a}].$
$\mathsf{c}_3[\mathsf{m} \rightarrow \mathsf{b}].$
$\mathsf{c}_2[\mathsf{m} \rightarrow \mathsf{c}] \ :- \ \mathsf{o}[\mathsf{m} \rightarrow \mathsf{a}].$

Fig. 12. Minimal Object Model

Example 6. Consider the KB in Figure 12 and the following two object models of the KB: $\mathcal{I}_1 = \langle \mathrm{P}_1; \mathrm{Q}_1 \rangle$, where

$$\mathrm{P}_1 = \{\mathsf{o}:\mathsf{c}_1, \mathsf{o}:\mathsf{c}_2, \mathsf{c}_2::\mathsf{c}_3, \mathsf{c}_1[\mathsf{m} \rightarrow \mathsf{a}]_{\mathrm{ex}}, \mathsf{c}_3[\mathsf{m} \rightarrow \mathsf{b}]_{\mathrm{ex}}\}$$
$$\mathrm{Q}_1 = \emptyset$$

and $\mathcal{I}_2 = \langle \mathrm{P}_2; \mathrm{Q}_2 \rangle$, where

$$\mathrm{P}_2 = \mathrm{P}_1$$
$$\mathrm{Q}_2 = \{\mathsf{o}[\mathsf{m} \rightarrow \mathsf{a}]^{\mathsf{c}_1}_{\mathrm{val}}, \mathsf{c}_2[\mathsf{m} \rightarrow \mathsf{c}]_{\mathrm{ex}}\}$$

$\mathcal{I}_1$ and $\mathcal{I}_2$ both agree on the atoms that are true. But in $\mathcal{I}_1$ both $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{a}]^{\mathsf{c}_1}_{\mathrm{val}}$ and $\mathsf{c}_2[\mathsf{m} \rightarrow \mathsf{c}]_{\mathrm{ex}}$ are false, whereas in $\mathcal{I}_2$ they are both underdefined. Clearly, $\mathcal{I}_1$ has more false atoms than $\mathcal{I}_2$ and so with the usual notion of minimality we would say $\mathcal{I}_1 \leq \mathcal{I}_2$. However, $\mathcal{I}_1$ is not as "tight" as it appears, because the additional false atoms in $\mathcal{I}_1$ are not automatically implied by the KB under our cautious object model semantics. Indeed, although $\mathsf{c}_3[\mathsf{m}]$ is a strong value inheritance candidate for o in $\mathcal{I}_1$, it is only a weak value inheritance candidate in $\mathcal{I}_2$. We can see that it is due to this spurious positive information about inheritance candidates that $\mathcal{I}_1$ can have additional false atoms (compared to $\mathcal{I}_2$) while the sets of the true atoms in these interpretations can remain the same. This anomaly is eliminated by the inheritance minimization built into Definition 35, which renders the two models incomparable, i.e., $\mathcal{I}_1 \not\leq \mathcal{I}_2$.

Theorem 5. *The cautious object model* $\mathcal{M}$ *of an F-logic KB* P *is minimal among those object models of* P *that satisfy the cautious ISA transitivity constraint and the cautious inheritance constraint.*

Proof. This proof is long and is relegated to Section A.2 of the Appendix.

10 Implementation

It turns out that the (unique) cautious object model of an F-logic KB P can be computed as the *well-founded model* of a certain general logic program with negation, which is obtained from P via rewriting. Before describing the rewriting procedure we first define a rewriting function that applies to all regular rules and template rules.

Definition 37. *Given an F-logic KB* P *and a literal* L *in* P*, the functions* ρ^h *and* ρ^b *that rewrite head and body literals in* P *are defined as follows:*

$$\rho^h(\mathsf{L}) = \begin{cases} isa(\mathsf{o},\mathsf{c}), & \textit{if } \mathsf{L} = \mathsf{o}:\mathsf{c} \\ sub(\mathsf{s},\mathsf{c}), & \textit{if } \mathsf{L} = \mathsf{s}::\mathsf{c} \\ exmv(\mathsf{s},\mathsf{m},\mathsf{v}), & \textit{if } \mathsf{L} = \mathsf{s}[\mathsf{m} \rightarrow \mathsf{v}] \end{cases} \qquad \rho^b(\mathsf{L}) = \begin{cases} isa(\mathsf{o},\mathsf{c}), & \textit{if } \mathsf{L} = \mathsf{o}:\mathsf{c} \\ sub(\mathsf{s},\mathsf{c}), & \textit{if } \mathsf{L} = \mathsf{s}::\mathsf{c} \\ mv(\mathsf{o},\mathsf{m},\mathsf{v}), & \textit{if } \mathsf{L} = \mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}] \\ \neg\,(\rho^b(\mathrm{G})), & \textit{if } \mathsf{L} = \neg\,\mathsf{G} \end{cases}$$

The rewriting function ρ *on regular rules and template rules in* P *is defined as follows:*

$$\rho(\mathsf{H}\ :-\ \mathsf{L}_1,\ldots,\mathsf{L}_n) = \rho^h(\mathsf{H})\ :-\ \rho^b(\mathsf{L}_1),\ldots,\rho^b(\mathsf{L}_n)$$
$$\rho(\mathsf{code(c)}\ @\mathsf{this}[\mathsf{m} \rightarrow \mathsf{v}]\ :-\ \mathsf{L}_1,\ldots,\mathsf{L}_n) = ins(\mathsf{O},\mathsf{m},\mathsf{v},\mathsf{c})\ :-\ \rho^b(\mathsf{B}_1),\ldots,\rho^b(\mathsf{B}_n)$$

where O *is a new variable that does not appear in* P *and, each* $\mathsf{B}_i = (\mathsf{L}_i)_{\|\mathsf{O}}$*, i.e.,* B_i *is obtained from* L_i *by substituting* O *for all occurrences of the template term* @this*. The predicates, isa, sub, exmv, mv, and ins, are auxiliary predicates introduced by the rewriting.*

Note that since literals in rule heads and bodies have different meanings, they are rewritten differently. Moreover, literals in the heads of regular rules and template rules are also rewritten differently. The rewriting procedure that transforms F-logic KBs into general logic programs is defined next.

Definition 38 (Well-Founded Rewriting). *The well-founded rewriting of an F-logic KB* P*, denoted* P^{wf}*, is a general logic program constructed by the following steps:*

(1) *For every regular rule* R *in* P*, add its rewriting* $\rho(\mathsf{R})$ *into* P^{wf}*;*
(2) *For every template rule* R *in* P*, which specifies an instance method* m *for a class* c*, add its rewriting* $\rho(\mathsf{R})$ *into* P^{wf}*. Moreover, add a fact* $codedef(\mathsf{c},\mathsf{m})$ *into* P^{wf}*;*
(3) *Include the trailer rules shown in Figure 13 to* P^{wf} *(note that uppercase letters denote variables in these trailer rules).*

$$
\begin{array}{rcl}
mv(\mathsf{O},\mathsf{M},\mathsf{V}) & :- & exmv(\mathsf{O},\mathsf{M},\mathsf{V}). \\
mv(\mathsf{O},\mathsf{M},\mathsf{V}) & :- & vamv(\mathsf{O},\mathsf{M},\mathsf{V},\mathsf{C}). \\
mv(\mathsf{O},\mathsf{M},\mathsf{V}) & :- & comv(\mathsf{O},\mathsf{M},\mathsf{V},\mathsf{C}). \\
sub(\mathsf{S},\mathsf{C}) & :- & sub(\mathsf{S},\mathsf{X}),\ sub(\mathsf{X},\mathsf{C}). \\
isa(\mathsf{O},\mathsf{C}) & :- & isa(\mathsf{O},\mathsf{S}),\ sub(\mathsf{S},\mathsf{C}). \\
vamv(\mathsf{O},\mathsf{M},\mathsf{V},\mathsf{C}) & :- & vacan(\mathsf{C},\mathsf{M},\mathsf{O}),\ exmv(\mathsf{C},\mathsf{M},\mathsf{V}),\ \neg\, ex(\mathsf{O},\mathsf{M}),\ \neg\, multi(\mathsf{C},\mathsf{M},\mathsf{O}). \\
comv(\mathsf{O},\mathsf{M},\mathsf{V},\mathsf{C}) & :- & cocan(\mathsf{C},\mathsf{M},\mathsf{O}),\ ins(\mathsf{O},\mathsf{M},\mathsf{V},\mathsf{C}),\ \neg\, ex(\mathsf{O},\mathsf{M}),\ \neg\, multi(\mathsf{C},\mathsf{M},\mathsf{O}). \\
vacan(\mathsf{C},\mathsf{M},\mathsf{O}) & :- & isa(\mathsf{O},\mathsf{C}),\ exmv(\mathsf{C},\mathsf{M},\mathsf{V}),\ \mathsf{C}\neq\mathsf{O},\ \neg\, override(\mathsf{C},\mathsf{M},\mathsf{O}). \\
cocan(\mathsf{C},\mathsf{M},\mathsf{O}) & :- & isa(\mathsf{O},\mathsf{C}),\ codedef(\mathsf{C},\mathsf{M}),\ \mathsf{C}\neq\mathsf{O},\ \neg\, override(\mathsf{C},\mathsf{M},\mathsf{O}). \\
ex(\mathsf{O},\mathsf{M}) & :- & exmv(\mathsf{O},\mathsf{M},\mathsf{V}). \\
multi(\mathsf{C},\mathsf{M},\mathsf{O}) & :- & vacan(\mathsf{X},\mathsf{M},\mathsf{O}),\ \mathsf{X}\neq\mathsf{C}. \\
multi(\mathsf{C},\mathsf{M},\mathsf{O}) & :- & cocan(\mathsf{X},\mathsf{M},\mathsf{O}),\ \mathsf{X}\neq\mathsf{C}. \\
override(\mathsf{C},\mathsf{M},\mathsf{O}) & :- & sub(\mathsf{X},\mathsf{C}),\ isa(\mathsf{O},\mathsf{X}),\ exmv(\mathsf{X},\mathsf{M},\mathsf{V}),\ \mathsf{X}\neq\mathsf{C},\ \mathsf{X}\neq\mathsf{O}. \\
override(\mathsf{C},\mathsf{M},\mathsf{O}) & :- & sub(\mathsf{X},\mathsf{C}),\ isa(\mathsf{O},\mathsf{X}),\ codedef(\mathsf{X},\mathsf{M}),\ \mathsf{X}\neq\mathsf{C},\ \mathsf{X}\neq\mathsf{O}.
\end{array}
$$

Fig. 13. Trailer Rules for Well-Founded Rewriting

Note that while rewriting an F-logic KB into a general logic program, we need to output facts of the form $codedef(\mathsf{c},\mathsf{m})$ to remember that there is a template rule specifying instance method m for class c. Such facts are used to derive overriding and code inheritance candidacy information.

There is a unique well-founded model for any general logic program [15]. Next we will present a characterization of well-founded models based on the alternating fixpoint computation introduced in [42]. Given any general logic program P, we will use $\mathcal{HB}_{\mathrm{P}}$ to denote the Herbrand base of P, which consists of all possible atoms constructed using the predicate symbols and function symbols in P.

Definition 39. *Let* P *be a general logic program and* I *a subset of* $\mathcal{HB}_{\mathrm{P}}$. *The operator* $\mathbf{C}_{\mathrm{P,I}}$ *takes as input a set of atoms,* J, *and generates another set of atoms,* $\mathbf{C}_{\mathrm{P,I}}(\mathrm{J}) \subseteq \mathcal{HB}_{\mathrm{P}}$, *as follows:*

$$
\left\{ \mathsf{H} \;\middle|\; \begin{array}{l} \textit{There is } \mathsf{H} :- \mathsf{A}_1,\ldots,\mathsf{A}_\mathsf{m},\neg\,\mathsf{B}_1,\ldots,\neg\,\mathsf{B}_\mathsf{n} \in ground(\mathrm{P}),\ \ \mathsf{m}\geq 0, \\ \mathsf{n}\geq 0,\ \mathsf{A}_\mathsf{i}\ (1\leq \mathsf{i}\leq \mathsf{m}) \textit{ and } \mathsf{B}_\mathsf{j}\ (1\leq \mathsf{j}\leq \mathsf{n}) \textit{ are positive literals, and} \\ \mathsf{A}_\mathsf{i}\in \mathrm{J} \textit{ for all } 1\leq \mathsf{i}\leq \mathsf{m},\ \mathsf{B}_\mathsf{j}\notin \mathrm{I} \textit{ for all } 1\leq \mathsf{j}\leq \mathsf{n}. \end{array} \right\}
$$

Lemma 13. $\mathbf{C}_{\mathrm{P,I}}$ *is monotonic when* P *and* I *are fixed.*

It follows that $\mathbf{C}_{\mathrm{P,I}}$ has a unique least fixpoint. Having defined $\mathbf{C}_{\mathrm{P,I}}$ we can introduce two more operators, $\mathbf{S}_{\mathrm{P}}$ and $\mathbf{A}_{\mathrm{P}}$, as follows.

Definition 40. *Let* P *be a general logic program and* I *be a subset of* $\mathcal{HB}_{\mathrm{P}}$. *Then:* $\mathbf{S}_{\mathrm{P}}(\mathrm{I}) \stackrel{\mathrm{def}}{=} lfp(\mathbf{C}_{\mathrm{P,I}})$, $\mathbf{A}_{\mathrm{P}}(\mathrm{I}) \stackrel{\mathrm{def}}{=} \mathbf{S}_{\mathrm{P}}(\mathbf{S}_{\mathrm{P}}(\mathrm{I}))$.

Lemma 14. $\mathbf{S}_{\mathrm{P}}$ *is antimonotonic and* $\mathbf{A}_{\mathrm{P}}$ *is monotonic when* P *is fixed.*

It follows that $\mathbf{A}_{\mathrm{P}}$ has a unique least fixpoint, denoted $\mathrm{lfp}(\mathbf{A}_{\mathrm{P}})$. The following lemma from [42] explains how well-founded models can be defined in terms of alternating fixpoints.

Lemma 15. *The well-founded model, $\langle \mathrm{T}; \mathrm{U} \rangle$, of a general logic program* P, *where* T *is the set of atoms that are true and* U *is the set of atoms that are underdefined, can be computed as follows:* $\mathrm{T} = lfp(\mathbf{A}_{\mathrm{P}})$, $\mathrm{U} = \mathbf{S}_{\mathrm{P}}(lfp(\mathbf{A}_{\mathrm{P}})) - lfp(\mathbf{A}_{\mathrm{P}})$.

Given the well-founded rewriting, P^{wf}, of an F-logic KB P, the Herbrand base of P^{wf}, denoted $\mathcal{HB}_{\mathrm{P}^{wf}}$, consists of atoms of the following forms: $isa/2$, $sub/2$, $exmv/3$, $vamv/4$, $ins/4$, $codedef/2$, $comv/4$, $mv/3$, $vacan/3$, $cocan/3$, $ex/2$, $multi/3$, and $override/3$. We can establish an isomorphism between interpretations of P^{wf} and P as follows.

Definition 41 (Isomorphism). *Let* P^{wf} *be the well-founded rewriting of an F-logic KB* P, $\mathcal{HB}_{\mathrm{P}^{wf}}$ *the Herbrand base of* P^{wf}, $\widehat{\mathcal{HB}}_{\mathrm{P}}$ *the extended Herbrand base of* P, I^{wf} *a subset of* $\mathcal{HB}_{\mathrm{P}^{wf}}$, *and* $\widehat{\mathrm{I}}$ *a subset of* $\widehat{\mathcal{HB}}_{\mathrm{P}}$. *We will say that* I^{wf} *is isomorphic to* $\widehat{\mathrm{I}}$, *if all of the following conditions hold:*

(1) *for all* o, c: $isa(\mathsf{o}, \mathsf{c}) \in \mathrm{I}^{wf}$ *iff* $\mathsf{o} : \mathsf{c} \in \mathrm{I}$
(2) *for all* s, c: $sub(\mathsf{s}, \mathsf{c}) \in \mathrm{I}^{wf}$ *iff* $\mathsf{s} :: \mathsf{c} \in \mathrm{I}$
(3) *for all* $\mathsf{s}, \mathsf{m}, \mathsf{v}$: $exmv(\mathsf{s}, \mathsf{m}, \mathsf{v}) \in \mathrm{I}^{wf}$ *iff* $\mathsf{s}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}} \in \mathrm{I}$
(4) *for all* $\mathsf{o}, \mathsf{m}, \mathsf{v}, \mathsf{c}$: $vamv(\mathsf{o}, \mathsf{m}, \mathsf{v}, \mathsf{c}) \in \mathrm{I}^{wf}$ *iff* $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}} \in \mathrm{I}$
(5) *for all* $\mathsf{o}, \mathsf{m}, \mathsf{v}, \mathsf{c}$: $comv(\mathsf{o}, \mathsf{m}, \mathsf{v}, \mathsf{c}) \in \mathrm{I}^{wf}$ *iff* $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\mathrm{code}} \in \mathrm{I}$
(6) *for all* $\mathsf{c}, \mathsf{m}, \mathsf{o}$: $vacan(\mathsf{c}, \mathsf{m}, \mathsf{o}) \in \mathrm{I}^{wf}$ *iff* $\mathsf{c}[\mathsf{m}] \overset{val}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{I}}$
(7) *for all* $\mathsf{c}, \mathsf{m}, \mathsf{o}$: $cocan(\mathsf{c}, \mathsf{m}, \mathsf{o}) \in \mathrm{I}^{wf}$ *iff* $\mathsf{c}[\mathsf{m}] \overset{code}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{I}}$
(8) *for all* o, m: $ex(\mathsf{o}, \mathsf{m}) \in \mathrm{I}^{wf}$ *iff* $ec(\mathsf{o}, \mathsf{m}) \in \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{I}})$
(9) *for all* $\mathsf{c}, \mathsf{m}, \mathsf{o}$: $multi(\mathsf{c}, \mathsf{m}, \mathsf{o}) \in \mathrm{I}^{wf}$ *iff* $mc(\mathsf{c}, \mathsf{m}, \mathsf{o}) \in \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{I}})$
(10) *for all* $\mathsf{c}, \mathsf{m}, \mathsf{o}$: $override(\mathsf{c}, \mathsf{m}, \mathsf{o}) \in \mathrm{I}^{wf}$ *iff* $ov(\mathsf{c}, \mathsf{m}, \mathsf{o}) \in \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{I}})$

Let $\mathcal{M}^{wf}$ *be the well-founded model of* P^{wf} *and* $\mathcal{M}$ *the cautious object model of* P, $\mathcal{M}^{wf} = \langle \mathrm{T}^{wf}; \mathrm{U}^{wf} \rangle$, $\mathcal{M} = \langle \pi(\widehat{\mathrm{T}}_{\infty}); \pi(\widehat{\mathrm{U}}_{\infty} - \widehat{\mathrm{T}}_{\infty}) \rangle$. *We will say that* $\mathcal{M}^{wf}$ *is isomorphic to* $\mathcal{M}$, *if* T^{wf} *and* U^{wf} *are isomorphic to* $\widehat{\mathrm{T}}_{\infty}$ *and* $\widehat{\mathrm{U}}_{\infty} - \widehat{\mathrm{T}}_{\infty}$, *respectively.*

Note that the definition above includes atoms which are not in any interpretation of an F-logic KB P. However, if we can show that the well-founded model of P^{wf} is isomorphic (according to the above definition) to the cautious object model $\mathcal{M}$ of P, we can then establish a one-to-one correspondence between $isa(\mathsf{o}, \mathsf{c}) \in \mathcal{M}^{wf}$ and $\mathsf{o} : \mathsf{c} \in \mathcal{M}$, between $sub(\mathsf{s}, \mathsf{c}) \in \mathcal{M}^{wf}$ and $\mathsf{s} :: \mathsf{c} \in \mathcal{M}$, between $exmv(\mathsf{s}, \mathsf{m}, \mathsf{v}) \in \mathcal{M}^{wf}$ and $\mathsf{s}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}} \in \mathcal{M}$, between $comv(\mathsf{o}, \mathsf{m}, \mathsf{v}, \mathsf{c}) \in \mathcal{M}^{wf}$ and $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\mathrm{code}} \in \mathcal{M}$, and between $vamv(\mathsf{o}, \mathsf{m}, \mathsf{v}, \mathsf{c}) \in \mathcal{M}^{wf}$ and $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}} \in \mathcal{M}$, taking into account the truth values of atoms. Thus the cautious object model of P can be effectively computed by the well-founded model of P^{wf}.

Definition 42. *Let* P^{wf} *be the well-founded rewriting of an F-logic KB* P *and* I^{wf} *a subset of* $\mathcal{HB}_{\mathrm{P}^{wf}}$. *We will say that* I^{wf} *is in normal form, if for all* $\mathsf{o}, \mathsf{m}, \mathsf{v}$: $mv(\mathsf{o}, \mathsf{m}, \mathsf{v}) \in \mathrm{I}^{wf}$ *iff* $exmv(\mathsf{o}, \mathsf{m}, \mathsf{v}) \in \mathrm{I}^{wf}$, *or there is some class* c *such that* $vamv(\mathsf{o}, \mathsf{m}, \mathsf{v}, \mathsf{c}) \in \mathrm{I}^{wf}$ *or* $comv(\mathsf{o}, \mathsf{m}, \mathsf{v}, \mathsf{c}) \in \mathrm{I}^{wf}$.

In the following we introduce notations to represent the intermediate results during the computation of the well-founded model of a general logic program. These notations are used in the proof of the main theorem of this section.

Definition 43. *Let* P^{wf} *be the well-founded rewriting of an F-logic KB* P*. Define:*

$$
\begin{array}{lll}
\mathrm{T}_0^{wf} = \emptyset & \mathrm{U}_0^{wf} = \mathbf{S}_{\mathrm{P}^{wf}}(\mathrm{T}_0^{wf}) & \textit{for limit ordinal } 0 \\
\mathrm{T}_\alpha^{wf} = \mathbf{S}_{\mathrm{P}^{wf}}(\mathrm{U}_{\alpha-1}^{wf}) & \mathrm{U}_\alpha^{wf} = \mathbf{S}_{\mathrm{P}^{wf}}(\mathrm{T}_\alpha^{wf}) & \textit{for successor ordinal } \alpha \\
\mathrm{T}_\alpha^{wf} = \bigcup_{\beta<\alpha} \mathrm{T}_\beta^{wf} & \mathrm{U}_\alpha^{wf} = \mathbf{S}_{\mathrm{P}^{wf}}(\mathrm{T}_\alpha^{wf}) & \textit{for limit ordinal } \alpha \neq 0 \\
\mathrm{T}_\infty^{wf} = \bigcup_{\alpha} \mathrm{T}_\alpha^{wf} & \mathrm{U}_\infty^{wf} = \mathbf{S}_{\mathrm{P}^{wf}}(\mathrm{T}_\infty^{wf}) &
\end{array}
$$

Now we are ready to present the main theorem of this section. This theorem relies on a number of lemmas and propositions whose proofs are quite long; they can be found in Section A.3 of the Appendix.

Theorem 6. *Given the well-founded rewriting* P^{wf} *of an F-logic KB* P*, the well-founded model of* P^{wf} *is isomorphic to the cautious object model of* P*.*

Proof. Let $\mathcal{M}^{wf} = \langle\, \mathrm{T}^{wf}; \mathrm{U}^{wf}\,\rangle$ be the well-founded model of P^{wf}. Then by Lemma 15, $\mathrm{T}^{wf} = \mathrm{T}_\infty^{wf}$ and $\mathrm{U}^{wf} = \mathrm{U}_\infty^{wf} - \mathrm{T}_\infty^{wf}$. Let $\mathcal{M} = \langle\, \mathrm{T}; \mathrm{U}\,\rangle$ be the cautious object model of P. Then by Lemma 10, $\mathrm{T} = \pi(\widehat{\mathrm{T}}_\infty)$ and $\mathrm{U} = \pi(\widehat{\mathrm{U}}_\infty - \widehat{\mathrm{T}}_\infty)$. Therefore, by Definition 41, to show that $\mathcal{M}^{wf}$ is isomorphic to $\mathcal{M}$, it suffices to show that T_∞^{wf} is isomorphic to $\widehat{\mathrm{T}}_\infty$ and U_∞^{wf} is isomorphic to $\widehat{\mathrm{U}}_\infty$.

First note that T_α^{wf} and U_α^{wf} are in normal form for any ordinal α, by Proposition 9. Now we will prove by transfinite induction that T_α^{wf} is isomorphic to $\widehat{\mathrm{T}}_\alpha$ and U_α^{wf} is isomorphic to $\widehat{\mathrm{U}}_\alpha$, for any ordinal α. There are three cases to consider:

(1) $\alpha = 0$.
The claim is vacuously true for T_0^{wf} and $\widehat{\mathrm{T}}_0$. $\mathrm{U}_0^{wf} = \mathbf{S}_{\mathrm{P}^{wf}}(\mathrm{T}_0^{wf}) = \mathrm{lfp}(\mathbf{C}_{\mathrm{P}^{wf},\mathrm{T}_0^{wf}})$, by Definitions 43 and 40, and $\widehat{\mathrm{U}}_0 = \Psi_{\mathrm{P}}(\widehat{\mathrm{T}}_0) = \mathrm{lfp}(\mathbf{T}_{\mathrm{P},\widehat{\mathrm{T}}_0})$, by Definitions 32 and 27. It follows that U_0^{wf} is isomorphic to $\widehat{\mathrm{U}}_0$, by Proposition 8.

(2) α is a successor ordinal.
Then $\mathrm{T}_\alpha^{wf} = \mathbf{S}_{\mathrm{P}^{wf}}(\mathrm{U}_{\alpha-1}^{wf}) = \mathrm{lfp}(\mathbf{C}_{\mathrm{P}^{wf},\mathrm{U}_{\alpha-1}^{wf}})$, by Definitions 43 and 40, and $\widehat{\mathrm{T}}_\alpha = \Psi_{\mathrm{P}}(\widehat{\mathrm{U}}_{\alpha-1}) = \mathrm{lfp}(\mathbf{T}_{\mathrm{P},\widehat{\mathrm{U}}_{\alpha-1}})$, by Definitions 32 and 27. Moreover, $\mathrm{U}_{\alpha-1}^{wf}$ is isomorphic to $\widehat{\mathrm{U}}_{\alpha-1}$ by the induction hypothesis. It follows that U_α^{wf} is isomorphic to $\widehat{\mathrm{U}}_\alpha$, by Proposition 8. Similarly to (1), we can also show that U_α^{wf} is isomorphic to $\widehat{\mathrm{U}}_\alpha$.

(3) $\alpha \neq 0$ is a limit ordinal.
Then $\mathrm{T}_\alpha^{wf} = \bigcup_{\beta<\alpha} \mathrm{T}_\beta^{wf}$ and $\widehat{\mathrm{T}}_\alpha = \bigcup_{\beta<\alpha} \widehat{\mathrm{T}}_\beta$. Clearly, T_α^{wf} is isomorphic to $\widehat{\mathrm{T}}_\alpha$, because T_β^{wf} is isomorphic to $\widehat{\mathrm{T}}_\beta$ for all $\beta < \alpha$, by the induction hypothesis. Similarly to (1), we can also show that U_α^{wf} is isomorphic to $\widehat{\mathrm{U}}_\alpha$.

Note that $\mathrm{T}_\infty^{wf} = \bigcup_\alpha \mathrm{T}_\alpha^{wf}$ and $\widehat{\mathrm{T}}_\infty = \bigcup_\alpha \widehat{\mathrm{T}}_\alpha$. Therefore, it follows that T_∞^{wf} is isomorphic to $\widehat{\mathrm{T}}_\infty$, because T_α^{wf} is isomorphic to $\widehat{\mathrm{T}}_\alpha$, for any ordinal α. Moreover, $\mathrm{U}_\infty^{wf} = \mathbf{S}_{\mathrm{P}^{wf}}(\mathrm{T}_\infty^{wf}) = \mathrm{lfp}(\mathbf{C}_{\mathrm{P}^{wf},\mathrm{T}_\infty^{wf}})$, by Definitions 43 and 40, and $\widehat{\mathrm{U}}_\infty = \Psi_{\mathrm{P}}(\widehat{\mathrm{T}}_\infty) = \mathrm{lfp}(\mathbf{T}_{\mathrm{P},\widehat{\mathrm{T}}_\infty})$, by Definitions 32 and 27. Thus U_∞^{wf} is isomorphic to $\widehat{\mathrm{U}}_\infty$, by Proposition 8.

It is easy to see that P^{wf} can be computed in time linear in the size of the original F-logic KB P. Also, the trailer rules in Figure 13 are fixed and do not depend on P. Therefore, the size of P^{wf} is also linear relative to the size of the original F-logic KB P. This observation, combined with Theorem 6, leads to the following claim about the data complexity [43] of our inheritance semantics.

Theorem 7. *The data complexity of the cautious object model semantics for function-free F-logic KBs is polynomial-time.*

11 Related Work

Inheritance is one of the key aspects in frame-based knowledge representation. This problem has been studied quite extensively in the AI and database literature. To make our comparison with the related work concrete, we first list the main features of inheritance that we view should be supported by a general frame-based knowledge system:

- Inference by default inheritance and inference via rules.
- Intentional class hierarchies, *i.e.*, the ability to define both class membership and subclass relationship via rules.
- Data-dependent and data-independent inheritance. As we have shown, value inheritance is data-dependent, and this is the type of inheritance generally considered in AI. Code inheritance is data-independent and is of the kind that is common in imperative programming languages like C^{++} and Java.
- Overriding of inheritance from more general classes by more specific classes. Note that this also needs to take into account the interactions between data-dependent and data-independent inheritance.
- Nonmonotonic inheritance from multiple superclasses. Some proposals avoid this problem by imposing syntactic restrictions on rules. To this end, these proposals do not support nonmonotonic multiple inheritance.
- Introspection, by which variables can range over both class and method names.
- Late binding. This feature is common in imperative object-oriented languages such as C^{++} and Java. Supporting late binding requires resolving method names at runtime, when the class from which the instance method definitions are inherited is decided.

There is a large body of work based on Touretzky's framework of Inheritance Nets [41]. On one hand, the overriding mechanism in this framework is more sophisticated than what is typically considered in the knowledge base context. On the other hand, this framework supports neither deductive inference via rules nor intensional class hierarchies, which makes it too weak for many applications of knowledge bases. A survey on several different approaches to computing inheritance semantics based on Inheritance Nets can be found in [26].

There is also vast literature on extending traditional relational database systems with object-oriented features. However, these proposals do not support deduction via inference rules, which makes them mostly orthogonal to the problems addressed in this paper. For a comprehensive survey on this subject we refer the readers to [23].

The original F-logic [21,22] resolved many semantic and proof-theoretic issues in rule-based frame systems. However, the original semantics for inheritance in F-logic was problematic. It was defined through a nondeterministic inflationary fixpoint and was not backed by a corresponding model theory. This semantics was known to produce questionable results (cf. Section 3) when default inheritance and inference via rules interact. In addition, only value inheritance was considered in the original F-logic.

Ordered Logic [25] incorporates some aspects of the object-oriented paradigm. In this framework, both positive and negative literals are allowed in rule heads, and inference rules are grouped into a set of modules that collectively form a static class hierarchy. Although Ordered Logic supports overriding and propagation of rules among different modules, the idea of late binding is not built into the logic. Since it is primarily committed to resolving inconsistency between positive and negative literals, its semantics has a strong value-based value inheritance flavor. Furthermore, this approach permits only fixed class hierarchies and it does not support introspection.

Abiteboul et al. proposed a framework for implementing inheritance that is based on program rewriting using Datalog with negation [1]. Our implementation of the new F-logic semantics is close in spirit to their approach. However, their proposal is not backed by an independent model-theoretic formalization. Their framework further excludes nonmonotonic multiple inheritance and makes a very strong assumption that the rewritten knowledge base must have a total (two-valued) well-founded model. This latter assumption does not generally hold without strong syntactic restrictions that force stratification of the knowledge base. The framework is also limited to value inheritance.

In [12], Dobbie and Topor developed a model theory for *monotonic* code inheritance in their object-oriented deductive language Gulog. A special feature of their language is that all the variables in a rule must be explicitly typed according to a separate signature declaration. However, this language does not support any kind of nonmonotonic, data-dependent, or multiple inheritance. In [11], Dobbie further extends this approach to allow nonmonotonic inheritance. However, even this extension disallows interaction between inheritance and deduction and does not support multiple inheritance (the user must disambiguate inheritance conflicts manually).

Liu et. al. [27] modified the original F-logic to support code inheritance. However, to achieve that they had to throw out data-dependent inheritance, much of introspection, and intensional class hierarchies.

Bugliesi and Jamil proposed a model-theoretic semantics for value and code inheritance with overriding [5], which bears close resemblance to two-valued stable models [16]. However, their semantics applies only to negation-free knowledge bases (a severe limitation) and does not handle multiple inheritance conflicts. Instead, it makes multiple inheritance behave additively. (Such behavior can be easily simulated via rules.) In addition, their framework does not support data-dependent value inheritance, intensional class hierarchies, and more importantly, it does not provide an algorithm to compute a canonical model under their semantics.

May et al. [32] applied the ideas behind the well-founded semantics to F-logic. However, inheritance is still dealt with in the same way as in the original F-logic. Deduction and inheritance are computed in two separate stages and so the computation process has an inflationary fixpoint flavor. As mentioned in Section 3, this semantics is known to produce counter-intuitive results when intensional class hierarchies interact with overriding and multiple inheritance. Code inheritance is also not handled by this semantics.

In [19,20], Jamil introduced a series of techniques to tackle the inheritance problem. Among these, the ideas of *locality* and *context*, which were proposed to resolve code inheritance and encapsulation in the language Datalog^{++}, have influenced our approach the most. However, this work does not come with a model-theoretic inheritance semantics and supports neither intensional class hierarchies nor introspection. The inheritance semantics in [19] is defined by program rewriting while in [20] the approach is proof-theoretic.

Finally, May and Kandzia [31] showed that the original F-logic semantics can be described using the *inflationary* extension of Reiter's Default Logic [37]. In their framework, inheritance semantics is encoded using *defaults*. However, their inheritance strategy is inflationary — once a fact is derived through inheritance, it is never undone. Therefore, a later inference might invalidate the original conditions (encoded as justifications of defaults) for inheritance (cf. Section 3). Moreover, nonmonotonic multiple inheritance is handled in such a way that when multiple incomparable inheritance sources exist, one of them is randomly selected for inheritance instead of none (as in our framework). Code inheritance is not considered in [31].

12 Conclusion and Future Work

We have developed a novel model theory and a computational framework for nonmonotonic multiple inheritance of value and code in rule-based frame systems. We have shown that this semantics is implementable using a deductive engine, such as XSB [7], that supports well-founded semantics [15]. The value inheritance part of this semantics has been implemented in $\mathcal{F}$LORA-2 [44,46], a knowledge representation system, which is built around F-logic, HiLog [6], and

Transaction Logic [4].[5] Adding code inheritance to $\mathcal{F}$LORA-2 is planned for the near future.

The semantics proposed in this paper can be extended — without adding to the syntax — to allow a new kind of inheritable methods in addition to the traditional instance methods. The idea is to allow the template term @this in a template rule to be instantiated not only with instances of the class for which the template rule is defined, but also with that class itself and its subclasses as well. Effectively this turns template rules into pieces of code that also define class methods and that are inheritable by subclasses. For instance, with such a modification, the template rule

$$\mathsf{code(employee)}\ @\mathsf{this[avgSalary} \rightarrow \mathsf{A]}\ :-\ \mathsf{A} = avg\{\mathsf{S|E}:@\mathsf{this}, \mathsf{E[salary} \rightarrow \mathsf{S]}\}.$$

where $avg\{...\}$ is the averaging aggregate function, could be instantiated with class employee itself to

$$\mathsf{employee[avgSalary} \rightarrow \mathsf{A]}\ :-\ \mathsf{A} = avg\{\mathsf{S|E}:\mathsf{employee}, \mathsf{E[salary} \rightarrow \mathsf{S]}\}.$$

and instantiated with one of employee's subclasses, secretary, to

$$\mathsf{secretary[avgSalary} \rightarrow \mathsf{A]}\ :-\ \mathsf{A} = avg\{\mathsf{S|E}:\mathsf{secretary}, \mathsf{E[salary} \rightarrow \mathsf{S]}\}.$$

Similar rules could be obtained for other subclasses of employee, such as engineer and faculty. The last two rules above define the class method, avgSalary, for classes employee and secretary. It returns the average salary of an employee and a secretary, respectively. This kind of methods is not possible using the earlier machinery of class and instance methods.

Our model-theoretic approach points to several future research directions. First, the proposed semantics for inheritance here can be viewed as *source-based*. This means that in determining whether a multiple inheritance conflict exists the semantics takes into account only whether the same method is *defined* at different inheritance sources. A conflict is declared even if they all return exactly the same set of values. A *content-based* inheritance policy would not view this as a conflict. Such content-based inheritance seems harder computationally, but is worth further investigation. Second, it has been observed that inheritance-like phenomena arise in many domains, such as discretionary access control and trust management [18], but they cannot be formalized using a single semantics. We are considering extensions to our framework to allow users to specify their own *ad hoc* inheritance policies in a programmable, yet declarative, way.

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[5] $\mathcal{F}$LORA-2 is freely available from http://flora.sourceforge.net

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A Appendix: Proofs

This appendix includes the proofs of all the main theorems, their supporting lemmas, and propositions found in the main body of the paper.

A.1 Lemmas and Propositions Supporting Theorem 1 in Section 8

Lemma 16. *Let* P *be an F-logic KB and* $\mathcal{I} = \langle\, \mathrm{T}; \mathrm{U}\, \rangle$ *a stable interpretation of* P*, then:*

$$\widehat{\mathrm{T}}_{\mathcal{I}} = \mathbf{RC}_{\mathrm{P},\widehat{\mathrm{U}}_{\mathcal{I}}}(\widehat{\mathrm{T}}_{\mathcal{I}}) \cup \mathbf{TC}_{\mathrm{P},\widehat{\mathrm{U}}_{\mathcal{I}}}(\widehat{\mathrm{T}}_{\mathcal{I}}) \cup \mathbf{IC}^{t}(\widehat{\mathrm{T}}_{\mathcal{I}}) \cup \mathbf{IC}^{c}_{\mathrm{P},\widehat{\mathrm{U}}_{\mathcal{I}}}(\widehat{\mathrm{T}}_{\mathcal{I}}) \cup \mathbf{IC}^{i}_{\mathrm{P},\widehat{\mathrm{U}}_{\mathcal{I}}}(\widehat{\mathrm{T}}_{\mathcal{I}})$$
$$\widehat{\mathrm{U}}_{\mathcal{I}} = \mathbf{RC}_{\mathrm{P},\widehat{\mathrm{T}}_{\mathcal{I}}}(\widehat{\mathrm{U}}_{\mathcal{I}}) \cup \mathbf{TC}_{\mathrm{P},\widehat{\mathrm{T}}_{\mathcal{I}}}(\widehat{\mathrm{U}}_{\mathcal{I}}) \cup \mathbf{IC}^{t}(\widehat{\mathrm{U}}_{\mathcal{I}}) \cup \mathbf{IC}^{c}_{\mathrm{P},\widehat{\mathrm{T}}_{\mathcal{I}}}(\widehat{\mathrm{U}}_{\mathcal{I}}) \cup \mathbf{IC}^{i}_{\mathrm{P},\widehat{\mathrm{T}}_{\mathcal{I}}}(\widehat{\mathrm{U}}_{\mathcal{I}})$$

Proposition 1. *Let* $\mathcal{I} = \langle\, \mathrm{T}; \mathrm{U}\, \rangle$ *be a stable interpretation of an F-logic KB* P*. Then* $\mathcal{I}$ *satisfies the regular rules of* P*.*

Proof. By contradiction.

Suppose, to the contrary, that $\mathcal{I}$ does not satisfy the regular rules of P. Then by Definitions 3 and 2, there is a ground regular rule, $\mathsf{H} :- \mathsf{L}_1, \ldots, \mathsf{L}_n$, in *ground*(P), such that $\mathcal{V}^{\mathbf{h}}_{\mathcal{I}}(\mathsf{H}) < \mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L}_1 \wedge \ldots \wedge \mathsf{L}_n)$. Thus it must be the case that $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L}_1 \wedge \ldots \wedge \mathsf{L}_n) = \mathbf{t}$ and $\mathcal{V}^{\mathbf{h}}_{\mathcal{I}}(\mathsf{H}) \neq \mathbf{t}$, or $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L}_1 \wedge \ldots \wedge \mathsf{L}_n) = \mathbf{u}$ and $\mathcal{V}^{\mathbf{h}}_{\mathcal{I}}(\mathsf{H}) = \mathbf{f}$.

(1) $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L}_1 \wedge \ldots \wedge \mathsf{L}_n) = \mathbf{t}$ and $\mathcal{V}^{\mathbf{h}}_{\mathcal{I}}(\mathsf{H}) \neq \mathbf{t}$
It follows that $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L}_i) = \mathbf{t}$ for all $\mathsf{L}_i, 1 \leq i \leq n$, by Definition 1. So by Lemma 1: (i) if L_i is a positive literal then $val^{\mathbf{b}}_{\widehat{\mathrm{T}}_{\mathcal{I}}}(\mathsf{L}_i) = \mathbf{t}$; and (ii) if L_i is a negative literal

then $val^{\mathbf{b}}_{\widehat{\mathrm{U}}_{\mathcal{I}}}(\mathsf{L_i}) = \mathbf{t}$. Therefore, for the atom $\mathsf{A} \in \mathcal{HB}_{\mathrm{P}}$ such that H matches A, it follows that $\mathsf{A} \in \mathbf{RC}_{\mathrm{P},\widehat{\mathrm{U}}_{\mathcal{I}}}(\widehat{\mathrm{T}}_{\mathcal{I}}) \subseteq \widehat{\mathrm{T}}_{\mathcal{I}}$, by Definition 22 and Lemma 16. Thus $\mathcal{I}(\mathsf{A}) = \mathbf{t}$, and so $\mathcal{V}^{\mathbf{h}}_{\mathcal{I}}(\mathsf{H}) = \mathcal{I}(\mathsf{A}) = \mathbf{t}$ by Definitions 21 and 1, a contradiction.

(2) $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L_1} \wedge \ \ldots\ \wedge \mathsf{L_n}) = \mathbf{u}$ and $\mathcal{V}^{\mathbf{h}}_{\mathcal{I}}(\mathsf{H}) = \mathbf{f}$
It follows that $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L_i}) \geq \mathbf{u}$ for all $\mathsf{L_i}, 1 \leq \mathsf{i} \leq \mathsf{n}$, by Definition 1. So by Lemma 1: (i) if $\mathsf{L_i}$ is a positive literal then $val^{\mathbf{b}}_{\widehat{\mathrm{U}}_{\mathcal{I}}}(\mathsf{L_i}) = \mathbf{t}$; and (2) if $\mathsf{L_i}$ is a negative literal then $val^{\mathbf{b}}_{\widehat{\mathrm{T}}_{\mathcal{I}}}(\mathsf{L_i}) = \mathbf{t}$. Therefore, for the atom $\mathsf{A} \in \mathcal{HB}_{\mathrm{P}}$ such that H matches A, it follows that $\mathsf{A} \in \mathbf{RC}_{\mathrm{P},\widehat{\mathrm{T}}_{\mathcal{I}}}(\widehat{\mathrm{U}}_{\mathcal{I}}) \subseteq \widehat{\mathrm{U}}_{\mathcal{I}}$, by Definition 22 and Lemma 16. Thus $\mathcal{I}(\mathsf{A}) \geq \mathbf{u}$, and so $\mathcal{V}^{\mathbf{h}}_{\mathcal{I}}(\mathsf{H}) = \mathcal{I}(\mathsf{A}) \geq \mathbf{u}$ by Definitions 21 and 1, a contradiction.

Proposition 2. *Let* $\mathcal{I} = \langle\, \mathrm{T}; \mathrm{U}\, \rangle$ *be a stable interpretation of an F-logic KB* P. *Then* $\mathcal{I}$ *satisfies the positive ISA transitivity constraint.*

Proof. By Definition 10, we need to show that the following conditions hold:

(1) for all s, c: if there is x such that $\mathcal{I}(\mathsf{s} :: \mathsf{x}) = \mathbf{t}$ and $\mathcal{I}(\mathsf{x} :: \mathsf{c}) = \mathbf{t}$, then $\mathcal{I}(\mathsf{s} :: \mathsf{c}) = \mathbf{t}$;
(2) for all o, c: if there is x such that $\mathcal{I}(\mathsf{o} : \mathsf{x}) = \mathbf{t}$ and $\mathcal{I}(\mathsf{x} :: \mathsf{c}) = \mathbf{t}$, then $\mathcal{I}(\mathsf{o} : \mathsf{c}) = \mathbf{t}$.

Note that for all s, c: $\mathcal{I}(\mathsf{s} :: \mathsf{c}) = \mathbf{t}$ iff $\mathsf{s} :: \mathsf{c} \in \mathrm{T} \subseteq \widehat{\mathrm{T}}_{\mathcal{I}}$ and, for all o, c: $\mathcal{I}(\mathsf{o} : \mathsf{c}) = \mathbf{t}$ iff $\mathsf{o} : \mathsf{c} \in \mathrm{T} \subseteq \widehat{\mathrm{T}}_{\mathcal{I}}$. Let $\mathsf{s} :: \mathsf{x} \in \mathrm{T} \subseteq \widehat{\mathrm{T}}_{\mathcal{I}}$ and $\mathsf{x} :: \mathsf{c} \in \mathrm{T} \subseteq \widehat{\mathrm{T}}_{\mathcal{I}}$. Then $\mathsf{s} :: \mathsf{c} \in \mathbf{IC}^t(\widehat{\mathrm{T}}_{\mathcal{I}})$ by Definition 25. It follows that $\mathsf{s} :: \mathsf{c} \in \mathbf{IC}^t(\widehat{\mathrm{T}}_{\mathcal{I}}) \subseteq \widehat{\mathrm{T}}_{\mathcal{I}}$, by Lemma 16. Similarly, if $\mathsf{o} : \mathsf{x} \in \widehat{\mathrm{T}}_{\mathcal{I}}$ and $\mathsf{x} :: \mathsf{c} \in \widehat{\mathrm{T}}_{\mathcal{I}}$, then $\mathsf{o} : \mathsf{c} \in \mathbf{IC}^t(\widehat{\mathrm{T}}_{\mathcal{I}}) \subseteq \widehat{\mathrm{T}}_{\mathcal{I}}$.

Proposition 3. *Let* $\mathcal{I} = \langle\, \mathrm{T}; \mathrm{U}\, \rangle$ *be a stable interpretation of an F-logic KB* P. *Then* $\mathcal{I}$ *satisfies the context consistency constraint.*

Proof. By Definition 11, we need to show that the following conditions hold:

(1) for all o, m, v: $\mathcal{I}(\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\,\mathsf{o}}_{\,\mathrm{val}}) = \mathbf{f}$ and $\mathcal{I}(\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\,\mathsf{o}}_{\,\mathrm{code}}) = \mathbf{f}$.
Note that $\mathcal{I}(\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\,\mathsf{o}}_{\,\mathrm{val}}) = \mathbf{f}$ iff $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\,\mathsf{o}}_{\,\mathrm{val}} \notin \mathrm{T} \cup \mathrm{U}$ iff $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\,\mathsf{o}}_{\,\mathrm{val}} \notin \widehat{\mathrm{U}}_{\mathcal{I}}$ by Definition 29. Similarly, $\mathcal{I}(\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\,\mathsf{o}}_{\,\mathrm{code}}) = \mathbf{f}$ iff $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\,\mathsf{o}}_{\,\mathrm{code}} \notin \widehat{\mathrm{U}}_{\mathcal{I}}$. Since $\mathcal{I}$ is a stable interpretation of P, we have $\widehat{\mathrm{U}}_{\mathcal{I}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$, by Definition 30. It then follows from Lemma 6 that $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\,\mathsf{o}}_{\,\mathrm{val}} \notin \widehat{\mathrm{U}}_{\mathcal{I}}$ and $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\,\mathsf{o}}_{\,\mathrm{code}} \notin \widehat{\mathrm{U}}_{\mathcal{I}}$, for all $\mathsf{o}, \mathsf{m}, \mathsf{v}$.

(2) for all c, m, v: if $\mathcal{I}(\mathsf{c}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}}) = \mathbf{f}$, then $\mathcal{I}(\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\,\mathsf{c}}_{\,\mathrm{val}}) = \mathbf{f}$ for all o.
Let $\mathcal{I}(\mathsf{c}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}}) = \mathbf{f}$. It follows that $\mathsf{c}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}} \notin \widehat{\mathrm{U}}_{\mathcal{I}}$. We need to show that $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\,\mathsf{c}}_{\,\mathrm{val}} \notin \widehat{\mathrm{U}}_{\mathcal{I}}$ for all o. Suppose, to the contrary, that there exists o such that $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\,\mathsf{c}}_{\,\mathrm{val}} \in \widehat{\mathrm{U}}_{\mathcal{I}}$. Because $\mathcal{I}$ is a stable interpretation of P, $\widehat{\mathrm{U}}_{\mathcal{I}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$. It follows that $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\,\mathsf{c}}_{\,\mathrm{val}} \in \mathbf{IC}^i_{\mathrm{P},\widehat{\mathrm{T}}_{\mathcal{I}}}(\widehat{\mathrm{U}}_{\mathcal{I}})$ by Lemma 6. So $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{U}}_{\mathcal{I}}$, by Definition 25. Thus $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \mathbf{IC}^c_{\mathrm{P},\widehat{\mathrm{T}}_{\mathcal{I}}}(\widehat{\mathrm{U}}_{\mathcal{I}})$ by Lemma 16. It follows that $\mathsf{c}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}} \in \widehat{\mathrm{U}}_{\mathcal{I}}$ by Definition 25, which contradicts the premise.

(3) for all c, m: if there is no template rule in *ground*(P) which specifies the instance method m for class c, then $\mathcal{I}(\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\text{code}}) = \mathbf{f}$ for all o, v.

Suppose, to the contrary, that there exist o, v such that $\mathcal{I}(\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\text{code}}) \neq \mathbf{f}$. Then $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\text{code}} \in \mathrm{T} \cup \mathrm{U} \subseteq \widehat{\mathrm{U}}_{\mathcal{I}}$. It follows that $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\text{code}} \in \mathbf{IC}^{i}_{\mathrm{P},\widehat{\mathrm{T}}_{\mathcal{I}}}(\widehat{\mathrm{U}}_{\mathcal{I}})$ by Lemma 6. Thus $\mathsf{c}[\mathsf{m}] \overset{code}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{U}}_{\mathcal{I}}$ by Definition 25 and so $\mathsf{c}[\mathsf{m}] \overset{code}{\rightsquigarrow} \mathsf{o} \in \mathbf{IC}^{c}_{\mathrm{P},\widehat{\mathrm{T}}_{\mathcal{I}}}(\widehat{\mathrm{U}}_{\mathcal{I}})$ by Lemma 16. Hence, by Definition 25, there must exist a template rule in *ground*(P) which specifies the instance method m for the class c, a contradiction.

(4) for all o, m: if $\mathsf{o}[\mathsf{m}]$ is a strong explicit definition, then $\mathcal{I}(\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\text{val}}) = \mathbf{f}$ and $\mathcal{I}(\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\text{code}}) = \mathbf{f}$ for all v, c.

Let $\mathsf{o}[\mathsf{m}]$ be a strong explicit definition. Then there must exist v such that $\mathsf{o}[\mathsf{m} \to \mathsf{v}]_{\text{ex}} \in \mathrm{T} \subseteq \widehat{\mathrm{T}}_{\mathcal{I}}$ by Definition 4. So $ec(\mathsf{o}, \mathsf{m}) \in \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$ by Definition 23. Suppose, to the contrary, that there exist v, c such that $\mathcal{I}(\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\text{val}}) \neq \mathbf{f}$. Then $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\text{val}} \in \mathrm{T} \cup \mathrm{U} \subseteq \widehat{\mathrm{U}}_{\mathcal{I}}$. It follows that $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\text{val}} \in \mathbf{IC}^{i}_{\mathrm{P},\widehat{\mathrm{T}}_{\mathcal{I}}}(\widehat{\mathrm{U}}_{\mathcal{I}})$ by Lemma 6. Thus $ec(\mathsf{o}, \mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$ by Definition 25, a contradiction. Similarly, we can show that $\mathcal{I}(\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\text{code}}) = \mathbf{f}$ for all v, c.

Proposition 4. *Let* $\mathcal{I} = \langle \mathrm{T}; \mathrm{U} \rangle$ *be a stable interpretation of an F-logic KB* P*. Then* $\mathcal{I}$ *satisfies the unique source inheritance constraint.*

Proof. By Definition 12, we need to show that the following conditions hold:

(1) for all $\mathsf{o}, \mathsf{m}, \mathsf{v}, \mathsf{c}$: if $\mathcal{I}(\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\text{val}}) = \mathbf{t}$ or $\mathcal{I}(\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\text{code}}) = \mathbf{t}$, then for all z, x such that $\mathsf{x} \neq \mathsf{c}$, $\mathcal{I}(\mathsf{o}[\mathsf{m} \to \mathsf{z}]^{\mathsf{x}}_{\text{val}}) = \mathbf{f}$ and $\mathcal{I}(\mathsf{o}[\mathsf{m} \to \mathsf{z}]^{\mathsf{x}}_{\text{code}}) = \mathbf{f}$.

Because $\mathcal{I}$ is a stable interpretation of P, $\widehat{\mathrm{T}}_{\mathcal{I}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{U}}_{\mathcal{I}})$ and $\widehat{\mathrm{U}}_{\mathcal{I}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$, by Definition 30. If $\mathcal{I}(\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\text{val}}) = \mathbf{t}$, then $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\text{val}} \in \mathrm{T} \subseteq \widehat{\mathrm{T}}_{\mathcal{I}}$ by Definition 29. So $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\text{val}} \in \mathbf{IC}^{i}_{\mathrm{P},\widehat{\mathrm{U}}_{\mathcal{I}}}(\widehat{\mathrm{T}}_{\mathcal{I}})$ by Lemma 6. It follows that $\mathsf{c}[\mathsf{m}] \overset{val}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{T}}_{\mathcal{I}}$ by Definition 25. On the other hand, if $\mathcal{I}(\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\text{code}}) = \mathbf{t}$, then $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\text{code}} \in \mathrm{T} \subseteq \widehat{\mathrm{T}}_{\mathcal{I}}$ by Definition 29. So $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\text{code}} \in \mathbf{TC}_{\mathrm{P},\widehat{\mathrm{U}}_{\mathcal{I}}}(\widehat{\mathrm{T}}_{\mathcal{I}})$ by Lemma 6. Thus $\mathsf{c}[\mathsf{m}] \overset{code}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{T}}_{\mathcal{I}}$ by Definition 25. Therefore, $\mathsf{c}[\mathsf{m}] \overset{val}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{T}}_{\mathcal{I}}$ or $\mathsf{c}[\mathsf{m}] \overset{code}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{T}}_{\mathcal{I}}$.

Suppose, to the contrary, that there are z, x such that $\mathsf{x} \neq \mathsf{c}$, $\mathcal{I}(\mathsf{o}[\mathsf{m} \to \mathsf{z}]^{\mathsf{x}}_{\text{val}}) \geq \mathbf{u}$. Then $\mathsf{o}[\mathsf{m} \to \mathsf{z}]^{\mathsf{x}}_{\text{val}} \in \mathrm{T} \cup \mathrm{U} \subseteq \widehat{\mathrm{U}}_{\mathcal{I}}$ by Definition 29. So $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{x}}_{\text{val}} \in \mathbf{IC}^{i}_{\mathrm{P},\widehat{\mathrm{T}}_{\mathcal{I}}}(\widehat{\mathrm{U}}_{\mathcal{I}})$ by Lemma 6. Therefore, $mc(\mathsf{x}, \mathsf{m}, \mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$ by Definition 25. Since $\mathsf{x} \neq \mathsf{c}$, it follows that $\mathsf{c}[\mathsf{m}] \overset{val}{\rightsquigarrow} \mathsf{o} \notin \widehat{\mathrm{T}}_{\mathcal{I}}$ by Definition 23, which is a contradiction. Therefore, $\mathcal{I}(\mathsf{o}[\mathsf{m} \to \mathsf{z}]^{\mathsf{x}}_{\text{val}}) = \mathbf{f}$ for all z, x such that $\mathsf{x} \neq \mathsf{c}$. Similarly, we can also show that $\mathcal{I}(\mathsf{o}[\mathsf{m} \to \mathsf{z}]^{\mathsf{x}}_{\text{code}}) = \mathbf{f}$ for all z, x such that $\mathsf{x} \neq \mathsf{c}$.

(2) for all c, m, o: if $\mathsf{c}[\mathsf{m}] \overset{s.val}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$ or $\mathsf{c}[\mathsf{m}] \overset{s.code}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$, then $\mathcal{I}(\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{x}}_{\text{val}}) = \mathbf{f}$ and $\mathcal{I}(\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{x}}_{\text{code}}) = \mathbf{f}$, for all v, x such that $\mathsf{x} \neq \mathsf{c}$.

Let $\mathsf{c}[\mathsf{m}] \overset{s.val}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$ or $\mathsf{c}[\mathsf{m}] \overset{s.code}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$. Suppose, to the contrary, that there exist v, x such that $\mathsf{x} \neq \mathsf{c}$, $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{x}}_{\text{val}} \neq \mathbf{f}$. Then $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{x}}_{\text{val}} \in \mathrm{T} \cup \mathrm{U} \subseteq \widehat{\mathrm{U}}_{\mathcal{I}}$ by Definition 29. Because $\mathcal{I}$ is a stable interpretation of P, therefore $\widehat{\mathrm{U}}_{\mathcal{I}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$ by Definition 30. Thus $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{x}}_{\text{val}} \in \mathbf{IC}^{i}_{\mathrm{P},\widehat{\mathrm{T}}_{\mathcal{I}}}(\widehat{\mathrm{U}}_{\mathcal{I}})$ by Lemma 6 and so $mc(\mathsf{x}, \mathsf{m}, \mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$

by Definition 25. However, by Definition 29, $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{T}}_{\mathcal{I}}$ or $\mathsf{c}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o} \in \widehat{\mathrm{T}}_{\mathcal{I}}$. Note that $\mathsf{x} \neq \mathsf{c}$. It follows that $mc(\mathsf{x},\mathsf{m},\mathsf{o}) \in \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$ by Definition 23, which is a contradiction. Therefore, $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\,\mathsf{x}}_{\,\mathrm{val}}) = \mathbf{f}$ for all v,x such that $\mathsf{x} \neq \mathsf{c}$. Similarly, we can also show that $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\,\mathsf{x}}_{\,\mathrm{code}}) = \mathbf{f}$ for all v,x such that $\mathsf{x} \neq \mathsf{c}$.

(3) for all o, m, v, c: $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\,\mathsf{c}}_{\,\mathrm{val}}) = \mathbf{t}$ iff
 (i) $\mathsf{o}[\mathsf{m}]$ is neither a strong nor a weak explicit definition; and
 (ii) $\mathsf{c}[\mathsf{m}] \overset{s.val}{\leadsto}_{\mathcal{I}} \mathsf{o}$; and
 (iii) $\mathcal{I}(\mathsf{c}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}}) = \mathbf{t}$; and
 (iv) there is no x such that $\mathsf{x} \neq \mathsf{c}$ and $\mathsf{x}[\mathsf{m}] \leadsto_{\mathcal{I}} \mathsf{o}$.

"$\Longrightarrow$". Since $\mathcal{I}$ is a stable interpretation of P, $\widehat{\mathrm{T}}_{\mathcal{I}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{U}}_{\mathcal{I}})$ by Definition 30. Because $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\,\mathsf{c}}_{\,\mathrm{val}}) = \mathbf{t}$, $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\,\mathsf{c}}_{\,\mathrm{val}} \in \mathrm{T} \subseteq \widehat{\mathrm{T}}_{\mathcal{I}}$ by Definition 29. Thus $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\,\mathsf{c}}_{\,\mathrm{val}} \in \mathbf{IC}^{i}_{\mathrm{P},\widehat{\mathrm{U}}_{\mathcal{I}}}(\widehat{\mathrm{T}}_{\mathcal{I}})$ by Lemma 6, and $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{T}}_{\mathcal{I}}$, $\mathsf{c}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}} \in \widehat{\mathrm{T}}_{\mathcal{I}}$, $ec(\mathsf{o},\mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{U}}_{\mathcal{I}})$, and $mc(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{U}}_{\mathcal{I}})$, by Definition 25. Note that $ec(\mathsf{o},\mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{U}}_{\mathcal{I}})$. It follows that $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{x}]_{\mathrm{ex}} \notin \widehat{\mathrm{U}}_{\mathcal{I}}$ for all x, by Definition 23. Thus $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{x}]_{\mathrm{ex}}) = \mathbf{f}$ for all x and so $\mathsf{o}[\mathsf{m}]$ is neither a strong nor a weak explicit definition, by Definition 4. Note that $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{T}}_{\mathcal{I}}$. It follows that $\mathsf{c}[\mathsf{m}] \overset{s.val}{\leadsto}_{\mathcal{I}} \mathsf{o}$ by Definition 29. $\mathsf{c}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}} \in \widehat{\mathrm{T}}_{\mathcal{I}}$ implies $\mathcal{I}(\mathsf{c}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}}) = \mathbf{t}$. Because $mc(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{U}}_{\mathcal{I}})$, therefore there is no $\mathsf{x} \neq \mathsf{c}$ such that $\mathsf{x}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{U}}_{\mathcal{I}}$ or $\mathsf{x}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o} \in \widehat{\mathrm{U}}_{\mathcal{I}}$, by Definition 23. So there is no x such that $\mathsf{x} \neq \mathsf{c}$ and $\mathsf{x}[\mathsf{m}] \leadsto_{\mathcal{I}} \mathsf{o}$, by Definition 29.
"$\Longleftarrow$". Since $\mathsf{o}[\mathsf{m}]$ is neither a strong nor a weak explicit definition, it follows that $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{x}]_{\mathrm{ex}}) = \mathbf{f}$ for all x, by Definition 4. So $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{x}]_{\mathrm{ex}} \notin \mathrm{T} \cup \mathrm{U}$ for all x, and $ec(\mathsf{o},\mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{U}}_{\mathcal{I}})$, by Definitions 29 and 23. Because $\mathsf{c}[\mathsf{m}] \overset{s.val}{\leadsto}_{\mathcal{I}} \mathsf{o}$, therefore $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{T}}_{\mathcal{I}}$ by Definition 29. Since $\mathcal{I}(\mathsf{c}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}}) = \mathbf{t}$, it follows that $\mathsf{c}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}} \in \mathrm{T} \subseteq \widehat{\mathrm{T}}_{\mathcal{I}}$. Because $\mathcal{I}$ is a stable interpretation of P, therefore $\widehat{\mathrm{T}}_{\mathcal{I}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{U}}_{\mathcal{I}})$, by Definition 30. So if we can show that $mc(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{U}}_{\mathcal{I}})$, then it follows that $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\,\mathsf{c}}_{\,\mathrm{val}} \in \mathbf{IC}^{i}_{\mathrm{P},\widehat{\mathrm{U}}_{\mathcal{I}}}(\widehat{\mathrm{T}}_{\mathcal{I}}) \subseteq \widehat{\mathrm{T}}_{\mathcal{I}}$, by Definition 25 and Lemma 16. Suppose, to the contrary, that $mc(\mathsf{c},\mathsf{m},\mathsf{o}) \in \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{U}}_{\mathcal{I}})$. Then, by Definition 23, there is $\mathsf{x} \neq \mathsf{c}$ such that $\mathsf{x}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{U}}_{\mathcal{I}}$ or $\mathsf{x}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o} \in \widehat{\mathrm{U}}_{\mathcal{I}}$. It follows that $\mathsf{x}[\mathsf{m}] \leadsto_{\mathcal{I}} \mathsf{o}$ by Definition 29, which contradicts the premise. Therefore, $mc(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{U}}_{\mathcal{I}})$, and so $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\,\mathsf{c}}_{\,\mathrm{val}} \in \widehat{\mathrm{T}}_{\mathcal{I}}$, $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\,\mathsf{c}}_{\,\mathrm{val}}) = \mathbf{t}$.

Proposition 5. *Let* $\mathcal{I} = \langle \mathrm{T}; \mathrm{U} \rangle$ *be a stable interpretation of an F-logic KB* P*. Then* $\mathcal{I}$ *satisfies the template rules of* P*.*

Proof. By contradiction. Because $\mathcal{I}$ is a stable interpretation of P, $\widehat{\mathrm{T}}_{\mathcal{I}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{U}}_{\mathcal{I}})$ and $\widehat{\mathrm{U}}_{\mathcal{I}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$ by Definition 30. Suppose, to the contrary, that $\mathcal{I}$ does not satisfy the template rules of P. Then, by Definition 19, *ground*(P) has an object $\mathsf{o} \in \mathcal{HU}_{\mathrm{P}}$ and a template rule R either of the form $\mathsf{code(c)}$ $\mathsf{@this}[\mathsf{m} \rightarrow \mathsf{v}]$:− B or of the form $\mathsf{code(c)}$ $\mathsf{@this}[\mathsf{m} \rightarrow \mathsf{v}]$, such that $\mathcal{I}(\mathsf{R}_{\|\mathsf{o}}) = \mathbf{f}$. Let us assume that $\mathsf{R} \equiv \mathsf{code(c)}$ $\mathsf{@this}[\mathsf{m} \rightarrow \mathsf{v}]$:− B (the case in which $\mathsf{R} \equiv \mathsf{code(c)}$ $\mathsf{@this}[\mathsf{m} \rightarrow \mathsf{v}]$ is similar). By Definition 18, there are three possible cases to consider:

(1) $imode_{\mathcal{I}}(\mathsf{R}_{\|\mathsf{o}}) = \mathbf{t}$ and $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\text{code}}) < \mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{B}_{\|\mathsf{o}})$
Because $imode_{\mathcal{I}}(\mathsf{R}_{\|\mathsf{o}}) = \mathbf{t}$, therefore by Definition 16: (i) $\mathsf{c}[\mathsf{m}] \overset{s.code}{\leadsto}_{\mathcal{I}} \mathsf{o}$ and so $\mathsf{c}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o} \in \widehat{\mathrm{T}}_{\mathcal{I}}$ by Definition 29; (ii) $ec(\mathsf{o}, \mathsf{m})$ is neither a strong nor a weak explicit definition. So $ec(\mathsf{o}, \mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{U}}_{\mathcal{I}})$ by Definitions 23 and 4; and (iii) there is no $\mathsf{x} \neq \mathsf{c}$ such that $\mathsf{x}[\mathsf{m}] \leadsto_{\mathcal{I}} \mathsf{o}$. It follows that there is no $\mathsf{x} \neq \mathsf{c}$ such that $\mathsf{x}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{U}}_{\mathcal{I}}$ or $\mathsf{x}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o} \in \widehat{\mathrm{U}}_{\mathcal{I}}$ by Definition 29. Thus $mc(\mathsf{c}, \mathsf{m}, \mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{U}}_{\mathcal{I}})$. Since $\widehat{\mathrm{T}}_{\mathcal{I}} \subseteq \widehat{\mathrm{U}}_{\mathcal{I}}$, it also follows that $\mathsf{c}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o} \in \widehat{\mathrm{U}}_{\mathcal{I}}$, $ec(\mathsf{o}, \mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$, and $mc(\mathsf{c}, \mathsf{m}, \mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$, by the monotonicity of $\mathbf{IB}_{\mathrm{P}}$.

First let us assume $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{B}_{\|\mathsf{o}}) = \mathbf{t}$. It follows that $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\text{code}}) \neq \mathbf{t}$. Since $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{B}_{\|\mathsf{o}}) = \mathbf{t}$, it follows that $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L}) = \mathbf{t}$ for all $\mathsf{L} \in \mathsf{B}_{\|\mathsf{o}}$, by Definition 1. So by Lemma 1: (i) if L is a positive literal then $val^{\mathbf{b}}_{\widehat{\mathrm{T}}_{\mathcal{I}}}(\mathsf{L}) = \mathbf{t}$; and (ii) if L is a negative literal then $val^{\mathbf{b}}_{\widehat{\mathrm{U}}_{\mathcal{I}}}(\mathsf{L}) = \mathbf{t}$. Therefore, $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\text{code}} \in \mathbf{TC}_{\mathrm{P},\widehat{\mathrm{U}}_{\mathcal{I}}}(\widehat{\mathrm{T}}_{\mathcal{I}}) \subseteq \widehat{\mathrm{T}}_{\mathcal{I}}$, by Definition 24 and Lemma 16. Thus $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\text{code}}) = \mathbf{t}$, a contradiction. On the other hand, if $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{B}_{\|\mathsf{o}}) = \mathbf{u}$, then $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\text{code}}) = \mathbf{f}$. Since $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{B}_{\|\mathsf{o}}) = \mathbf{u}$, it follows that $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L}) \geq \mathbf{u}$ for all $\mathsf{L} \in \mathsf{B}_{\|\mathsf{o}}$, by Definition 1. So by Lemma 1: (i) if L is a positive literal then $val^{\mathbf{b}}_{\widehat{\mathrm{U}}_{\mathcal{I}}}(\mathsf{L}) = \mathbf{t}$; and (2) if L is a negative literal then $val^{\mathbf{b}}_{\widehat{\mathrm{T}}_{\mathcal{I}}}(\mathsf{L}) = \mathbf{t}$. Therefore, $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\text{code}} \in \mathbf{TC}_{\mathrm{P},\widehat{\mathrm{T}}_{\mathcal{I}}}(\widehat{\mathrm{U}}_{\mathcal{I}}) \subseteq \widehat{\mathrm{U}}_{\mathcal{I}}$, by Definition 22 and Lemma 16. Thus $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\text{code}}) \geq \mathbf{u}$, a contradiction.

(2) $imode_{\mathcal{I}}(\mathsf{R}_{\|\mathsf{o}}) = \mathbf{u}$, $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\text{code}}) = \mathbf{f}$, and $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{B}_{\|\mathsf{o}}) \geq \mathbf{u}$
Because $imode_{\mathcal{I}}(\mathsf{R}_{\|\mathsf{o}}) = \mathbf{u}$, by Definition 17: (i) $\mathsf{c}[\mathsf{m}] \overset{s.code}{\leadsto}_{\mathcal{I}} \mathsf{o}$ or $\mathsf{c}[\mathsf{m}] \overset{w.code}{\leadsto}_{\mathcal{I}} \mathsf{o}$, and so $\mathsf{c}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o} \in \widehat{\mathrm{U}}_{\mathcal{I}}$ by Definition 29; (ii) $ec(\mathsf{o}, \mathsf{m})$ is not a strong explicit definition and so $ec(\mathsf{o}, \mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$ by Definitions 23 and 4; and (iii) there is no $\mathsf{x} \neq \mathsf{c}$ such that $\mathsf{x}[\mathsf{m}] \overset{s.val}{\leadsto}_{\mathcal{I}} \mathsf{o}$ or $\mathsf{x}[\mathsf{m}] \overset{s.code}{\leadsto}_{\mathcal{I}} \mathsf{o}$. It follows that there is no $\mathsf{x} \neq \mathsf{c}$ such that $\mathsf{x}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{T}}_{\mathcal{I}}$ or $\mathsf{x}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o} \in \widehat{\mathrm{T}}_{\mathcal{I}}$ by Definition 29. Thus $mc(\mathsf{c}, \mathsf{m}, \mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$. Since $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{B}_{\|\mathsf{o}}) \geq \mathbf{u}$, it follows that $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L}) \geq \mathbf{u}$ for all $\mathsf{L} \in \mathsf{B}_{\|\mathsf{o}}$, by Definition 1. So by Lemma 1: (i) if L is a positive literal then $val^{\mathbf{b}}_{\widehat{\mathrm{U}}_{\mathcal{I}}}(\mathsf{L}) = \mathbf{t}$; and (2) if L is a negative literal then $val^{\mathbf{b}}_{\widehat{\mathrm{T}}_{\mathcal{I}}}(\mathsf{L}) = \mathbf{t}$. Thus $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\text{code}} \in \mathbf{TC}_{\mathrm{P},\widehat{\mathrm{T}}_{\mathcal{I}}}(\widehat{\mathrm{U}}_{\mathcal{I}}) \subseteq \widehat{\mathrm{U}}_{\mathcal{I}}$, by Definition 22 and Lemma 16. So, $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\text{code}}) \geq \mathbf{u}$, a contradiction.

(3) $imode_{\mathcal{I}}(\mathsf{R}_{\|\mathsf{o}}) = \mathbf{f}$ and $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\text{code}}) \geq \mathbf{u}$
Because $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\text{code}}) \geq \mathbf{u}$, so $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\text{code}} \in \mathrm{U} \subseteq \widehat{\mathrm{U}}_{\mathcal{I}}$. It follows that $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\text{code}} \in \mathbf{TC}_{\mathrm{P},\widehat{\mathrm{T}}_{\mathcal{I}}}(\widehat{\mathrm{U}}_{\mathcal{I}})$ by Lemma 6. So by Definition 24, $\mathsf{c}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o} \in \widehat{\mathrm{U}}_{\mathcal{I}}$, $ec(\mathsf{o}, \mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$, and $mc(\mathsf{c}, \mathsf{m}, \mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$. Because $\mathsf{c}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o} \in \widehat{\mathrm{U}}_{\mathcal{I}}$, so $\mathsf{c}[\mathsf{m}] \overset{s.code}{\leadsto}_{\mathcal{I}} \mathsf{o}$ or $\mathsf{c}[\mathsf{m}] \overset{w.code}{\leadsto}_{\mathcal{I}} \mathsf{o}$, by Definition 29. Since $ec(\mathsf{o}, \mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$, therefore $ec(\mathsf{o}, \mathsf{m})$ is not a strong explicit definition, by Definitions 23 and 4. Because $mc(\mathsf{c}, \mathsf{m}, \mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$, so there is no $\mathsf{x} \neq \mathsf{c}$ such that $\mathsf{x}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{T}}_{\mathcal{I}}$ or $\mathsf{x}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{T}}_{\mathcal{I}}$. It follows that there is no $\mathsf{x} \neq \mathsf{c}$ such that $\mathsf{x}[\mathsf{m}] \overset{s.val}{\leadsto}_{\mathcal{I}} \mathsf{o}$ or $\mathsf{x}[\mathsf{m}] \overset{s.code}{\leadsto}_{\mathcal{I}} \mathsf{o}$, by Definition 29. Thus o must either weakly or strongly inherit R, by Definitions 17 and 16. Therefore, $imode_{\mathcal{I}}(\mathsf{R}_{\|\mathsf{o}}) \geq \mathbf{u}$, a contradiction.

Proposition 6. *Let* $\mathcal{I} = \langle \mathrm{T}; \mathrm{U} \rangle$ *be a stable interpretation of an F-logic KB* P. *Then* $\mathcal{I}$ *satisfies the cautious ISA transitivity constraint.*

Proof. By Definition 13, we need to show that the following conditions hold:

(1) for all s, c: if there is x such that $\mathcal{I}(\mathsf{s} :: \mathsf{x} \wedge \mathsf{x} :: \mathsf{c}) = \mathbf{u}$ and $\mathcal{I}(\mathsf{s} :: \mathsf{c}) \neq \mathbf{t}$, then $\mathcal{I}(\mathsf{s} :: \mathsf{c}) = \mathbf{u}$;

(2) for all o, c: if there is x such that $\mathcal{I}(\mathsf{o} : \mathsf{x} \wedge \mathsf{x} :: \mathsf{c}) = \mathbf{u}$ and $\mathcal{I}(\mathsf{o} : \mathsf{c}) \neq \mathbf{t}$, then $\mathcal{I}(\mathsf{o} : \mathsf{c}) = \mathbf{u}$.

Suppose $\mathcal{I}(\mathsf{s} :: \mathsf{x} \wedge \mathsf{x} :: \mathsf{c}) = \mathbf{u}$. Then $\mathsf{s} :: \mathsf{x} \in \mathrm{T} \cup \mathrm{U}$ and $\mathsf{x} :: \mathsf{c} \in \mathrm{T} \cup \mathrm{U}$. It follows that $\mathsf{s} :: \mathsf{x} \in \widehat{\mathrm{U}}_{\mathcal{I}}$ and $\mathsf{x} :: \mathsf{c} \in \widehat{\mathrm{U}}_{\mathcal{I}}$, by Definition 29. So $\mathsf{s} :: \mathsf{c} \in \mathbf{IC}^t(\widehat{\mathrm{U}}_{\mathcal{I}})$ by Definition 25. Since $\widehat{\mathrm{U}}_{\mathcal{I}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$ by Definition 30, it follows that $\mathsf{s} :: \mathsf{c} \in \mathbf{IC}^t(\widehat{\mathrm{U}}_{\mathcal{I}}) \subseteq \widehat{\mathrm{U}}_{\mathcal{I}}$, by Lemma 16. Thus $\mathcal{I}(\mathsf{s} :: \mathsf{c}) \geq \mathbf{u}$. But $\mathcal{I}(\mathsf{s} :: \mathsf{c}) \neq \mathbf{t}$. It follows that $\mathcal{I}(\mathsf{s} :: \mathsf{c}) = \mathbf{u}$. Similarly, if $\mathcal{I}(\mathsf{o} : \mathsf{x} \wedge \mathsf{x} :: \mathsf{c}) = \mathbf{u}$ and $\mathcal{I}(\mathsf{o} : \mathsf{c}) \neq \mathbf{t}$, then $\mathcal{I}(\mathsf{o} : \mathsf{c}) = \mathbf{u}$.

Proposition 7. *Let* $\mathcal{I} = \langle \mathrm{T}; \mathrm{U} \rangle$ *be a stable interpretation of an F-logic KB* P. *Then* $\mathcal{I}$ *satisfies the cautious inheritance constraint.*

Proof. By Definition 14, we need to show for all o, m, v, c: $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}}) = \mathbf{u}$ iff the following conditions hold:

(i) o[m] is not a strong explicit definition;

(ii) $\mathsf{c}[\mathsf{m}] \overset{s.val}{\leadsto}_{\mathcal{I}} \mathsf{o}$ or $\mathsf{c}[\mathsf{m}] \overset{w.val}{\leadsto}_{\mathcal{I}} \mathsf{o}$;

(iii) $\mathcal{I}(\mathsf{c}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}}) \geq \mathbf{u}$;

(iv) there is no $\mathsf{x} \neq$ such that $\mathsf{x}[\mathsf{m}] \overset{s.val}{\leadsto}_{\mathcal{I}} \mathsf{o}$ or $\mathsf{x}[\mathsf{m}] \overset{s.code}{\leadsto}_{\mathcal{I}} \mathsf{o}$;

(v) $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}}) \neq \mathbf{t}$.

"$\Longrightarrow$". Because $\mathcal{I}$ is a stable interpretation of P, therefore $\widehat{\mathrm{U}}_{\mathcal{I}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$ by Definition 30. Because $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}} = \mathbf{u}$, therefore $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}} \in \mathrm{T} \cup \mathrm{U} \subseteq \widehat{\mathrm{U}}_{\mathcal{I}}$ by Definition 29. Thus $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}} \in \mathbf{IC}^i_{\mathrm{P},\widehat{\mathrm{T}}_{\mathcal{I}}}(\widehat{\mathrm{U}}_{\mathcal{I}})$, by Lemma 6. So $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{U}}_{\mathcal{I}}$, $\mathsf{c}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}} \in \widehat{\mathrm{U}}_{\mathcal{I}}$, $ec(\mathsf{o}, \mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$, and $mc(\mathsf{c}, \mathsf{m}, \mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$, according to Definition 25. Because $ec(\mathsf{o}, \mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$, it follows that $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{x}]_{\mathrm{ex}} \notin \widehat{\mathrm{T}}_{\mathcal{I}}$ for all x, by Definition 23. So $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{x}]_{\mathrm{ex}}) \neq \mathbf{t}$ for all x. Thus o[m] is not a strong explicit definition by Definition 4. Because $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{U}}_{\mathcal{I}}$, it follows that $\mathsf{c}[\mathsf{m}] \overset{s.val}{\leadsto}_{\mathcal{I}} \mathsf{o}$ or $\mathsf{c}[\mathsf{m}] \overset{w.val}{\leadsto}_{\mathcal{I}} \mathsf{o}$, by Definition 29. $\mathsf{c}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}} \in \widehat{\mathrm{U}}_{\mathcal{I}}$ implies $\mathcal{I}(\mathsf{c}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}}) \geq \mathbf{u}$. Because $mc(\mathsf{c}, \mathsf{m}, \mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$, it follows that there is no $\mathsf{x} \neq \mathsf{c}$ such that $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{T}}_{\mathcal{I}}$ or $\mathsf{c}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o} \in \widehat{\mathrm{T}}_{\mathcal{I}}$. So there is no $\mathsf{x} \neq \mathsf{c}$ such that $\mathsf{c}[\mathsf{m}] \overset{s.val}{\leadsto}_{\mathcal{I}} \mathsf{o}$ or $\mathsf{c}[\mathsf{m}] \overset{s.code}{\leadsto}_{\mathcal{I}} \mathsf{o}$, by Definition 29.

"$\Longleftarrow$". Because o[m] is not a strong explicit definition, $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{x}]_{\mathrm{ex}}) \neq \mathbf{t}$ for all x, by Definition 4. It follows that $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{x}]_{\mathrm{ex}} \notin \mathrm{T}$ for all x, and so $ec(\mathsf{o}, \mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$, by Definitions 29 and 23. Because $\mathsf{c}[\mathsf{m}] \overset{s.val}{\leadsto}_{\mathcal{I}} \mathsf{o}$ or $\mathsf{c}[\mathsf{m}] \overset{w.val}{\leadsto}_{\mathcal{I}} \mathsf{o}$, therefore $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{U}}_{\mathcal{I}}$ by Definition 29. Since $\mathcal{I}(\mathsf{c}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}}) \geq \mathbf{u}$, it follows that $\mathsf{c}[\mathsf{m} \rightarrow \mathsf{v}]_{\mathrm{ex}} \in \mathrm{T} \cup \mathrm{U} \subseteq \widehat{\mathrm{U}}_{\mathcal{I}}$. Because $\mathcal{I}$ is a stable interpretation of P, therefore $\widehat{\mathrm{U}}_{\mathcal{I}} = \Psi_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$, by Definition 30. So if we can show $mc(\mathsf{c}, \mathsf{m}, \mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$, then it follows that $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}} \in \mathbf{IC}^i_{\mathrm{P},\widehat{\mathrm{T}}_{\mathcal{I}}}(\widehat{\mathrm{U}}_{\mathcal{I}}) \subseteq \widehat{\mathrm{U}}_{\mathcal{I}}$, by Definition 25 and Lemma 16. Suppose, to the contrary, that $mc(\mathsf{c}, \mathsf{m}, \mathsf{o}) \in \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$. Then there is $\mathsf{x} \neq \mathsf{c}$ such that $\mathsf{x}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{T}}_{\mathcal{I}}$ or $\mathsf{x}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o} \in \widehat{\mathrm{T}}_{\mathcal{I}}$ by Definition 23. It follows that $\mathsf{x}[\mathsf{m}] \overset{s.val}{\leadsto}_{\mathcal{I}} \mathsf{o}$ or $\mathsf{x}[\mathsf{m}] \overset{s.code}{\leadsto}_{\mathcal{I}} \mathsf{o}$, by Definition 29, which contradicts the premise. Therefore,

$mc(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\mathcal{I}})$, and so $\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}} \in \widehat{\mathrm{U}}_{\mathcal{I}}$, $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}}) \geq \mathbf{u}$. But $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}}) \neq \mathbf{t}$. So $\mathcal{I}(\mathsf{o}[\mathsf{m} \rightarrow \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}}) = \mathbf{u}$.

A.2 Proof of Theorem 5 in Section 9

Theorem 5. *The cautious object model $\mathcal{M}$ of an F-logic KB* P *is minimal among those object models of* P *that satisfy the cautious ISA transitivity constraint and the cautious inheritance constraint.*

Proof. Recall that $\mathcal{M} = \langle\, \pi(\widehat{\mathrm{T}}_\infty); \pi(\widehat{\mathrm{U}}_\infty) - \pi(\widehat{\mathrm{T}}_\infty)\,\rangle$. Let $\mathcal{I} = \langle\, \mathrm{T}; \mathrm{U}\,\rangle$ be an object model of P that satisfies the cautious ISA transitivity constraint and the cautious inheritance constraint. Moreover, $\mathcal{I} \leq \mathcal{M}$. To show that $\mathcal{M}$ is minimal, it suffices to show that $\mathrm{T} = \widehat{\mathrm{T}}_\infty$ and $\mathrm{T} \cup \mathrm{U} = \widehat{\mathrm{U}}_\infty$. By Definition 35: (i) $\mathrm{T} \subseteq \widehat{\mathrm{T}}_\infty$; (ii) $\mathrm{T} \cup \mathrm{U} \subseteq \widehat{\mathrm{U}}_\infty$; (iii) for all $\mathsf{c},\mathsf{m},\mathsf{o}$: $\mathsf{c}[\mathsf{m}] \overset{s.val}{\leadsto}_{\mathcal{I}} \mathsf{o}$ implies $\mathsf{c}[\mathsf{m}] \overset{s.val}{\leadsto}_{\mathcal{M}} \mathsf{o}$; and (iv) for all $\mathsf{c},\mathsf{m},\mathsf{o}$: $\mathsf{c}[\mathsf{m}] \overset{s.code}{\leadsto}_{\mathcal{I}} \mathsf{o}$ implies $\mathsf{c}[\mathsf{m}] \overset{s.code}{\leadsto}_{\mathcal{M}} \mathsf{o}$. Let $\mathcal{J} = \langle\, \mathrm{T}; \emptyset\,\rangle$ and $\mathcal{K} = \langle\, \mathrm{T} \cup \mathrm{U}; \emptyset\,\rangle$.

Suppose, to the contrary, that $\mathrm{T} \subset \widehat{\mathrm{T}}_\infty$. Since $\widehat{\mathrm{T}}_\infty = \bigcup_\gamma \widehat{\mathrm{T}}_\gamma$ by Definition 32 and $\{\widehat{\mathrm{T}}_\gamma\}$ is an increasing sequence by Lemma 9, let α be the first ordinal such that $\mathrm{T} \subset \widehat{\mathrm{T}}_\alpha$ and $\mathrm{T} \supseteq \widehat{\mathrm{T}}_\gamma$ for all $\gamma < \alpha$. Clearly, α must be a successor ordinal. Thus $\widehat{\mathrm{T}}_\alpha = \mathrm{lfp}(\mathbf{T}_{\mathrm{P},\widehat{\mathrm{U}}_{\alpha-1}})$, by Definitions 32 and 27. Since $\mathbf{T}_{\mathrm{P},\widehat{\mathrm{U}}_{\alpha-1}}$ is monotonic by Lemma 5, it follows that the ordinal powers of $\mathbf{T}_{\mathrm{P},\widehat{\mathrm{U}}_{\alpha-1}}$ is an increasing sequence. Denote $\widehat{\mathrm{J}}_\gamma = \mathbf{T}^\gamma_{\mathrm{P},\widehat{\mathrm{U}}_{\alpha-1}}$ for all ordinal γ. Let β be the first ordinal such that $\mathrm{T} \subset \widehat{\mathrm{J}}_\beta$ and $\mathrm{T} \supseteq \widehat{\mathrm{J}}_\gamma$ for all $\gamma < \beta$. Clearly, β must be a successor ordinal. Let A be any atom in $\mathcal{HB}_{\mathrm{P}}$ such that $\mathsf{A} \notin \mathrm{T}$ and $\mathsf{A} \in \widehat{\mathrm{J}}_\beta$. By Definitions 26 and 25, we have:

$$\begin{aligned}\widehat{\mathrm{J}}_\beta = {} & \mathbf{RC}_{\mathrm{P},\widehat{\mathrm{U}}_{\alpha-1}}(\widehat{\mathrm{J}}_{\beta-1}) \cup \mathbf{TC}_{\mathrm{P},\widehat{\mathrm{U}}_{\alpha-1}}(\widehat{\mathrm{J}}_{\beta-1}) \cup \\ & \mathbf{IC}^t(\widehat{\mathrm{J}}_{\beta-1}) \cup \mathbf{IC}^c_{\mathrm{P},\widehat{\mathrm{U}}_{\alpha-1}}(\widehat{\mathrm{J}}_{\beta-1}) \cup \mathbf{IC}^i_{\mathrm{P},\widehat{\mathrm{U}}_{\alpha-1}}(\widehat{\mathrm{J}}_{\beta-1})\end{aligned}$$

There are four cases to consider:

(1) $\mathsf{A} \in \mathbf{RC}_{\mathrm{P},\widehat{\mathrm{U}}_{\alpha-1}}(\widehat{\mathrm{J}}_{\beta-1})$

By Definition 22, there must exist a regular rule, $\mathsf{H} \coloneq \mathsf{L}_1, \ldots, \mathsf{L}_n$, in *ground*(P), such that H matches A, and for all $\mathsf{L}_i, 1 \leq i \leq n$: (i) if L_i is a positive literal, then $val^{\mathbf{b}}_{\widehat{\mathrm{J}}_{\beta-1}}(\mathsf{L}_i) = \mathbf{t}$; and (ii) if L_i is a negative literal, then $val^{\mathbf{b}}_{\widehat{\mathrm{U}}_{\alpha-1}}(\mathsf{L}_i) = \mathbf{t}$. Next we show that for all $\mathsf{L}_i, 1 \leq i \leq n$, $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L}_i) = \mathbf{t}$. If L_i is a positive literal, since $\widehat{\mathrm{J}}_{\beta-1} \subseteq \mathrm{T}$ and $val^{\mathbf{b}}_{\widehat{\mathrm{J}}_{\beta-1}}(\mathsf{L}_i) = \mathbf{t}$, then it follows that $\mathcal{V}^{\mathbf{b}}_{\mathcal{J}}(\mathsf{L}_i) = \mathbf{t}$, by Lemma 2. Thus $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L}_i) = \mathbf{t}$ by Lemma 1. Note that $\widehat{\mathrm{U}}_\infty \subseteq \widehat{\mathrm{U}}_{\alpha-1}$ by Lemma 9. It follows that $\mathrm{T} \cup \mathrm{U} \subseteq \widehat{\mathrm{U}}_\infty \subseteq \widehat{\mathrm{U}}_{\alpha-1}$. Therefore, if L_i is a negative literal, since $val^{\mathbf{b}}_{\widehat{\mathrm{U}}_{\alpha-1}}(\mathsf{L}_i) = \mathbf{t}$, then it follows that $\mathcal{V}^{\mathbf{b}}_{\mathcal{K}}(\mathsf{L}_i) = \mathbf{t}$, by Lemma 2. Thus $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L}_i) = \mathbf{t}$ by Lemma 1. Because $\mathcal{I}$ satisfies P, it follows that $\mathcal{I}(\mathsf{A}) = \mathcal{V}^{\mathbf{h}}_{\mathcal{I}}(\mathsf{H}) = \mathbf{t}$. Thus $\mathsf{A} \in \mathrm{T}$, a contradiction.

(2) $\mathsf{A} \in \mathbf{TC}_{\mathrm{P},\widehat{\mathrm{U}}_{\alpha-1}}(\widehat{\mathrm{J}}_{\beta-1})$

It must be true that $\mathsf{A} = \mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\text{code}}$. So, by Definition 24, $\mathsf{c}[\mathsf{m}] \overset{code}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{J}}_{\beta-1}$, $ec(\mathsf{o},\mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{U}}_{\alpha-1})$, $mc(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{U}}_{\alpha-1})$, and there is a template rule, $\mathsf{R} \equiv \mathsf{code}(\mathsf{c})\ @\mathsf{this}[\mathsf{m} \to \mathsf{v}] \ :-\ \mathsf{B}$, in $ground(\mathrm{P})$ such that for every literal $\mathsf{L} \in \mathsf{B}_{\|\mathsf{o}}$: (i) if L is a positive literal then $val^{\mathbf{b}}_{\widehat{\mathrm{J}}_{\beta-1}}(\mathsf{L}) = \mathbf{t}$; and (ii) if L is a negative literal then $val^{\mathbf{b}}_{\widehat{\mathrm{U}}_{\alpha-1}}(\mathsf{L}) = \mathbf{t}$.

Because $\mathsf{c}[\mathsf{m}] \overset{code}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{J}}_{\beta-1}$, there must exist a successor ordinal $\rho \le \beta-1 < \beta$, such that $\mathsf{c}[\mathsf{m}] \overset{code}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{J}}_{\rho}$. It follows that $\mathsf{c}[\mathsf{m}] \overset{code}{\rightsquigarrow} \mathsf{o} \in \mathbf{IC}^{c}_{\mathrm{P},\widehat{\mathrm{U}}_{\alpha-1}}(\widehat{\mathrm{J}}_{\rho-1})$. Thus $\mathsf{c} \neq \mathsf{o}$, $\mathsf{o}:\mathsf{c} \in \widehat{\mathrm{J}}_{\rho-1}$, and $ov(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{U}}_{\alpha-1})$, by Definition 25. Note that $\widehat{\mathrm{J}}_{\rho-1} \subseteq \mathrm{T}$ and $\mathrm{T} \cup \mathrm{U} \subseteq \widehat{\mathrm{U}}_{\infty} \subseteq \widehat{\mathrm{U}}_{\alpha-1}$. So, $\mathsf{o}:\mathsf{c} \in \mathrm{T}$ and $ov(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\mathrm{T} \cup \mathrm{U})$. Thus $\mathsf{c}[\mathsf{m}] \overset{s.code}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$ by Lemma 4. Because $ec(\mathsf{o},\mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{U}}_{\alpha-1})$, it follows that $ec(\mathsf{o},\mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\mathrm{T} \cup \mathrm{U})$ by the monotonicity of $\mathbf{IB}_{\mathrm{P}}$. It follows that $\mathsf{o}[\mathsf{m}]$ is neither a strong nor a weak explicit definition in $\mathcal{I}$, by Definitions 23 and 4.
Next we show that there is no x such that $\mathsf{x} \neq \mathsf{c}$, $\mathsf{x}[\mathsf{m}] \rightsquigarrow_{\mathcal{I}} \mathsf{o}$. Suppose, to the contrary, that there is $\mathsf{x} \neq \mathsf{c}$ such that $\mathsf{x}[\mathsf{m}] \rightsquigarrow_{\mathcal{I}} \mathsf{o}$. Then $\mathsf{x} \neq \mathsf{o}$, $\mathsf{o}:\mathsf{x} \in \mathrm{T} \cup \mathrm{U}$, $\mathsf{x}[\mathsf{m} \to \mathsf{y}]_{\text{ex}} \in \mathrm{T} \cup \mathrm{U}$ for some y or there is a template rule in $ground(\mathrm{P})$ that specifies the instance method m for class c, and $ov(\mathsf{x},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\mathrm{T})$, by Lemma 4. Since $\mathrm{T} \cup \mathrm{U} \subseteq \widehat{\mathrm{U}}_{\infty} \subseteq \widehat{\mathrm{U}}_{\alpha-1}$ and $\mathrm{T} \supseteq \widehat{\mathrm{T}}_{\alpha-1}$, so $\mathsf{o}:\mathsf{x} \in \widehat{\mathrm{U}}_{\alpha-1}$, $\mathsf{x}[\mathsf{m} \to \mathsf{y}]_{\text{ex}} \in \widehat{\mathrm{U}}_{\alpha-1}$ for some y or there is a template rule that specifies the instance method m for class c, and $ov(\mathsf{x},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\alpha-1})$. It follows that $\mathsf{x}[\mathsf{m}] \overset{val}{\rightsquigarrow} \mathsf{o} \in \mathbf{IC}^{c}_{\mathrm{P},\widehat{\mathrm{T}}_{\alpha-1}}(\widehat{\mathrm{U}}_{\alpha-1})$ or $\mathsf{x}[\mathsf{m}] \overset{code}{\rightsquigarrow} \mathsf{o} \in \mathbf{IC}^{c}_{\mathrm{P},\widehat{\mathrm{T}}_{\alpha-1}}(\widehat{\mathrm{U}}_{\alpha-1})$, by Definition 25. Thus $mc(\mathsf{c},\mathsf{m},\mathsf{o}) \in \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{U}}_{\alpha-1})$, by Definition 23, which contradicts the fact that $mc(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{U}}_{\alpha-1})$.

So far we have shown that $\mathsf{o}[\mathsf{m}]$ is neither a strong nor a weak explicit definition in $\mathcal{I}$, $\mathsf{c}[\mathsf{m}] \overset{s.code}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$, and there is no x such that $\mathsf{x} \neq \mathsf{c}$ and $\mathsf{x}[\mathsf{m}] \rightsquigarrow_{\mathcal{I}} \mathsf{o}$. Therefore, o strongly inherits R in $\mathcal{I}$, by Definition 16. So $imode_{\mathcal{I}}(\mathsf{R}_{\|\mathsf{o}}) = \mathbf{t}$. We already know that for every literal $\mathsf{L} \in \mathsf{B}_{\|\mathsf{o}}$: (i) if L is positive then $val^{\mathbf{b}}_{\widehat{\mathrm{J}}_{\beta-1}}(\mathsf{L}) = \mathbf{t}$; and (ii) if L is negative then $val^{\mathbf{b}}_{\widehat{\mathrm{U}}_{\alpha-1}}(\mathsf{L}) = \mathbf{t}$. Now we will show that for all $\mathsf{L} \in \mathsf{B}_{\|\mathsf{o}}$, $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L}) = \mathbf{t}$. If L is a positive literal, since $\widehat{\mathrm{J}}_{\beta-1} \subseteq \mathrm{T}$ and $val^{\mathbf{b}}_{\widehat{\mathrm{J}}_{\beta-1}}(\mathsf{L}) = \mathbf{t}$, then it follows that $\mathcal{V}^{\mathbf{b}}_{\mathcal{J}}(\mathsf{L}) = \mathbf{t}$, by Lemma 2. Thus $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L}) = \mathbf{t}$ by Lemma 1. Note that $\widehat{\mathrm{U}}_{\infty} \subseteq \widehat{\mathrm{U}}_{\alpha-1}$ by Lemma 9. It follows that $\mathrm{T} \cup \mathrm{U} \subseteq \widehat{\mathrm{U}}_{\infty} \subseteq \widehat{\mathrm{U}}_{\alpha-1}$. Therefore, if L is a negative literal, since $val^{\mathbf{b}}_{\widehat{\mathrm{U}}_{\alpha-1}}(\mathsf{L}) = \mathbf{t}$, then it follows that $\mathcal{V}^{\mathbf{b}}_{\mathcal{K}}(\mathsf{L}) = \mathbf{t}$, by Lemma 2. Thus $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L}) = \mathbf{t}$ by Lemma 1.

Therefore, $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L}) = \mathbf{t}$ for every literal $\mathsf{L} \in \mathsf{B}_{\|\mathsf{o}}$. It follows that $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{B}_{\|\mathsf{o}}) = \mathbf{t}$. In the above we have shown that $imode_{\mathcal{I}}(\mathsf{R}_{\|\mathsf{o}}) = \mathbf{t}$. Since $\mathcal{I}$ is an object model of P, so $\mathcal{I}$ should satisfy $\mathsf{R}_{\|\mathsf{o}}$. Thus $\mathcal{I}(\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\text{code}}) = \mathbf{t}$, by Definition 18. It follows that $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\text{code}} \in \mathrm{T}$, a contradiction.

(3) $\mathsf{A} \in \mathbf{IC}^{t}(\widehat{\mathrm{J}}_{\beta-1})$

If $\mathsf{A} = \mathsf{o}:\mathsf{c}$, then there exists x, such that $\mathsf{o}:\mathsf{x} \in \widehat{\mathrm{J}}_{\beta-1}$ and $\mathsf{x}::\mathsf{c} \in \widehat{\mathrm{J}}_{\beta-1}$, by Definition 25. Since $\widehat{\mathrm{J}}_{\beta-1} \subseteq \mathrm{T}$, it follows that $\mathsf{o}:\mathsf{x} \in \mathrm{T}$ and $\mathsf{x}::\mathsf{c} \in \mathrm{T}$. So $\mathcal{I}(\mathsf{o}:\mathsf{x}) = \mathbf{t}$ and $\mathcal{I}(\mathsf{x}::\mathsf{c}) = \mathbf{t}$. Because $\mathcal{I}$ is an object model of P and so satisfies the positive ISA transitivity constraint, therefore $\mathcal{I}(\mathsf{o}:\mathsf{c}) = \mathbf{t}$ by Definition 10. It follows that $\mathsf{o}:\mathsf{c} \in \mathrm{T}$, a contradiction. Similarly, if $\mathsf{A} = \mathsf{s}::\mathsf{c}$, then we can also show that $\mathsf{s}::\mathsf{c} \in \mathrm{T}$, which is a contradiction.

(4) $\mathsf{A} \in \mathbf{IC}^{i}_{\mathrm{P},\widehat{\mathrm{U}}_{\alpha-1}}(\widehat{\mathrm{J}}_{\beta-1})$

It must be the case $\mathsf{A} = \mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}}$. Thus, by Definition 25, $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{J}}_{\beta-1}$, $\mathsf{c}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}} \in \widehat{\mathrm{J}}_{\beta-1}$, $ec(\mathsf{o},\mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{U}}_{\alpha-1})$, and $mc(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{U}}_{\alpha-1})$. Because $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{J}}_{\beta-1}$, there must exist a successor ordinal $\rho \le \beta - 1 < \beta$, such that $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{J}}_{\rho}$. It follows that $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \mathbf{IC}^{c}_{\mathrm{P},\widehat{\mathrm{U}}_{\alpha-1}}(\widehat{\mathrm{J}}_{\rho-1})$. So, $\mathsf{c} \ne \mathsf{o}$, $\mathsf{o}:\mathsf{c} \in \widehat{\mathrm{J}}_{\rho-1}$, $\mathsf{c}[\mathsf{m} \to \mathsf{z}]_{\mathrm{ex}} \in \widehat{\mathrm{J}}_{\rho-1}$ for some z, and $ov(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{U}}_{\alpha-1})$, by Definition 25. Since $\widehat{\mathrm{J}}_{\rho-1} \subseteq \widehat{\mathrm{J}}_{\beta-1} \subseteq \mathrm{T}$ and $\mathrm{T} \cup \mathrm{U} \subseteq \widehat{\mathrm{U}}_{\infty} \subseteq \widehat{\mathrm{U}}_{\alpha-1}$, it follows that $\mathsf{o}:\mathsf{c} \in \mathrm{T}$, $\mathsf{c}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}} \in \mathrm{T}$, and $ov(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\mathrm{T} \cup \mathrm{U})$. Thus $\mathsf{c}[\mathsf{m}] \overset{s.val}{\leadsto}_{\mathcal{I}} \mathsf{o}$ by Lemma 4. Because $ec(\mathsf{o},\mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{U}}_{\alpha-1})$, so $ec(\mathsf{o},\mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\mathrm{T} \cup \mathrm{U})$ by the monotonicity of $\mathbf{IB}_{\mathrm{P}}$. It follows that $\mathsf{o}[\mathsf{m}]$ is neither a strong nor a weak explicit definition in $\mathcal{I}$, by Definitions 23 and 4.

Next we show that there is no x such that $\mathsf{x} \ne \mathsf{c}$, $\mathsf{x}[\mathsf{m}] \leadsto_{\mathcal{I}} \mathsf{o}$. Suppose, to the contrary, that there is $\mathsf{x} \ne \mathsf{c}$ such that $\mathsf{x}[\mathsf{m}] \leadsto_{\mathcal{I}} \mathsf{o}$. Then $\mathsf{x} \ne \mathsf{o}$, $\mathsf{o}:\mathsf{x} \in \mathrm{T} \cup \mathrm{U}$, $\mathsf{x}[\mathsf{m} \to \mathsf{y}]_{\mathrm{ex}} \in \mathrm{T} \cup \mathrm{U}$ for some y or there is a template rule in *ground*(P) which specifies the instance method m for class c, and $ov(\mathsf{x},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\mathrm{T})$, by Lemma 4. Since $\mathrm{T} \cup \mathrm{U} \subseteq \widehat{\mathrm{U}}_{\infty} \subseteq \widehat{\mathrm{U}}_{\alpha-1}$ and $\mathrm{T} \supseteq \widehat{\mathrm{T}}_{\alpha-1}$, it follows that $\mathsf{o}:\mathsf{x} \in \widehat{\mathrm{U}}_{\alpha-1}$, $\mathsf{x}[\mathsf{m} \to \mathsf{y}]_{\mathrm{ex}} \in \widehat{\mathrm{U}}_{\alpha-1}$ for some y or there is a template rule that specifies the instance method m for class c, and $ov(\mathsf{x},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\alpha-1})$. It follows that $\mathsf{x}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \mathbf{IC}^{c}_{\mathrm{P},\widehat{\mathrm{T}}_{\alpha-1}}(\widehat{\mathrm{U}}_{\alpha-1})$ or $\mathsf{x}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o} \in \mathbf{IC}^{c}_{\mathrm{P},\widehat{\mathrm{T}}_{\alpha-1}}(\widehat{\mathrm{U}}_{\alpha-1})$, by Definition 25. Therefore, $mc(\mathsf{c},\mathsf{m},\mathsf{o}) \in \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{U}}_{\alpha-1})$, by Definition 23, which contradicts the fact that $mc(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{U}}_{\alpha-1})$.

So far we have shown that $\mathsf{o}[\mathsf{m}]$ is neither a strong nor a weak explicit definition in $\mathcal{I}$, $\mathsf{c}[\mathsf{m}] \overset{s.val}{\leadsto}_{\mathcal{I}} \mathsf{o}$, $\mathcal{I}(\mathsf{c}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}}) = \mathbf{t}$, and there is no x such that $\mathsf{x} \ne \mathsf{c}$ and $\mathsf{x}[\mathsf{m}] \leadsto_{\mathcal{I}} \mathsf{o}$. Because $\mathcal{I}$ is an object model of P and so satisfies the unique source inheritance constraint, therefore $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}} \in \mathrm{T}$ by Definition 12, a contradiction.

Therefore, if $\mathrm{T} \subset \widehat{\mathrm{T}}_{\infty}$, then we can derive a contradiction in all four possible cases. So it must be true that $\mathrm{T} = \widehat{\mathrm{T}}_{\infty}$. It remains to show that $\mathrm{T} \cup \mathrm{U} = \widehat{\mathrm{U}}_{\infty}$. We know that $\mathrm{T} \cup \mathrm{U} \subseteq \widehat{\mathrm{U}}_{\infty}$, because $\mathcal{I} \le \mathcal{M}$. Therefore, if we can show that $\mathrm{T} \cup \mathrm{U} \supseteq \widehat{\mathrm{U}}_{\infty}$, then $\mathrm{T} \cup \mathrm{U} = \widehat{\mathrm{U}}_{\infty}$. By Definitions 32 and 27, $\widehat{\mathrm{U}}_{\infty} = \mathrm{lfp}(\mathbf{T}_{\mathrm{P},\widehat{\mathrm{T}}_{\infty}})$. Since $\mathbf{T}_{\mathrm{P},\widehat{\mathrm{T}}_{\infty}}$ is monotonic, the ordinal powers of $\mathbf{T}_{\mathrm{P},\widehat{\mathrm{T}}_{\infty}}$ is an increasing sequence. Denote $\widehat{\mathrm{K}}_{\gamma} = \mathbf{T}^{\gamma}_{\mathrm{P},\widehat{\mathrm{T}}_{\infty}}$ for all ordinal γ. We will prove by transfinite induction that $\mathrm{T} \cup \mathrm{U} \supseteq \widehat{\mathrm{K}}_{\alpha}$ for all ordinal α, thus complete the proof.

The case for a limit ordinal α is trivial. If $\alpha = 0$, then $\widehat{\mathrm{K}}_{0} = \emptyset \subseteq \mathrm{T} \cup \mathrm{U}$. If $\alpha \ne 0$, then $\widehat{\mathrm{K}}_{\alpha} = \bigcup_{\beta<\alpha} \widehat{\mathrm{K}}_{\beta}$. By the induction hypothesis we know that $\mathrm{T} \cup \mathrm{U} \supseteq \widehat{\mathrm{K}}_{\beta}$ for all $\beta < \alpha$. So $\mathrm{T} \cup \mathrm{U} \supseteq \widehat{\mathrm{K}}_{\alpha}$.

Let α be a successor ordinal and A any atom in $\mathcal{HB}_{\mathrm{P}}$ such that $\mathsf{A} \in \widehat{\mathrm{K}}_{\alpha}$. We will show $\mathsf{A} \in \mathrm{T} \cup \mathrm{U}$. By Definitions 26 and 25, we have:

$$\widehat{\mathrm{K}}_{\alpha} = \mathbf{RC}_{\mathrm{P},\widehat{\mathrm{T}}_{\infty}}(\widehat{\mathrm{K}}_{\alpha-1}) \cup \mathbf{TC}_{\mathrm{P},\widehat{\mathrm{T}}_{\infty}}(\widehat{\mathrm{K}}_{\alpha-1}) \cup \mathbf{IC}^{t}(\widehat{\mathrm{K}}_{\alpha-1}) \cup \mathbf{IC}^{c}_{\mathrm{P},\widehat{\mathrm{T}}_{\infty}}(\widehat{\mathrm{K}}_{\alpha-1}) \cup \mathbf{IC}^{i}_{\mathrm{P},\widehat{\mathrm{T}}_{\infty}}(\widehat{\mathrm{K}}_{\alpha-1})$$

There are four cases to consider:

(1) $\mathsf{A} \in \mathbf{RC}_{\mathrm{P},\widehat{\mathrm{T}}_{\infty}}(\widehat{\mathrm{K}}_{\alpha-1})$
By Definition 22, there must exist a regular rule, $\mathsf{H} :- \mathsf{L}_1, \ldots, \mathsf{L}_n$, in *ground*(P), such that H matches A, and for all $\mathsf{L}_i, 1 \le i \le n$: (i) if L_i is a positive literal, then $val^{\mathbf{b}}_{\widehat{\mathrm{K}}_{\alpha-1}}(\mathsf{L}_i) = \mathbf{t}$; and (ii) if L_i is a negative literal, then $val^{\mathbf{b}}_{\widehat{\mathrm{T}}_{\infty}}(\mathsf{L}_i) = \mathbf{t}$. Next we show that for all $\mathsf{L}_i, 1 \le i \le n, \mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L}_i) \ge \mathbf{u}$. If L_i is a positive literal, since $\widehat{\mathrm{K}}_{\alpha-1} \subseteq \mathrm{T} \cup \mathrm{U}$ by the induction hypothesis and $val^{\mathbf{b}}_{\widehat{\mathrm{K}}_{\alpha-1}}(\mathsf{L}_i) = \mathbf{t}$, then it follows that $\mathcal{V}^{\mathbf{b}}_{\mathcal{K}}(\mathsf{L}_i) = \mathbf{t}$, by Lemma 2. Thus $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L}_i) \ge \mathbf{u}$ by Lemma 1. We have proved that $\mathrm{T} = \widehat{\mathrm{T}}_{\infty}$. Therefore, if L_i is a negative literal, then $\mathcal{V}^{\mathbf{b}}_{\mathcal{J}}(\mathsf{L}_i) = \mathcal{V}^{\mathbf{b}}_{\widehat{\mathrm{T}}_{\infty}}(\mathsf{L}_i) = \mathbf{t}$. Thus $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L}_i) \ge \mathbf{u}$ by Lemma 1. Because $\mathcal{I}$ satisfies P, so $\mathcal{I}(\mathsf{A}) = \mathcal{V}^{\mathbf{h}}_{\mathcal{I}}(\mathsf{H}) \ge \mathbf{u}$. Thus $\mathsf{A} \in \mathrm{T} \cup \mathrm{U}$.

(2) $\mathsf{A} \in \mathbf{TC}_{\mathrm{P},\widehat{\mathrm{T}}_{\infty}}(\widehat{\mathrm{K}}_{\alpha-1})$
It must be true that $\mathsf{A} = \mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{code}}$. So, by Definition 24, $\mathsf{c}[\mathsf{m}] \overset{code}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{K}}_{\alpha-1}$, $ec(\mathsf{o},\mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\infty})$, $mc(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\infty})$, and *ground*(P) has a template rule, $\mathsf{R} \equiv \mathsf{code(c)}\ @\mathsf{this}[\mathsf{m} \to \mathsf{v}] :- \mathsf{B}$, such that for every literal $\mathsf{L} \in \mathsf{B}_{\|\mathsf{o}}$: (i) if L is a positive literal then $val^{\mathbf{b}}_{\widehat{\mathrm{K}}_{\alpha-1}}(\mathsf{L}) = \mathbf{t}$; and (ii) if L is a negative literal then $val^{\mathbf{b}}_{\widehat{\mathrm{T}}_{\infty}}(\mathsf{L}) = \mathbf{t}$.

Because $\mathsf{c}[\mathsf{m}] \overset{val}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{K}}_{\alpha-1}$, there must exist a successor ordinal $\rho \le \alpha - 1 < \alpha$, such that $\mathsf{c}[\mathsf{m}] \overset{val}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{K}}_{\rho}$. It follows that $\mathsf{c}[\mathsf{m}] \overset{val}{\rightsquigarrow} \mathsf{o} \in \mathbf{IC}^{c}_{\mathrm{P},\widehat{\mathrm{T}}_{\infty}}(\widehat{\mathrm{K}}_{\rho-1})$. Therefore, $\mathsf{c} \ne \mathsf{o}$, $\mathsf{o}:\mathsf{c} \in \widehat{\mathrm{K}}_{\rho-1}$, and $ov(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\infty})$, by Definition 25. Note that $\widehat{\mathrm{K}}_{\rho-1} \subseteq \mathrm{T} \cup \mathrm{U}$, by the induction hypothesis. We have proved that $\mathrm{T} = \widehat{\mathrm{T}}_{\infty}$. It follows that $\mathsf{o}:\mathsf{c} \in \mathrm{T} \cup \mathrm{U}$ and $ov(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\mathrm{T})$. Therefore $\mathsf{c}[\mathsf{m}] \overset{s.code}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$ or $\mathsf{c}[\mathsf{m}] \overset{w.code}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$, by Lemma 4. Because $ec(\mathsf{o},\mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\infty})$, so $ec(\mathsf{o},\mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\mathrm{T})$. Thus $\mathsf{o}[\mathsf{m}]$ is not a strong explicit definition, by Definitions 23 and 4. Because $\mathsf{c}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}} \in \widehat{\mathrm{K}}_{\alpha-1} \subseteq \mathrm{T} \cup \mathrm{U}$, it follows that $\mathcal{I}(\mathsf{c}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}}) \ge \mathbf{u}$.

Next we show that there is no $\mathsf{x} \ne \mathsf{c}$ such that $\mathsf{x}[\mathsf{m}] \overset{s.val}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$ or $\mathsf{x}[\mathsf{m}] \overset{s.code}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$. Suppose, to the contrary, that there exists $\mathsf{x} \ne \mathsf{c}$ such that $\mathsf{x}[\mathsf{m}] \overset{s.val}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$ or $\mathsf{x}[\mathsf{m}] \overset{s.code}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$. Then $\mathsf{x}[\mathsf{m}] \overset{s.val}{\rightsquigarrow}_{\mathcal{M}} \mathsf{o}$ or $\mathsf{x}[\mathsf{m}] \overset{s.code}{\rightsquigarrow}_{\mathcal{M}} \mathsf{o}$, because $\mathcal{I} \le \mathcal{M}$. Thus $\mathsf{x}[\mathsf{m}] \overset{val}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{T}}_{\infty}$ or $\mathsf{x}[\mathsf{m}] \overset{code}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{T}}_{\infty}$, by Lemma 12. Thus $mc(\mathsf{c},\mathsf{m},\mathsf{o}) \in \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\infty})$ by Definition 23, which contradicts the fact that $mc(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\infty})$.

So far we have shown that $\mathsf{c}[\mathsf{m}] \overset{s.code}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$ or $\mathsf{c}[\mathsf{m}] \overset{w.code}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$, $\mathsf{o}[\mathsf{m}]$ is not a strong explicit definition, and there is no $\mathsf{x} \ne \mathsf{c}$ such that $\mathsf{x}[\mathsf{m}] \overset{s.val}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$ or $\mathsf{x}[\mathsf{m}] \overset{s.code}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$. Therefore, o must either strongly or weakly inherit R in $\mathcal{I}$, by Definitions 16 and 16. So $imode_{\mathcal{I}}(\mathsf{R}_{\|\mathsf{o}}) \ge \mathbf{u}$. We already know that for every literal $\mathsf{L} \in \mathsf{B}_{\|\mathsf{o}}$:

(i) if L is a positive literal then $val^{\mathbf{b}}_{\widehat{\mathrm{K}}_{\alpha-1}}(\mathsf{L}) = \mathbf{t}$; and (ii) if L is a negative literal then $val^{\mathbf{b}}_{\widehat{\mathrm{T}}_{\infty}}(\mathsf{L}) = \mathbf{t}$. Now we show that for all $\mathsf{L}_i, 1 \le i \le n, \mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L}_i) \ge \mathbf{u}$. If L_i is a positive literal, since $\widehat{\mathrm{K}}_{\alpha-1} \subseteq \mathrm{T} \cup \mathrm{U}$ by the induction hypothesis and $val^{\mathbf{b}}_{\widehat{\mathrm{K}}_{\alpha-1}}(\mathsf{L}_i) = \mathbf{t}$, then it follows that $\mathcal{V}^{\mathbf{b}}_{\mathcal{K}}(\mathsf{L}_i) = \mathbf{t}$, by Lemma 2. Thus $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L}_i) \ge \mathbf{u}$ by Lemma 1. Note that $\mathrm{T} = \widehat{\mathrm{T}}_{\infty}$. Therefore, if L_i is a negative literal, then $\mathcal{V}^{\mathbf{b}}_{\mathcal{J}}(\mathsf{L}_i) = \mathcal{V}^{\mathbf{b}}_{\widehat{\mathrm{T}}_{\infty}}(\mathsf{L}_i) = \mathbf{t}$. Thus $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L}_i) \ge \mathbf{u}$ by Lemma 1. Therefore, $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{L}) \ge \mathbf{u}$ for every literal $\mathsf{L} \in \mathsf{B}_{\|\mathsf{o}}$. It follows that $\mathcal{V}^{\mathbf{b}}_{\mathcal{I}}(\mathsf{B}_{\|\mathsf{o}}) \ge \mathbf{u}$. Moreover, $imode_{\mathcal{I}}(\mathsf{R}_{\|\mathsf{o}}) \ge \mathbf{u}$. Because $\mathcal{I}$ is an object model of P, so $\mathcal{I}$ should satisfy $\mathsf{R}_{\|\mathsf{o}}$. It follows that $\mathcal{I}(\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{code}}) \ge \mathbf{u}$, by Definition 18. Thus $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{code}} \in \mathrm{T} \cup \mathrm{U}$.

(3) $\mathsf{A} \in \mathbf{IC}^{t}(\widehat{\mathrm{K}}_{\alpha-1})$

If $\mathsf{A} = \mathsf{o}:\mathsf{c}$, then there exists x such that $\mathsf{o}:\mathsf{x} \in \widehat{\mathrm{K}}_{\alpha-1}$ and $\mathsf{x}::\mathsf{c} \in \widehat{\mathrm{K}}_{\alpha-1}$, by Definition 25. Since $\widehat{\mathrm{K}}_{\alpha-1} \subseteq \mathrm{T} \cup \mathrm{U}$ by the induction hypothesis, it follows that $\mathsf{o}:\mathsf{x} \in \mathrm{T}\cup\mathrm{U}$ and $\mathsf{x}::\mathsf{c} \in \mathrm{T}\cup\mathrm{U}$. So $\mathcal{I}(\mathsf{o}:\mathsf{x}) \ge \mathbf{u}$ and $\mathcal{I}(\mathsf{x}::\mathsf{c}) \ge \mathbf{u}$. Because $\mathcal{I}$ satisfies the cautious ISA transitivity constraint, therefore $\mathcal{I}(\mathsf{o}:\mathsf{c}) \ge \mathbf{u}$ by Definitions 13 and 10. It follows that $\mathsf{o}:\mathsf{c} \in \mathrm{T}\cup\mathrm{U}$. Similarly, if $\mathsf{A} = \mathsf{s}::\mathsf{c}$, then we can also show that $\mathsf{s}::\mathsf{c} \in \mathrm{T} \cup \mathrm{U}$.

(4) $\mathsf{A} \in \mathbf{IC}^{i}_{\mathrm{P},\widehat{\mathrm{T}}_{\infty}}(\widehat{\mathrm{K}}_{\alpha-1})$

It must be the case that $\mathsf{A} = \mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}}$. It follows that $\mathsf{c}[\mathsf{m}] \overset{val}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{K}}_{\alpha-1}$, $\mathsf{c}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}} \in \widehat{\mathrm{K}}_{\alpha-1}$, $ec(\mathsf{o},\mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\infty})$, and $mc(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\infty})$, by Definition 25. Because $\mathsf{c}[\mathsf{m}] \overset{val}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{K}}_{\alpha-1}$, so there must exist a successor ordinal $\rho \le \alpha - 1 < \alpha$, such that $\mathsf{c}[\mathsf{m}] \overset{val}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{K}}_{\rho}$. Thus $\mathsf{c}[\mathsf{m}] \overset{val}{\rightsquigarrow} \mathsf{o} \in \mathbf{IC}^{c}_{\mathrm{P},\widehat{\mathrm{T}}_{\infty}}(\widehat{\mathrm{K}}_{\rho-1})$. Therefore, by Definition 25, $\mathsf{c} \ne \mathsf{o}$, $\mathsf{o}:\mathsf{c} \in \widehat{\mathrm{K}}_{\rho-1}$, $\mathsf{c}[\mathsf{m} \to \mathsf{z}]_{\mathrm{ex}} \in \widehat{\mathrm{K}}_{\rho-1}$ for some z, and $ov(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\infty})$. Since $\widehat{\mathrm{K}}_{\rho-1} \subseteq \widehat{\mathrm{K}}_{\alpha-1} \subseteq \mathrm{T} \cup \mathrm{U}$ by the induction hypothesis and $\mathrm{T} = \widehat{\mathrm{T}}_{\infty}$, it follows that $\mathsf{o}:\mathsf{c} \in \mathrm{T} \cup \mathrm{U}$, $\mathsf{c}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}} \in \mathrm{T} \cup \mathrm{U}$, and $ov(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\mathrm{T})$. Thus $\mathsf{c}[\mathsf{m}] \overset{s.val}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$ or $\mathsf{c}[\mathsf{m}] \overset{w.val}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$ by Lemma 4. Because $ec(\mathsf{o},\mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\infty})$, it follows that $ec(\mathsf{o},\mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\mathrm{T})$, and so $\mathsf{o}[\mathsf{m}]$ is not a strong explicit definition, by Definitions 23 and 4. Because $\mathsf{c}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}} \in \mathrm{T}\cup\mathrm{U}$, it follows that $\mathcal{I}(\mathsf{c}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}}) \ge \mathbf{u}$.

Now we show that there is no $\mathsf{x} \ne \mathsf{c}$ such that $\mathsf{x}[\mathsf{m}] \overset{s.val}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$ or $\mathsf{x}[\mathsf{m}] \overset{s.code}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$. Suppose, to the contrary, that there exists $\mathsf{x} \ne \mathsf{c}$ such that $\mathsf{x}[\mathsf{m}] \overset{s.val}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$ or $\mathsf{x}[\mathsf{m}] \overset{s.code}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$. Then $\mathsf{x}[\mathsf{m}] \overset{s.val}{\rightsquigarrow}_{\mathcal{M}} \mathsf{o}$ or $\mathsf{x}[\mathsf{m}] \overset{s.code}{\rightsquigarrow}_{\mathcal{M}} \mathsf{o}$, because $\mathcal{I} \le \mathcal{M}$. It follows that $\mathsf{x}[\mathsf{m}] \overset{val}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{T}}_{\infty}$ or $\mathsf{x}[\mathsf{m}] \overset{code}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{T}}_{\infty}$, by Lemma 12. Thus $mc(\mathsf{c},\mathsf{m},\mathsf{o}) \in \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\infty})$ by Definition 23, which contradicts the fact that $mc(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{T}}_{\infty})$.

So far we have shown that $\mathsf{o}[\mathsf{m}]$ is not a strong explicit definition, $\mathsf{c}[\mathsf{m}] \overset{s.val}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$ or $\mathsf{c}[\mathsf{m}] \overset{w.val}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$, $\mathcal{I}(\mathsf{c}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}}) \ge \mathbf{u}$, and there is no $\mathsf{x} \ne \mathsf{c}$ such that $\mathsf{x}[\mathsf{m}] \overset{s.val}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$ or $\mathsf{x}[\mathsf{m}] \overset{s.code}{\rightsquigarrow}_{\mathcal{I}} \mathsf{o}$. Because $\mathcal{I}$ satisfies the cautious inheritance constraint, therefore $\mathcal{I}(\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}}) \ge \mathbf{u}$, by Definition 14. So $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}} \in \mathrm{T} \cup \mathrm{U}$.

We have shown that in all four possible cases, if $\mathsf{A} \in \widehat{\mathrm{K}}_{\alpha}$, then $\mathsf{A} \in \mathrm{T} \cup \mathrm{U}$. It follows that $\mathrm{T} \cup \mathrm{U} \supseteq \widehat{\mathrm{K}}_{\alpha}$. This completes the induction step.

A.3 Lemmas and Propositions Supporting Theorem 6 in Section 10

Lemma 17. *Let* P^{wf} *be the well-founded rewriting of an F-logic KB* P *and* I^{wf} *be a subset of* $\mathcal{HB}_{\mathrm{P}^{wf}}$*. Then* $lfp(\mathbf{C}_{\mathrm{P}^{wf},\mathrm{I}^{wf}})$ *is in normal form.*

Lemma 18. *Let* P^{wf} *be the well-founded rewriting of an F-logic KB* P, $\widehat{\mathrm{I}}$ *a subset of* $\widehat{\mathcal{HB}}_{\mathrm{P}}$, I^{wf} *a subset of* $\mathcal{HB}_{\mathrm{P}^{wf}}$ *which is isomorphic to* $\widehat{\mathrm{I}}$ *and is in normal form, and* G *a ground positive literal. Then* $val^{\mathbf{b}}_{\widehat{\mathrm{I}}}(\neg\,\mathsf{G}) = \mathbf{t}$ *iff* $\rho^b(\mathsf{G}) \notin \mathrm{I}^{wf}$.

Proposition 8. *Let* P^{wf} *be the well-founded rewriting of an F-logic KB* P, I^{wf} *a subset of* $\mathcal{HB}_{\mathrm{P}^{wf}}$ *which is in normal form, and* $\widehat{\mathrm{I}}$ *a subset of* $\widehat{\mathcal{HB}}_{\mathrm{P}}$*. If* I^{wf} *is isomorphic to* $\widehat{\mathrm{I}}$*, then* $lfp(\mathbf{C}_{\mathrm{P}^{wf},\mathrm{I}^{wf}})$ *is isomorphic to* $lfp(\mathbf{T}_{\mathrm{P},\widehat{\mathrm{I}}})$.

Proof. Let $\mathrm{J}^{wf} = \mathrm{lfp}(\mathbf{C}_{\mathrm{P}^{wf},\mathrm{I}^{wf}})$ and $\widehat{\mathrm{J}} = \mathrm{lfp}(\mathbf{T}_{\mathrm{P},\widehat{\mathrm{I}}})$. First we will show that all of the following conditions are true:

(1) for all o,c: $isa(\mathsf{o},\mathsf{c}) \in \mathrm{J}^{wf}$ iff $\mathsf{o}:\mathsf{c} \in \widehat{\mathrm{J}}$
(2) for all s,c: $sub(\mathsf{s},\mathsf{c}) \in \mathrm{J}^{wf}$ iff $\mathsf{s}::\mathsf{c} \in \widehat{\mathrm{J}}$
(3) for all $\mathsf{s},\mathsf{m},\mathsf{v}$: $exmv(\mathsf{s},\mathsf{m},\mathsf{v}) \in \mathrm{J}^{wf}$ iff $\mathsf{s}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}} \in \widehat{\mathrm{J}}$
(4) for all $\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c}$: $vamv(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c}) \in \mathrm{J}^{wf}$ iff $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}} \in \widehat{\mathrm{J}}$
(5) for all $\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c}$: $comv(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c}) \in \mathrm{J}^{wf}$ iff $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{code}} \in \widehat{\mathrm{J}}$
(6) for all $\mathsf{c},\mathsf{m},\mathsf{o}$: $vacan(\mathsf{c},\mathsf{m},\mathsf{o}) \in \mathrm{J}^{wf}$ iff $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{J}}$
(7) for all $\mathsf{c},\mathsf{m},\mathsf{o}$: $cocan(\mathsf{c},\mathsf{m},\mathsf{o}) \in \mathrm{J}^{wf}$ iff $\mathsf{c}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o} \in \widehat{\mathrm{J}}$

I. $\Rightarrow$

Let us define:

$$\begin{array}{lll}
\mathrm{S}^{wf}_0 = \emptyset & \widehat{\mathrm{S}}_0 = \emptyset & \text{for limit ordinal } 0 \\
\mathrm{S}^{wf}_\alpha = \mathbf{C}_{\mathrm{P}^{wf},\mathrm{I}^{wf}}(\mathrm{S}^{wf}_{\alpha-1}) & \widehat{\mathrm{S}}_\alpha = \mathbf{T}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{S}}_{\alpha-1}) & \text{for successor ordinal } \alpha \\
\mathrm{S}^{wf}_\alpha = \bigcup_{\beta<\alpha} \mathrm{S}^{wf}_\beta & \widehat{\mathrm{S}}_\alpha = \bigcup_{\beta<\alpha} \widehat{\mathrm{S}}_\beta & \text{for limit ordinal } \alpha \neq 0 \\
\mathrm{S}^{wf}_\infty = \bigcup_\alpha \mathrm{S}^{wf}_\alpha & \widehat{\mathrm{S}}_\infty = \bigcup_\alpha \widehat{\mathrm{S}}_\alpha &
\end{array}$$

Then $\mathrm{S}^{wf}_\infty = \mathrm{lfp}(\mathbf{C}_{\mathrm{P}^{wf},\mathrm{I}^{wf}})$ and $\widehat{\mathrm{S}}_\infty = \mathrm{lfp}(\mathbf{T}_{\mathrm{P},\widehat{\mathrm{I}}})$. We will prove by transfinite induction that for any ordinal α and for all $\mathsf{o},\mathsf{s},\mathsf{c},\mathsf{m},\mathsf{v}$, the following conditions are true:

(1) if $isa(\mathsf{o},\mathsf{c}) \in \mathrm{S}^{wf}_\alpha$ then $\mathsf{o}:\mathsf{c} \in \widehat{\mathrm{S}}_\alpha$
(2) if $sub(\mathsf{s},\mathsf{c}) \in \mathrm{S}^{wf}_\alpha$ then $\mathsf{s}::\mathsf{c} \in \widehat{\mathrm{S}}_\alpha$
(3) if $exmv(\mathsf{s},\mathsf{m},\mathsf{v}) \in \mathrm{S}^{wf}_\alpha$ then $\mathsf{s}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}} \in \widehat{\mathrm{S}}_\alpha$
(4) if $vamv(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c}) \in \mathrm{S}^{wf}_\alpha$ then $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}} \in \widehat{\mathrm{S}}_\alpha$
(5) if $comv(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c}) \in \mathrm{S}^{wf}_\alpha$ then $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{code}} \in \widehat{\mathrm{S}}_\alpha$
(6) if $vacan(\mathsf{c},\mathsf{m},\mathsf{o}) \in \mathrm{S}^{wf}_\alpha$ then $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{S}}_\alpha$
(7) if $cocan(\mathsf{c},\mathsf{m},\mathsf{o}) \in \mathrm{S}^{wf}_\alpha$ then $\mathsf{c}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o} \in \widehat{\mathrm{S}}_\alpha$

The case for a limit ordinal α is trivial. Now let α be a successor ordinal. So $S^{wf}_{\alpha} = \mathbf{C}_{P^{wf},I^{wf}}(S^{wf}_{\alpha-1})$. First we show that for any ground positive literal L, if $\rho^b(\mathsf{L}) \in S^{wf}_{\alpha-1}$, then $val^{\mathbf{b}}_{\widehat{S}_{\alpha-1}}(\mathsf{L}) = \mathbf{t}$: (i) If $\mathsf{L} = \mathsf{o}:\mathsf{c}$, then $\rho^b(\mathsf{L}) = isa(\mathsf{o},\mathsf{c})$. It follows that $\mathsf{o}:\mathsf{c} \in \widehat{S}_{\alpha-1}$ by the induction hypothesis. Thus $val^{\mathbf{b}}_{\widehat{S}_{\alpha-1}}(\mathsf{o}:\mathsf{c}) = \mathbf{t}$; (ii) Similarly, we can show if $\rho^b(\mathsf{L}) = sub(\mathsf{s},\mathsf{c}) \in S^{wf}_{\alpha-1}$, then $val^{\mathbf{b}}_{\widehat{S}_{\alpha-1}}(\mathsf{s}::\mathsf{c}) = \mathbf{t}$; (iii) If $\mathsf{L} = \mathsf{o}[\mathsf{m} \to \mathsf{v}]$, then $\rho^b(\mathsf{L}) = mv(\mathsf{o},\mathsf{m},\mathsf{v})$. Note that $S^{wf}_{\gamma} \subseteq S^{wf}_{\alpha-1}$ for all $\gamma \leq \alpha - 1$. Therefore, there must exist a successor ordinal $\rho \leq \alpha - 1$ such that $mv(\mathsf{s},\mathsf{m},\mathsf{v}) \in S^{wf}_{\rho} = \mathbf{C}_{P^{wf},I^{wf}}(S^{wf}_{\rho-1})$. It follows that $exmv(\mathsf{o},\mathsf{m},\mathsf{v}) \in S^{wf}_{\rho-1}$, or there is c such that $vamv(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c}) \in S^{wf}_{\rho-1}$ or $comv(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c}) \in S^{wf}_{\rho-1}$, according to the trailer rules in Definition 38. Thus $\mathsf{o}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}} \in \widehat{S}_{\rho-1}$, or there is c such that $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}} \in \widehat{S}_{\rho-1}$ or $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{code}} \in \widehat{S}_{\rho-1}$, by the induction hypothesis. Clearly, $\widehat{S}_{\rho-1} \subseteq \widehat{S}_{\alpha-1}$. Thus $val^{\mathbf{b}}_{\widehat{S}_{\alpha-1}}(\mathsf{o}[\mathsf{m} \to \mathsf{v}]) = \mathbf{t}$.

Now consider the following cases:

(1) $isa(\mathsf{o},\mathsf{c}) \in S^{wf}_{\alpha}$ and $isa(\mathsf{o},\mathsf{c})$ is derived via a rule $\mathsf{R}^{wf} \in ground(P^{wf})$ which is rewritten from a regular rule $\mathsf{R} \in ground(P)$.
Then $\mathsf{R}^{wf} \equiv isa(\mathsf{o},\mathsf{c})$:- $\rho^b(\mathsf{C}_1), \ldots, \rho^b(\mathsf{C}_m), \neg\, \rho^b(\mathsf{G}_1), \ldots, \neg\, \rho^b(\mathsf{G}_n)$ must be the rewriting of $\mathsf{R} \equiv \mathsf{o}:\mathsf{c}$:- $\mathsf{C}_1, \ldots, \mathsf{C}_m, \neg\, \mathsf{G}_1, \ldots, \neg\, \mathsf{G}_n$, where C_i, $1 \leq i \leq m$, and G_j, $1 \leq j \leq n$, are positive literals. By Definition 39, each $\rho^b(\mathsf{C}_i) \in S^{wf}_{\alpha-1}$ and each $\rho^b(\mathsf{G}_j) \notin I^{wf}$. Following the above claim, $val^{\mathbf{b}}_{\widehat{S}_{\alpha-1}}(\mathsf{C}_i) = \mathbf{t}$ for all $1 \leq i \leq m$. Moreover, I^{wf} is isomorphic to $\widehat{I}$ and is in normal form, therefore $val^{\mathbf{b}}_{\widehat{I}}(\neg\, \mathsf{G}_j) = \mathbf{t}$ for all $1 \leq j \leq n$, by Lemma 18. So $\mathsf{o}:\mathsf{c} \in \mathbf{RC}_{P,\widehat{I}}(\widehat{S}_{\alpha-1}) \subseteq \mathbf{T}_{P,\widehat{I}}(\widehat{S}_{\alpha-1}) = \widehat{S}_{\alpha}$, by Definitions 22 and 26.

(2) $isa(\mathsf{o},\mathsf{c}) \in S^{wf}_{\alpha}$ and $isa(\mathsf{o},\mathsf{c})$ is derived via a trailer rule R^{wf} in $ground(P^{wf})$.
Then there exists s such that $\mathsf{R}^{wf} = isa(\mathsf{o},\mathsf{c})$:- $isa(\mathsf{o},\mathsf{s}), sub(\mathsf{s},\mathsf{c})$. It follows that $isa(\mathsf{o},\mathsf{s}) \in S^{wf}_{\alpha-1}$ and $sub(\mathsf{s},\mathsf{c}) \in S^{wf}_{\alpha-1}$. Thus $\mathsf{o}:\mathsf{s} \in \widehat{S}_{\alpha-1}$ and $\mathsf{s}::\mathsf{c} \in \widehat{S}_{\alpha-1}$ by the induction hypothesis. So $\mathsf{o}:\mathsf{c} \in \mathbf{IC}^t(\widehat{S}_{\alpha-1}) \subseteq \mathbf{T}_{P,\widehat{I}}(\widehat{S}_{\alpha-1}) = \widehat{S}_{\alpha}$, by Definitions 25 and 26.

(3) $sub(\mathsf{s},\mathsf{c}) \in S^{wf}_{\alpha}$ and $sub(\mathsf{s},\mathsf{c})$ is derived via a rule $\mathsf{R}^{wf} \in ground(P^{wf})$ which is rewritten from a regular rule $\mathsf{R} \in ground(P)$.
Similarly to (1), we can show that $\mathsf{s}::\mathsf{c} \in \widehat{S}_{\alpha}$.

(4) $sub(\mathsf{s},\mathsf{c}) \in S^{wf}_{\alpha}$ and $sub(\mathsf{s},\mathsf{c})$ is derived via a trailer rule R^{wf} in $ground(P^{wf})$.
Similarly to (2), we can show that $\mathsf{s}::\mathsf{c} \in \widehat{S}_{\alpha}$.

(5) $exmv(\mathsf{s},\mathsf{m},\mathsf{v}) \in S^{wf}_{\alpha}$
Then $exmv(\mathsf{s},\mathsf{m},\mathsf{v})$ must be derived via a rule $\mathsf{R}^{wf} \in ground(P^{wf})$ which is rewritten from a regular rule $\mathsf{R} \in ground(P)$. Similarly to (1), we can also show that $\mathsf{s}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}} \in \widehat{S}_{\alpha}$.

(6) $vamv(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c}) \in S^{wf}_{\alpha}$
By Definition 38, $vamv(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c})$ should be derived via a trailer rule from $ground(P^{wf})$. So $vacan(\mathsf{c},\mathsf{m},\mathsf{o}) \in S^{wf}_{\alpha-1}$, $exmv(\mathsf{c},\mathsf{m},\mathsf{v}) \in S^{wf}_{\alpha-1}$, $ex(\mathsf{o},\mathsf{m}) \notin I^{wf}$,

and $multi(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathrm{I}^{wf}$. Thus $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \widehat{\mathrm{S}}_{\alpha-1}$ and $\mathsf{c}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}} \in \widehat{\mathrm{S}}_{\alpha-1}$, by the induction hypothesis. Since I^{wf} is isomorphic to $\widehat{\mathrm{I}}$, $ec(\mathsf{o},\mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{I}})$ and $mc(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{I}})$. Thus $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}} \in \mathbf{IC}^{i}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{S}}_{\alpha-1}) \subseteq \mathbf{T}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{S}}_{\alpha-1}) = \widehat{\mathrm{S}}_{\alpha}$, by Definitions 25 and 26.

(7) $comv(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c}) \in \mathrm{S}^{wf}_{\alpha}$
Then $comv(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c})$ must be derived via a trailer rule. So $cocan(\mathsf{c},\mathsf{m},\mathsf{o}) \in \mathrm{S}^{wf}_{\alpha-1}$, $ins(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c}) \in \mathrm{S}^{wf}_{\alpha-1}$, $ex(\mathsf{o},\mathsf{m}) \notin \mathrm{I}^{wf}$, and $multi(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathrm{I}^{wf}$. By the induction hypothesis, $\mathsf{c}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o} \in \widehat{\mathrm{S}}_{\alpha-1}$. Since I^{wf} is isomorphic to $\widehat{\mathrm{I}}$, it follows that $ec(\mathsf{o},\mathsf{m}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{I}})$, $mc(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{I}})$. Clearly, $ins(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c})$ must be derived via a rule $\mathsf{R}^{wf} \equiv ins(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c}) :- \rho^b(\mathsf{C}_1),\ldots,\rho^b(\mathsf{C}_\mathsf{m}), \neg\,\rho^b(\mathsf{G}_1),\ldots,\neg\,\rho^b(\mathsf{G}_\mathsf{n})$, in $ground(\mathrm{P}^{wf})$, which is rewritten from the following template rule in $ground(\mathrm{P})$, $\mathsf{R} \equiv \mathsf{code}(\mathsf{c})\ @\mathsf{this}[\mathsf{m} \to \mathsf{v}] :- \mathsf{B}_1,\ldots,\mathsf{B}_\mathsf{m}, \neg\,\mathsf{F}_1,\ldots,\neg\,\mathsf{F}_\mathsf{n}$, where B_i and F_j are positive literals, $\mathsf{C}_\mathsf{i} = (\mathsf{B}_\mathsf{i})_{\|\mathsf{o}}$ and $\mathsf{G}_\mathsf{j} = (\mathsf{F}_\mathsf{j})_{\|\mathsf{o}}$, for all $1 \le \mathsf{i} \le \mathsf{m}$ and $1 \le \mathsf{j} \le \mathsf{n}$. Similarly to (1), we can show that $val^{\mathbf{b}}_{\widehat{\mathrm{S}}_{\alpha-1}}((\mathsf{B}_\mathsf{i})_{\|\mathsf{o}}) = \mathbf{t}$ for all $1 \le \mathsf{i} \le \mathsf{m}$ and $val^{\mathbf{b}}_{\widehat{\mathrm{I}}}(\neg\,(\mathsf{F}_\mathsf{j})_{\|\mathsf{o}}) = \mathbf{t}$ for all $1 \le \mathsf{j} \le \mathsf{n}$. It follows that by Definitions 24 and 26, $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}} \in \mathbf{TC}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{S}}_{\alpha-1}) \subseteq \mathbf{T}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{S}}_{\alpha-1}) = \widehat{\mathrm{S}}_{\alpha}$.

(8) $vacan(\mathsf{c},\mathsf{m},\mathsf{o}) \in \mathrm{S}^{wf}_{\alpha}$
Then $vacan(\mathsf{c},\mathsf{m},\mathsf{o})$ must be derived via a trailer rule in $ground(\mathrm{P}^{wf})$. It follows that $isa(\mathsf{o},\mathsf{c}) \in \mathrm{S}^{wf}_{\alpha-1}$, $exmv(\mathsf{c},\mathsf{m},\mathsf{v}) \in \mathrm{S}^{wf}_{\alpha-1}$, $\mathsf{c} \neq \mathsf{o}$, and $override(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathrm{I}^{wf}$, by Definition 38. So $\mathsf{o}:\mathsf{c} \in \widehat{\mathrm{S}}_{\alpha-1}$ and $\mathsf{c}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}} \in \widehat{\mathrm{S}}_{\alpha-1}$, by the induction hypothesis. Moreover, $ov(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{I}})$, since I^{wf} is isomorphic to $\widehat{\mathrm{I}}$. Thus $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o} \in \mathbf{IC}^{c}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{S}}_{\alpha-1}) \subseteq \mathbf{T}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{S}}_{\alpha-1}) = \widehat{\mathrm{S}}_{\alpha}$, by Definitions 25 and 26.

(9) $cocan(\mathsf{c},\mathsf{m},\mathsf{o}) \in \mathrm{S}^{wf}_{\alpha}$
Then $cocan(\mathsf{c},\mathsf{m},\mathsf{o})$ must be derived via a trailer rule in $ground(\mathrm{P}^{wf})$. It follows that $isa(\mathsf{o},\mathsf{c}) \in \mathrm{S}^{wf}_{\alpha-1}$, $codedef(\mathsf{c},\mathsf{m}) \in \mathrm{S}^{wf}_{\alpha-1}$, $\mathsf{c} \neq \mathsf{o}$, and $override(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathrm{I}^{wf}$, by Definition 38. Note that $\mathsf{o}:\mathsf{c} \in \widehat{\mathrm{S}}_{\alpha-1}$, by the induction hypothesis, and $ov(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{I}})$, because I^{wf} is isomorphic to $\widehat{\mathrm{I}}$. Since $codedef(\mathsf{c},\mathsf{m}) \in \mathrm{S}^{wf}_{\alpha-1}$, there is a template rule in P which specifies the instance method m for class c, by Definition 38. It follows that $\mathsf{c}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o} \in \mathbf{IC}^{c}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{S}}_{\alpha-1}) \subseteq \mathbf{T}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{S}}_{\alpha-1}) = \widehat{\mathrm{S}}_{\alpha}$, by Definitions 25 and 26.

II. $\Leftarrow$

Let us construct an extended atom set $\widehat{\mathrm{K}}$ from J^{wf} as follows: generate one $\mathsf{o}:\mathsf{c}$ in $\widehat{\mathrm{K}}$ for every $isa(\mathsf{o},\mathsf{c})$ in J^{wf}, one $\mathsf{s}::\mathsf{c}$ in $\widehat{\mathrm{K}}$ for every $sub(\mathsf{s},\mathsf{c})$ in J^{wf}, one $\mathsf{s}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}}$ in $\widehat{\mathrm{K}}$ for every $exmv(\mathsf{s},\mathsf{m},\mathsf{v})$ in J^{wf}, one $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}}$ in $\widehat{\mathrm{K}}$ for every $vamv(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c})$ in J^{wf}, one $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{code}}$ in $\widehat{\mathrm{K}}$ for every $comv(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c})$ in J^{wf}, one $\mathsf{c}[\mathsf{m}] \overset{val}{\leadsto} \mathsf{o}$ in $\widehat{\mathrm{K}}$ for every $vacan(\mathsf{c},\mathsf{m},\mathsf{o})$ in J^{wf}, and one $\mathsf{c}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o}$ in $\widehat{\mathrm{K}}$ for every $cocan(\mathsf{c},\mathsf{m},\mathsf{o})$ in J^{wf}. Clearly, to prove that the conditions are true, it suffices to show that $\widehat{\mathrm{K}} \supseteq \widehat{\mathrm{J}}$.

Because $\widehat{\mathrm{J}} = \mathrm{lfp}(\mathbf{T}_{\mathrm{P},\widehat{\mathrm{I}}})$, therefore, to show that $\widehat{\mathrm{K}} \supseteq \widehat{\mathrm{J}}$, it suffices to show that $\mathbf{T}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{K}}) \subseteq \widehat{\mathrm{K}}$ according to the conventional fixpoint theory [28]. Recall that by Definitions 26 and 25,

$$\mathbf{T}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{K}}) = \mathbf{RC}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{K}}) \cup \mathbf{TC}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{K}}) \cup \mathbf{IC}^{t}(\widehat{\mathrm{K}}) \cup \mathbf{IC}^{c}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{K}}) \cup \mathbf{IC}^{i}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{K}})$$

Let A be any atom in $\mathbf{T}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{K}})$. There are five possible cases to consider:

(1) $\mathsf{A} \in \mathbf{RC}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{K}})$
Then there is a regular rule $\mathsf{R} \equiv \mathsf{H} :- \mathsf{C}_1, \ldots, \mathsf{C}_m, \neg\, \mathsf{G}_1, \ldots, \neg\, \mathsf{G}_n$ in *ground*(P), such that H matches A, C_i $(1 \le i \le m)$ and G_j $(1 \le j \le n)$ are positive literals, $val^{\mathbf{b}}_{\widehat{\mathrm{K}}}(\mathsf{C}_i) = \mathbf{t}$ for all $1 \le i \le m$ and $val^{\mathbf{b}}_{\widehat{\mathrm{I}}}(\neg\, \mathsf{G}_j) = \mathbf{t}$ for all $1 \le j \le n$. Consider the rewriting R^{wf} of R, $\rho^h(\mathsf{H}) :- \rho^b(\mathsf{C}_1), \ldots, \rho^b(\mathsf{C}_m), \neg\, \rho^b(\mathsf{G}_1), \ldots, \neg\, \rho^b(\mathsf{G}_n)$. First we show $\rho^b(\mathsf{C}_i) \in \mathrm{J}^{wf}$ for all $1 \le i \le m$: (i) If $\mathsf{C}_i = \mathsf{o}:\mathsf{c}$, then $\rho^b(\mathsf{C}_i) = isa(\mathsf{o},\mathsf{c})$. Since $val^{\mathbf{b}}_{\widehat{\mathrm{K}}}(\mathsf{o}:\mathsf{c}) = \mathbf{t}$, it follows that $\mathsf{o}:\mathsf{c} \in \widehat{\mathrm{K}}$ by Definition 1. Therefore, $isa(\mathsf{o},\mathsf{c}) \in \mathrm{J}^{wf}$, by the construction of $\widehat{\mathrm{K}}$; (ii) Similarly, we can show if $\mathsf{C}_i = \mathsf{s}::\mathsf{c}$, then $\rho^b(\mathsf{C}_i) = sub(\mathsf{s},\mathsf{c}) \in \mathrm{J}^{wf}$; (iii) If $\mathsf{C}_i = \mathsf{s}[\mathsf{m} \to \mathsf{v}]$, then $\rho^b(\mathsf{C}_i) = mv(\mathsf{s},\mathsf{m},\mathsf{v})$. Since $val^{\mathbf{b}}_{\widehat{\mathrm{K}}}(\mathsf{s}[\mathsf{m} \to \mathsf{v}]) = \mathbf{t}$, so $\mathsf{s}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}} \in \widehat{\mathrm{K}}$, or there exists c such that $\mathsf{s}[\mathsf{m} \to \mathsf{v}]^{\,\mathsf{c}}_{\,\mathrm{val}} \in \widehat{\mathrm{K}}$ or $\mathsf{s}[\mathsf{m} \to \mathsf{v}]^{\,\mathsf{c}}_{\,\mathrm{code}} \in \widehat{\mathrm{K}}$. So $exmv(\mathsf{s},\mathsf{m},\mathsf{v}) \in \mathrm{J}^{wf}$, or there exists c such that $vamv(\mathsf{s},\mathsf{m},\mathsf{v},\mathsf{c}) \in \mathrm{J}^{wf}$ or $comv(\mathsf{s},\mathsf{m},\mathsf{v},\mathsf{c}) \in \mathrm{J}^{wf}$, by the construction of $\widehat{\mathrm{K}}$. Because $\mathrm{J}^{wf} = \mathbf{C}_{\mathrm{P}^{wf},\mathrm{I}^{wf}}(\mathrm{J}^{wf})$, therefore $mv(\mathsf{s},\mathsf{m},\mathsf{v}) \in \mathrm{J}^{wf}$, according to the trailer rules in Definition 38. By Lemma 18, $\rho^b(\mathsf{G}_j) \notin \mathrm{I}^{wf}$ for all $1 \le j \le n$. So $\rho^h(\mathsf{H}) \in \mathbf{C}_{\mathrm{P}^{wf},\mathrm{I}^{wf}}(\mathrm{J}^{wf}) = \mathrm{J}^{wf}$, by Definition 39. It follows that: (i) If $\mathsf{A} = \mathsf{o}:\mathsf{c}$, then $\mathsf{H} = \mathsf{o}:\mathsf{c}$. So $\rho^h(\mathsf{H}) = isa(\mathsf{o},\mathsf{c}) \in \mathrm{J}^{wf}$, thus $\mathsf{o}:\mathsf{c} \in \widehat{\mathrm{K}}$; (ii) Similarly, if $\mathsf{A} = \mathsf{s}::\mathsf{c}$, then $\mathsf{s}::\mathsf{c} \in \widehat{\mathrm{K}}$; (iii) If $\mathsf{A} = \mathsf{s}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}}$, then $\mathsf{H} = \mathsf{s}[\mathsf{m} \to \mathsf{v}]$. So $\rho^h(\mathsf{H}) = exmv(\mathsf{s},\mathsf{m},\mathsf{v}) \in \mathrm{J}^{wf}$, thus $\mathsf{s}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}} \in \widehat{\mathrm{K}}$.

(2) $\mathsf{A} \in \mathbf{TC}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{K}})$
It must be the case that $\mathsf{A} = \mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\,\mathsf{c}}_{\,\mathrm{code}}$. It follows that $mc(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{I}})$, $ec(\mathsf{o},\mathsf{c}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{I}})$, $\mathsf{c}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o} \in \widehat{\mathrm{K}}$, and *ground*(P) has a template rule, R, of the form $\mathsf{code}(\mathsf{c})$ $@\mathsf{this}[\mathsf{m} \to \mathsf{v}] :- \mathsf{C}_1, \ldots, \mathsf{C}_m, \neg\, \mathsf{G}_1, \ldots, \neg\, \mathsf{G}_n$, where C_i $(1 \le i \le m)$ and G_j $(1 \le j \le n)$ are positive literals, $val^{\mathbf{b}}_{\widehat{\mathrm{K}}}((\mathsf{C}_i)_{\|\mathsf{o}}) = \mathbf{t}$ for all $1 \le i \le m$ and $val^{\mathbf{b}}_{\widehat{\mathrm{I}}}(\neg\,(\mathsf{G}_j)_{\|\mathsf{o}}) = \mathbf{t}$ for all $1 \le j \le n$. Consider the rewriting R^{wf} of R, such that $\mathsf{R}^{wf} \equiv ins(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c}) :- \rho^b(\mathsf{B}_1), \ldots, \rho^b(\mathsf{B}_m), \neg\, \rho^b(\mathsf{F}_1), \ldots, \neg\, \rho^b(\mathsf{F}_n)$, where $\mathsf{B}_i = (\mathsf{C}_i)_{\|\mathsf{o}}$ for all $1 \le i \le m$ and $\mathsf{F}_j = (\mathsf{G}_j)_{\|\mathsf{o}}$ for all $1 \le j \le n$. Similarly to (1), we can also show that $\rho^b(\mathsf{B}_i) \in \mathrm{J}^{wf}$ for all $1 \le i \le m$. By Lemma 18, $\rho^b(\mathsf{F}_j) \notin \mathrm{I}^{wf}$ for all $1 \le j \le n$. So $ins(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c}) \in \mathbf{C}_{\mathrm{P}^{wf},\mathrm{I}^{wf}}(\mathrm{J}^{wf}) = \mathrm{J}^{wf}$, by Definition 39. Because $\mathsf{c}[\mathsf{m}] \overset{code}{\leadsto} \mathsf{o} \in \widehat{\mathrm{K}}$, therefore $cocan(\mathsf{c},\mathsf{m},\mathsf{o}) \in \mathrm{J}^{wf}$, by the construction of $\widehat{\mathrm{K}}$. Note that $ec(\mathsf{o},\mathsf{c}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{I}})$ and $mc(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{I}})$. Since I^{wf} is isomorphic to $\widehat{\mathrm{I}}$, it follows that $ex(\mathsf{o},\mathsf{c}) \notin \mathrm{I}^{wf}$ and $multi(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathrm{I}^{wf}$. So $comv(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c}) \in \mathbf{C}_{\mathrm{P}^{wf},\mathrm{I}^{wf}}(\mathrm{J}^{wf}) = \mathrm{J}^{wf}$, according to the trailer rules of P^{wf} and Definition 39. It follows that $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\,\mathsf{c}}_{\,\mathrm{code}} \in \widehat{\mathrm{K}}$.

(3) $\mathsf{A} \in \mathbf{IC}^{t}(\widehat{\mathrm{K}})$
If $\mathsf{A} = \mathsf{o}:\mathsf{c}$, then there is x such that $\mathsf{o}:\mathsf{x} \in \widehat{\mathrm{K}}$, $\mathsf{x}::\mathsf{c} \in \widehat{\mathrm{K}}$. So $isa(\mathsf{o},\mathsf{x}) \in \mathrm{J}^{wf}$ and $sub(\mathsf{x},\mathsf{c}) \in \mathrm{J}^{wf}$, by the construction of $\widehat{\mathrm{K}}$. Thus $isa(\mathsf{o},\mathsf{c}) \in \mathbf{C}_{\mathrm{P}^{wf},\mathrm{I}^{wf}}(\mathrm{J}^{wf}) = \mathrm{J}^{wf}$,

by Definition 39 and the trailer rules of P^{wf}. Thus $\mathsf{o}:\mathsf{c} \in \widehat{\mathrm{K}}$. Similarly, we can show that if $\mathsf{A} = \mathsf{s}::\mathsf{c}$, then $\mathsf{s}::\mathsf{c} \in \widehat{\mathrm{K}}$.

(4) $\mathsf{A} \in \mathbf{IC}^{c}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{K}})$

If $\mathsf{A} = \mathsf{c}[\mathsf{m}] \overset{val}{\rightsquigarrow} \mathsf{o}$, then $\mathsf{o}:\mathsf{c} \in \widehat{\mathrm{K}}$, $\mathsf{c} \neq \mathsf{o}$, $\mathsf{c}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}} \in \widehat{\mathrm{K}}$, and $ov(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \widehat{\mathrm{I}}$, by Definition 25. Because $\widehat{\mathrm{K}}$ is constructed from J^{wf} and I^{wf} is isomorphic to $\widehat{\mathrm{I}}$, it follows that $isa(\mathsf{o},\mathsf{c}) \in \mathrm{J}^{wf}$, $exmv(\mathsf{c},\mathsf{m},\mathsf{v}) \in \mathrm{J}^{wf}$, and $override(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathrm{I}^{wf}$. So $vacan(\mathsf{c},\mathsf{m},\mathsf{o}) \in \mathbf{C}_{\mathrm{P}^{wf},\mathrm{I}^{wf}}(\mathrm{J}^{wf}) = \mathrm{J}^{wf}$, by Definition 39 and the trailer rules of P^{wf}. Thus $\mathsf{c}[\mathsf{m}] \overset{val}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{K}}$. Similarly, if $\mathsf{A} = \mathsf{c}[\mathsf{m}] \overset{code}{\rightsquigarrow} \mathsf{o}$, we can also show that $\mathsf{c}[\mathsf{m}] \overset{code}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{K}}$.

(5) $\mathsf{A} \in \mathbf{IC}^{i}_{\mathrm{P},\widehat{\mathrm{I}}}(\widehat{\mathrm{K}})$

Then $\mathsf{A} = \mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}}$, and $\mathsf{c}[\mathsf{m}] \overset{val}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{K}}$, $\mathsf{c}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}} \in \widehat{\mathrm{K}}$, $ec(\mathsf{o},\mathsf{m}) \notin \widehat{\mathrm{I}}$, $mc(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \widehat{\mathrm{I}}$, by Definition 25. Because $\widehat{\mathrm{K}}$ is constructed from J^{wf}, and I^{wf} is isomorphic to $\widehat{\mathrm{I}}$, it follows that $vacan(\mathsf{c},\mathsf{m},\mathsf{o}) \in \mathrm{J}^{wf}$, $exmv(\mathsf{c},\mathsf{m},\mathsf{v}) \in \mathrm{J}^{wf}$, $ex(\mathsf{o},\mathsf{m}) \notin \mathrm{I}^{wf}$, $multi(\mathsf{c},\mathsf{m},\mathsf{o}) \notin \mathrm{I}^{wf}$. So by Definition 39 and the trailer rules of P^{wf}, $vamv(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c}) \in \mathbf{C}_{\mathrm{P}^{wf},\mathrm{I}^{wf}}(\mathrm{J}^{wf}) = \mathrm{J}^{wf}$. Thus $\mathsf{o}[\mathsf{m} \to \mathsf{v}]^{\mathsf{c}}_{\mathrm{val}} \in \widehat{\mathrm{K}}$.

Finally, to finish the proof for the claim that J^{wf} is isomorphic to $\widehat{\mathrm{J}}$, we still need to show that the following conditions are true:

(1) for all o,m: $ex(\mathsf{o},\mathsf{m}) \in \mathrm{J}^{wf}$ iff $ec(\mathsf{o},\mathsf{m}) \in \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{J}})$
(2) for all $\mathsf{c},\mathsf{m},\mathsf{o}$: $multi(\mathsf{c},\mathsf{m},\mathsf{o}) \in \mathrm{J}^{wf}$ iff $mc(\mathsf{c},\mathsf{m},\mathsf{o}) \in \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{J}})$
(3) for all $\mathsf{c},\mathsf{m},\mathsf{o}$: $override(\mathsf{c},\mathsf{m},\mathsf{o}) \in \mathrm{J}^{wf}$ iff $ov(\mathsf{c},\mathsf{m},\mathsf{o}) \in \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{J}})$

Note that $ex/2$, $multi/3$, and $override/3$ can only be derived via the trailer rules as defined in Definition 38. Moreover, $\mathrm{J}^{wf} = \mathbf{C}_{\mathrm{P}^{wf},\mathrm{I}^{wf}}(\mathrm{J}^{wf})$. It follows that:

(1) $ex(\mathsf{o},\mathsf{m}) \in \mathrm{J}^{wf}$, iff there exists v such that $exmv(\mathsf{o},\mathsf{m},\mathsf{v}) \in \mathrm{J}^{wf}$, iff there exits v such that $\mathsf{o}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}} \in \widehat{\mathrm{J}}$, iff $ec(\mathsf{o},\mathsf{m}) \in \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{J}})$.

(2) $multi(\mathsf{c},\mathsf{m},\mathsf{o}) \in \mathrm{J}^{wf}$, iff there exists $\mathsf{x} \neq \mathsf{c}$ such that $vacan(\mathsf{x},\mathsf{m},\mathsf{o}) \in \mathrm{J}^{wf}$ or $cocan(\mathsf{x},\mathsf{m},\mathsf{o}) \in \mathrm{J}^{wf}$, iff there is $\mathsf{x} \neq \mathsf{c}$ such that $\mathsf{x}[\mathsf{m}] \overset{val}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{J}}$ or $\mathsf{x}[\mathsf{m}] \overset{code}{\rightsquigarrow} \mathsf{o} \in \widehat{\mathrm{J}}$, iff $mc(\mathsf{c},\mathsf{m},\mathsf{o}) \in \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{J}})$.

(3) $override(\mathsf{c},\mathsf{m},\mathsf{o}) \in \mathrm{J}^{wf}$, iff there exists some x, such that $\mathsf{x} \neq \mathsf{c}$, $\mathsf{x} \neq \mathsf{o}$, $sub(\mathsf{x},\mathsf{c}) \in \mathrm{J}^{wf}$, $isa(\mathsf{o},\mathsf{x}) \in \mathrm{J}^{wf}$, and there is v such that $exmv(\mathsf{x},\mathsf{m},\mathsf{v}) \in \mathrm{J}^{wf}$ or $codedef(\mathsf{x},\mathsf{m}) \in \mathrm{J}^{wf}$, iff there is x such that $\mathsf{x} \neq \mathsf{c}$, $\mathsf{x} \neq \mathsf{o}$, $\mathsf{x}::\mathsf{c} \in \widehat{\mathrm{J}}$, $\mathsf{o}:\mathsf{x} \in \widehat{\mathrm{J}}$, and there is v such that $\mathsf{x}[\mathsf{m} \to \mathsf{v}]_{\mathrm{ex}} \in \widehat{\mathrm{J}}$ or there is a template rule in P which specifies the instance method m for class c, iff $ov(\mathsf{c},\mathsf{m},\mathsf{o}) \in \mathbf{IB}_{\mathrm{P}}(\widehat{\mathrm{J}})$.

Proposition 9. *Let α range over all ordinals, then T^{wf}_{α}, T^{wf}_{∞}, U^{wf}_{α}, and U^{wf}_{∞} are all in normal form. (The notations used here are from Definition 43.)*

Proof. First we show by transfinite induction that T^{wf}_{α} is in normal form for any ordinal α. The case is trivial for limit ordinal 0. If α is a successor ordinal, then $\mathrm{T}^{wf}_{\alpha} = \mathbf{S}_{\mathrm{P}^{wf}}(\mathrm{U}^{wf}_{\alpha-1}) = \mathrm{lfp}(\mathbf{C}_{\mathrm{P}^{wf},\mathrm{U}^{wf}_{\alpha-1}})$. It follows that T^{wf}_{α} is in normal form, by Lemma 17. Now suppose $\alpha \neq 0$ is a limit ordinal, $\mathrm{T}^{wf}_{\alpha} = \bigcup_{\beta<\alpha} \mathrm{T}^{wf}_{\beta}$. According to Definition 42, we need to show for all $\mathsf{o},\mathsf{m},\mathsf{v}$: $mv(\mathsf{o},\mathsf{m},\mathsf{v}) \in \mathrm{T}^{wf}_{\alpha}$

iff $exmv(\mathsf{o},\mathsf{m},\mathsf{v}) \in \mathrm{T}_\alpha^{wf}$, or there is c such that $vamv(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c}) \in \mathrm{T}_\alpha^{wf}$ or $comv(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c}) \in \mathrm{T}_\alpha^{wf}$.

(1) $\Rightarrow$
If $mv(\mathsf{o},\mathsf{m},\mathsf{v}) \in \mathrm{T}_\alpha^{wf}$, then there is $\beta < \alpha$ such that $mv(\mathsf{o},\mathsf{m},\mathsf{v}) \in \mathrm{T}_\beta^{wf}$. By the induction hypothesis, $\mathrm{T}_\beta^{wf} \subseteq \mathrm{T}_\alpha^{wf}$ is in normal form. Thus $exmv(\mathsf{o},\mathsf{m},\mathsf{v}) \in \mathrm{T}_\beta^{wf}$, or there is c such that $vamv(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c}) \in \mathrm{T}_\beta^{wf}$ or $comv(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c}) \in \mathrm{T}_\beta^{wf}$.

(2) $\Leftarrow$
If $exmv(\mathsf{o},\mathsf{m},\mathsf{v}) \in \mathrm{T}_\alpha^{wf}$, then there is $\beta < \alpha$ such that $exmv(\mathsf{o},\mathsf{m},\mathsf{v}) \in \mathrm{T}_\beta^{wf}$. It follows that $mv(\mathsf{o},\mathsf{m},\mathsf{v}) \in \mathrm{T}_\beta^{wf} \subseteq \mathrm{T}_\alpha^{wf}$, since T_β^{wf} is in normal form by the induction hypothesis. On the other hand, if there exists c such that $vamv(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c}) \in \mathrm{T}_\alpha^{wf}$ or $vamv(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c}) \in \mathrm{T}_\alpha^{wf}$, then there is $\gamma < \alpha$ such that $vamv(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c}) \in \mathrm{T}_\gamma^{wf}$ or $comv(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c}) \in \mathrm{T}_\gamma^{wf}$. It follows that $mv(\mathsf{o},\mathsf{m},\mathsf{v},\mathsf{c}) \in \mathrm{T}_\gamma^{wf} \subseteq \mathrm{T}_\alpha^{wf}$, since T_γ^{wf} is in normal form by the induction hypothesis.

Similarly, we can also prove that T_∞^{wf} is in normal form. Moreover, for any ordinal α, $\mathrm{U}_\alpha^{wf} = \mathbf{S}_{\mathrm{P}^{wf}}(\mathrm{T}_\alpha^{wf}) = \mathrm{lfp}(\mathbf{C}_{\mathrm{P}^{wf},\mathrm{T}_\alpha^{wf}})$. It follows that U_α^{wf} is in normal form, by Lemma 17. Similarly, we can show that U_∞^{wf} is in the normal form.