Given a graph, find an embedding s.t. greedy routing works

**Greedy embedding of a graph**
Greedy embedding

• Given a graph G, find an embedding of the vertices in $\mathbb{R}^d$, s.t. for each pair of nodes s, t, there is a neighbor of s closer to t than s itself.
Questions to ask

- We want to find a virtual coordinates such that greedy routing always works.

- Does there exist such a greedy embedding in $\mathbb{R}^2$?
- in $\mathbb{R}^3$?
- in Euclidean metric? Hyperbolic space?
- If it exists, how to compute?
Greedy embedding does not always exist

- $K_{1,6}$ does not have a greedy embedding in $\mathbb{R}^2$
A lemma

- Lemma: each node $t$ must have an edge to its closest (in terms of Euclidean distance) node $u$.

- Otherwise, $u$ has no neighbor that is closer to $t$ than itself.
Proof

- $K_{1,6}$ does not have a greedy embedding in $\mathbb{R}^2$

Proof:
1. One of the angles is less than $\pi/3$.
2. One of $ab_2$ and $ab_3$, say, $ab_2$, is longer than $b_2b_3$.
3. Then $b_2$ does not have edge with its closest point $b_3$. 
A conjecture

- Corollary: $K_{k,5k+1}$ does not have a greedy embedding in $\mathbb{R}^2$.
- Conjecture: Any planar 3-connected graph has a greedy embedding $\mathbb{R}^2$.

- Hint: this is tight.
- $K_{2,11}$ is planar but not 3-connected.
- $K_{3,16}$ is 3-connected but not planar. (it has $K_{3,3}$ minor).
- Planar 3-connected graph has a greedy embedding in $\mathbb{R}^3$
Polyhedral routing

Theorem: Any 3-connected planar graph has a greedy embedding \( e \) in \( \mathbb{R}^3 \), where the distance function is defined as \( d(u, v) = - e(u) \cdot e(v) \).

Proof:
1. Any 3-connected planar graph is the edge graph of a 3D convex polytope, with edges tangent to a sphere. [Steinitz 1922].
2. Each vertex has a supporting hyperplane with the normal being the 3D coordinate of the vertex.
Polyhedral routing

Proof: For any $s$, $t$, there is a neighbor $v$ of $s$, $d(v, t) < d(s, t)$.

1. $d(s, t) - d(v, t) = [e(v) - e(s)] \cdot e(t) > 0$.

2. Now suppose such neighbor $v$ does not exist, then $s$ is a reflex vertex, with all the neighbors pointing away from $t$.

3. This contradicts with the convexity of the polytope.
Discussions

• Papadimitriou’s conjecture: Any planar 3-connected graph has a greedy embedding $\mathbb{R}^2$. $\Rightarrow$ has been proved!

• The theorem only gives a sufficient condition, not necessary.
  – $K_{3,3}$ has a greedy embedding.
  – A graph with a Hamiltonian cycle has a greedy embedding on a line.

• Given a graph, can we tell whether it has a greedy embedding in $\mathbb{R}^2$? Is this problem hard? (Recall that many such embedding problems are hard…)

• More understanding of greedy embedding in $\mathbb{R}^2$, $\mathbb{R}^3$…
Follow-up work

- Dhandapani proved that any triangulation admits a greedy embedding (SODA’08).
- Leighton and Moitra proved the conjecture (FOCS’08).
- Independently, Angelini et al. also proved it (Graph Drawing’08).
- Goodrich and D. Strash improved the coordinates to be of size $O(\log n)$ (under submission).

- We briefly introduce the main idea.
Leighton and Moitra

- All 3-connected planar graph contain a spanning Christmas Cactus graph.
- All Christmas Cactus graphs admit a greedy embedding in the plane.
Leighton and Moitra

- A **cactus graph** is connected, each edge is in at most one simple cycle.
- A **Christmas Cactus graph** is a cactus graph for which the removal of any node disconnects into at most 2 pieces.
A Christmas Cactus
Example
Connection to graph labeling

• Given a graph, find a labeling of the nodes such that one can compute the (approximate) shortest path distance between any two vertices from their labels only.

• Tradeoff between approximation ratio and the label size.

• For shortest path distance, the maximum label size is $\Theta(n)$ for general graph, $O(n^{1/2})$ ($\Omega(n^{1/3})$) for planar graphs, and $\Theta(\log^2 n)$ for trees.

• General graph: $\exists$ a scheme with label size $O(kn^{1/k})$ and approximation ratio $2k-1$.

• Google “distance labeling” for the literature.
Approach II: Embed a spanning tree in polar coordinate system
Embed a tree in polar coordinate system

• Start from any node as root, flood to find the shortest path tree.

• Assign polar ranges to each node in the tree.
  – The range of a node is divided among its children.
  – The size of the range is proportional to the size of its subtree.

• Order the subtrees that align with the sensor connectivity.
Embed a tree in polar coordinate system

• Order the subtrees that align with the sensor connectivity.
  – Three reference nodes flood the network. Each node knows the hop count to each reference.
  – Each node embeds itself with respect to the references. (trilateration with hop counts)
  – A node’s position is defined as the center of mass of all the nodes in its subtree.
  – This will provide an angular ordering of all the children.
Routing on a tree

- Route to the common ancestor of the source and destination.
  - Check whether the destination range is included in the range of the current node.
  - If not, go to the parent.
  - Otherwise go to the corresponding child.
- Root is the bottleneck
- Path may be long.
**Routing on a tree**

- Be a little smarter: store a local routing table that keeps the ranges of up to k-hop neighbors. ➔ find shortcuts.
- **Virtual Polar Coordinate Routing**: check the neighborhood, find the node that is closer to the destination. ⇐ greedy forwarding in polar coordinates.

If the upper/lower bound is closer to the destination.

![Diagram of routing on a tree](image)
Load balancing

- Root is still the bottleneck even for smart routing.
Routing on spanning trees – in theory and in practice

- For any graph $G$ there is a spanning tree $T$, s.t. the average stretch of the shortest paths on $T$, compared with $G$, is $O((\log n \log \log n)^2)$. 