Number Systems

Reading:

Chapter 3 – Number Systems (except 3.7)

Base of a Number System

- **Base**: the number of different digits including zero in the number system
  - Example: Base 10 has 10 digits, 0 through 9
- **Decimal or base 10 number system**
  - Origin: counting on the fingers
  - “Digit” from the Latin word *digitus* meaning “finger”
- **Binary or base 2**
  - **Bit** (binary digit): 2 digits, 0 and 1
- **Octal or base 8**: 8 digits, 0 through 7
- **Hexadecimal or base 16**: 16 digits, 0-9, followed by A-F
  - Examples: \(10_{10} = A_{16}; \ 11_{10} = B_{16}\)

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### Base or Radix

- **Base:**
  - The number of different symbols required to represent any given number
  - The larger the base, the more numerals are required
  - Base 10: 0,1, 2,3,4,5,6,7,8,9
  - Base 2: 0,1
  - Base 8: 0,1,2, 3,4,5,6,7
  - Base 16: 0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F

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### Why Binary?

- Early computer design was decimal
  - Mark I and ENIAC
- John von Neumann proposed binary data processing (1945)
  - Simplified computer design
  - Used for both instructions and data
- Natural relationship between on/off switches and calculation using Boolean logic

<table>
<thead>
<tr>
<th>On</th>
<th>Off</th>
</tr>
</thead>
<tbody>
<tr>
<td>True</td>
<td>False</td>
</tr>
<tr>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
Keeping Track of the Bits

- Bits commonly stored and manipulated in groups
  - 8 bits = 1 byte
  - 4 bytes = 1 word (in many systems)
- Number of bits used in calculations
  - Affects accuracy of results
  - Limits size of numbers manipulated by the computer

Consider some primitive types used in Java: short, int, and long

Numbers: Physical Representation

- Different numerals, same number of oranges
  - Cave dweller: I III I I I
  - Roman: V
  - Arabic: 5
- Different bases, same number of oranges
  - 5\textsubscript{10}
  - 101\textsubscript{2}
  - 12\textsubscript{3}
Number System

- Roman: position independent
- Modern: based on positional notation (place value)
  - Decimal system: system of \textit{positional} notation based on powers of 10.
  - Binary system: system of \textit{positional} notation based powers of 2
  - Octal system: system of \textit{positional} notation based on powers of 8
  - Hexadecimal system: system of \textit{positional} notation based powers of 16

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Positional Notation: Base 10

\[ 527 = 5 	imes 10^2 + 2 	imes 10^1 + 7 	imes 10^0 \]

\begin{tabular}{|c|c|c|c|}
\hline
Place & $10^2$ & $10^1$ & $10^0$ \\
\hline
Value & 100 & 10 & 1 \\
\hline
Evaluate & $5 \times 100$ & $2 \times 10$ & $7 \times 1$ \\
\hline
Sum & 500 & 20 & 7 \\
\hline
\end{tabular}
Positional Notation: Octal

\[ 624_8 = 404_{10} \]

<table>
<thead>
<tr>
<th>Place</th>
<th>(8^2)</th>
<th>(8^1)</th>
<th>(8^0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>64</td>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>Evaluate</td>
<td>6 x 64</td>
<td>2 x 8</td>
<td>4 x 1</td>
</tr>
<tr>
<td>Sum for Base 10</td>
<td>384</td>
<td>16</td>
<td>4</td>
</tr>
</tbody>
</table>

Positional Notation: Hexadecimal

\[ 6704_{16} = 26,372_{10} \]

<table>
<thead>
<tr>
<th>Place</th>
<th>(16^3)</th>
<th>(16^2)</th>
<th>(16^1)</th>
<th>(16^0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>4,096</td>
<td>256</td>
<td>16</td>
<td>1</td>
</tr>
<tr>
<td>Evaluate</td>
<td>6 x 4,096</td>
<td>7 x 256</td>
<td>0 x 16</td>
<td>4 x 1</td>
</tr>
<tr>
<td>Sum for Base 10</td>
<td>24,576</td>
<td>1,792</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

In Web color notation, #0000FF is pure blue.
Positional Notation: Binary

$1101\ 0110_2 = 214_{10}$

<table>
<thead>
<tr>
<th>Place</th>
<th>$2^7$</th>
<th>$2^6$</th>
<th>$2^5$</th>
<th>$2^4$</th>
<th>$2^3$</th>
<th>$2^2$</th>
<th>$2^1$</th>
<th>$2^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>128</td>
<td>64</td>
<td>32</td>
<td>16</td>
<td>8</td>
<td>4</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Evaluate</td>
<td>1 x 128</td>
<td>1 x 64</td>
<td>0 x 32</td>
<td>1 x 16</td>
<td>0 x 8</td>
<td>1 x 4</td>
<td>1 x 2</td>
<td>0 x 1</td>
</tr>
<tr>
<td>Sum for Base 10</td>
<td>128</td>
<td>64</td>
<td>0</td>
<td>16</td>
<td>0</td>
<td>4</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Range of Possible Numbers

- $R = B^K$ where
  - $R = \text{range}$
  - $B = \text{base}$
  - $K = \text{number of digits}$
- Example #1: Base 10, 2 digits
  - $R = 10^2 = 100$ different numbers (0…99)
- Example #2: Base 2, 16 digits
  - $R = 2^{16} = 65,536$ or 64K
  - 16-bit PC can store 65,536 different number values

How many colors can be represented in a 24-bit RGB color scheme? Each base color (r, g, and b) uses 8 bits.
### Decimal Range for Bit Widths

<table>
<thead>
<tr>
<th>Bits</th>
<th>Digits</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0+</td>
<td>2 (0 and 1)</td>
</tr>
<tr>
<td>4</td>
<td>1+</td>
<td>16 (0 to 15)</td>
</tr>
<tr>
<td>8</td>
<td>2+</td>
<td>256</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>1,024 (1K)</td>
</tr>
<tr>
<td>16</td>
<td>4+</td>
<td>65,536 (64K)</td>
</tr>
<tr>
<td>20</td>
<td>6</td>
<td>1,048,576 (1M)</td>
</tr>
<tr>
<td>32</td>
<td>9+</td>
<td>4,294,967,296 (4G)</td>
</tr>
<tr>
<td>64</td>
<td>19+</td>
<td>Approx. $1.6 \times 10^{19}$</td>
</tr>
<tr>
<td>128</td>
<td>38+</td>
<td>Approx. $2.6 \times 10^{38}$</td>
</tr>
</tbody>
</table>

### Counting in Base 2

<table>
<thead>
<tr>
<th>Binary Number</th>
<th>Equivalent</th>
<th>Decimal Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0 x 2^0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1 x 2^0</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>1 x 2^1</td>
<td>2</td>
</tr>
<tr>
<td>11</td>
<td>1 x 2^1</td>
<td>3</td>
</tr>
<tr>
<td>100</td>
<td>1 x 2^2</td>
<td>4</td>
</tr>
<tr>
<td>101</td>
<td>1 x 2^2</td>
<td>5</td>
</tr>
<tr>
<td>110</td>
<td>1 x 2^2</td>
<td>6</td>
</tr>
<tr>
<td>111</td>
<td>1 x 2^2</td>
<td>7</td>
</tr>
<tr>
<td>1000</td>
<td>1 x 2^3</td>
<td>8</td>
</tr>
<tr>
<td>1001</td>
<td>1 x 2^3</td>
<td>9</td>
</tr>
<tr>
<td>1010</td>
<td>1 x 2^3</td>
<td>10</td>
</tr>
</tbody>
</table>
Binary Arithmetic

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 \\
+ & 1 & 0 & 1 & 1 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}
\]

Boolean Logic

- System for logical operations
- Named after George Boole
- Components
  - Elements as members of a set
  - Operations (not, and, or, exclusive or)
  - Subsets and supersets
- Set membership and boolean operations are often implemented with binary numbers and operations
Binary Arithmetic: Boolean Logic

- Boolean logic without performing arithmetic
  - EXCLUSIVE-OR
    - Output is “1” only if either input, but not both inputs, is a “1”
  - AND
    - Output is “1” if and only both inputs are a “1”

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 \\
\hline
\text{and} & 1 & 0 & 1 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}
\]

Sometimes referred to as a masking operation

Converting from Base 10

- Powers Table

<table>
<thead>
<tr>
<th>Power Base</th>
<th>Base 8</th>
<th>Base 7</th>
<th>Base 6</th>
<th>Base 5</th>
<th>Base 4</th>
<th>Base 3</th>
<th>Base 2</th>
<th>Base 1</th>
<th>Base 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>256</td>
<td>128</td>
<td>64</td>
<td>32</td>
<td>16</td>
<td>8</td>
<td>4</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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Conversion Calculator

- To quickly convert numbers, use www.translatorscafe.com/cafe/units-converter/numbers/calculator/

From Base 16 to Base 2

- The nibble approach
  - Hex easier to read and write than binary

<table>
<thead>
<tr>
<th>Base 16</th>
<th>1</th>
<th>F</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base 2</td>
<td>0001</td>
<td>1111</td>
<td>0110</td>
<td>0111</td>
</tr>
</tbody>
</table>

- Why hexadecimal?
  - Modern computer operating systems and networks present variety of troubleshooting data in hex format

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