The Logic of Quantified Statements

Lecture 06 / Spring 2015
State University of New York, Korea
Instructor: Dr. Ilchul Yoon

Adapted from slides by Paul Fodor

Administrivia

- Recitation class in B203!
- Homework #1 Due: 3/26
  - Submit by 12:40PM.
  - Expect penalty for late submissions.
- Quiz #1 on 3/26 (Thursday) in class.
- Mid-Term #1 on 4/2 (Thursday) in class.

- No class on 3/31 (Tuesday).
  - Makeup class: 4/3 (Friday) 11:00AM ~ 11:50AM.
  - Expect another half-hour make class.
Universal Conditional Statements

- Universal conditional statement:
  \[ \forall x, \text{if } P(x) \text{ then } Q(x) \]

- Example:
  If a real number is greater than 2 then its square is greater than 4.
  \[ \forall x \in \mathbb{R}, \text{if } x > 2 \text{ then } x^2 > 4 \]

Equivalent Forms of Universal and Existential Statements

- \[ \forall x \in U, \text{if } P(x) \text{ then } Q(x) \] can be rewritten in the form
  \[ \forall x \in D, Q(x) \] by narrowing \( U \) to be the domain \( D \) consisting of all values of the variable \( x \) that make \( P(x) \) true.
- Example: \( \forall x, \text{if } x \text{ is a square then } x \text{ is a rectangle} \)
  \[ \forall \text{ squares } x, x \text{ is a rectangle}. \]

- \[ \exists x \text{ such that } P(x) \text{ and } Q(x) \] can be rewritten in the form
  \[ \exists x \in D \text{ such that } Q(x) \text{ where } D \text{ consists of all values of the variable } x \text{ that make } P(x) \text{ true} \]
Implicit Quantification

- $P(x) \Rightarrow Q(x)$ means that every element in the truth set of $P(x)$ is in the truth set of $Q(x)$, or, equivalently, $\forall x, P(x) \rightarrow Q(x)$
  - Example: If a number is an integer, then it is a rational number.

- $P(x) \iff Q(x)$ means that $P(x)$ and $Q(x)$ have identical truth sets, or, equivalently, $\forall x, P(x) \leftrightarrow Q(x)$
  - Example: The number 24 can be written as a sum of two even integers.

Negations of Quantified Statements

- Negation of a Universal Statement:
  - The negation of a statement of the form $\forall x \in D, Q(x)$ is logically equivalent to a statement of the form
    \[
    \exists x \in D, \sim Q(x):
    \sim (\forall x \in D, Q(x)) \equiv \exists x \in D, \sim Q(x)
    \]
  - Example:
    - “All mathematicians wear glasses”
    - Its negation is: “There is at least one mathematician who does not wear glasses”
    - Its negation is NOT “No mathematicians wear glasses”
Negations of Quantified Statements

- Negation of an Existential Statement
  - The negation of a statement of the form \( \exists x \in D, Q(x) \) is logically equivalent to a statement of the form \( \forall x \in D, \sim Q(x) \):

  \[
  \sim (\exists x \in D, Q(x)) \equiv \forall x \in D, \sim Q(x)
  \]

- Example:
  - “Some snowflakes are the same.”
  - Its negation is: “No snowflakes are the same” \( \equiv \) “All snowflakes are different.”

Negations of Quantified Statements

- More Examples:
  - \( \sim (\forall \text{primes } p, p \text{ is odd}) \equiv \exists \text{a prime } p \text{ such that } p \text{ is not odd} \)
  - \( \sim (\exists \text{a triangle } T \text{ such that the sum of the angles of } T \text{ equals } 200^\circ) \equiv \forall \text{triangle } T, \text{ the sum of the angles of } T \text{ does not equal } 200^\circ \)
  - \( \sim (\forall \text{politicians } x, x \text{ is not honest}) \equiv \exists \text{a politician } x \text{ such that } x \text{ is honest (by double negation)} \)
  - \( \sim (\forall \text{computer programs } p, p \text{ is finite}) \equiv \exists \text{a computer program } p \text{ that is not finite} \)
Negations of Quantified Statements

- More Examples:
  - \( \neg (\exists \text{ a computer hacker } c, c \text{ is over } 40) \equiv \forall \text{ computer hacker } c, c \text{ is } 40 \text{ or under} \)
  - \( \neg (\exists \text{ an integer } n \text{ between } 1 \text{ and } 37 \text{ such that } 1,357 \text{ is divisible by } n ) \equiv \forall \text{ integers } n \text{ between } 1 \text{ and } 37, 1,357 \text{ is not divisible by } n \)

Negations of Universal Conditional Statements

- \( \neg (\forall x, P(x) \rightarrow Q(x)) \equiv \exists x \text{ such that } P(x) \land \neg Q(x) \)
- Proof:
  - \( \neg (\forall x, P(x) \rightarrow Q(x)) \equiv \exists x \text{ such that } \neg (P(x) \rightarrow Q(x)) \)
  - \( (P(x) \rightarrow Q(x)) \equiv (\neg P(x) \lor Q(x)) \equiv \exists x \text{ such that } P(x) \land \neg Q(x) \)
- Examples:
  - \( \neg (\forall \text{ people } p, \text{ if } p \text{ is blond then } p \text{ has blue eyes}) \equiv \exists \text{ a person } p \text{ such that } p \text{ is blond and } p \text{ does not have blue eyes} \)
  - \( \neg (\text{If a computer program has more than } 100,000 \text{ lines, then it contains a bug}) \equiv \exists \text{ a computer program that has more than } 100,000 \text{ lines and does not contain a bug} \)
The Relation among $\forall$, $\exists$, $\land$, and $\lor$

- $D = \{x_1, x_2, \ldots, x_n\}$ and $\forall x \in D$, $Q(x) \equiv Q(x_1) \land Q(x_2) \land \cdots \land Q(x_n)$

- $D = \{x_1, x_2, \ldots, x_n\}$ and $\exists x \in D$ such that $Q(x) \equiv Q(x_1) \lor Q(x_2) \lor \cdots \lor Q(x_n)$

Vacuous Truth of Universal Statements

All the balls in the bowl are blue

$\forall x$ in $D$, if $P(x)$ then $Q(x)$ is vacuously true or true by default if, and only if, $P(x)$ is false for every $x$ in $D$
Variants of Universal Conditional Statements

- Universal conditional statement: $\forall x \in D$, if $P(x)$ then $Q(x)$
- **Contrapositive**: $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$
  - $\forall x \in D$, if $P(x)$ then $Q(x) \equiv \forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$
  - Proof: for any $x$ in $D$ by the logical equivalence between statement and its contrapositive
- **Converse**: $\forall x \in D$, if $Q(x)$ then $P(x)$.
- **Inverse**: $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$.
- Example:
  - $\forall x \in \mathbb{R}$, if $x > 2$ then $x^2 > 4$
  - Contrapositive: $\forall x \in \mathbb{R}$, if $x^2 \leq 4$ then $x \leq 2$
  - Converse: $\forall x \in \mathbb{R}$, if $x^2 > 4$ then $x > 2$
  - Inverse: $\forall x \in \mathbb{R}$, if $x \leq 2$ then $x^2 \leq 4$

Necessary and Sufficient Conditions

- Necessary condition:
  - “$\forall x$, r (x) is a necessary condition for s(x)” means
  - “$\forall x$, if $\sim r (x)$ then $\sim s(x)$” $\equiv$ “$\forall x$, if s(x) then r(x)” $^{(*)}$
  - $^{(*)}$ (by contrapositive and double negation)

- Sufficient condition:
  - “$\forall x$, r (x) is a sufficient condition for s(x)” means
  - “$\forall x$, if r (x) then s(x)”
Necessary and Sufficient Conditions

- Examples:
  - Squareness is a **sufficient condition** for rectangularity;
    Formal statement: ∀x, if x is a square, then x is a rectangle
  - Being at least 35 years old is a **necessary condition** for being
    President of the United States
    ∀ people x, if x is younger than 35, then x cannot be President of
    the United States
    ≡
    ∀ people x, if x is President of the United States then x is at least
    35 years old (by contrapositive)

Only If

- Only If:
  “∀x, r(x) only if s(x)” means
  “∀x, if ∼s(x) then ∼r (x)” ≡ “∀x, if r(x) then s(x).”

- Example:
  - A product of two numbers is 0 only if one of the numbers is 0.
    If neither of two numbers is 0, then the product of the numbers
    is not 0 ≡
    If a product of two numbers is 0, then one of the numbers is 0
    (by contrapositive)
Statements with Multiple Quantifiers

- Example:
  - “There is a person supervising every detail of the production process”
  - What is the meaning?
    - “There is one single person who supervises all the details of the production process”?
    - OR
      - “For any particular production detail, there is a person who supervises that detail, but there might be different supervisors for different details”?

NATURAL LANGUAGE IS AMBIGUOUS
LOGIC IS CLEAR

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Statements with Multiple Quantifiers

- Quantifiers are performed in the order in which the quantifiers occur:
- Example:
  - $\forall x \text{ in set } D, \exists y \text{ in set } E \text{ such that } x \text{ and } y \text{ satisfy property } P(x, y)$

- Informal language examples
  - There is a smallest positive integer.
  - There is no smallest positive real number.
- Example in calculus
  - The definition of limit of a sequence
Tarski’s World

- Blocks of various sizes, shapes, and colors located on a grid

- \(\forall t, \text{Triangle}(t) \rightarrow \text{Blue}(t)\)
  - TRUE

- \(\forall x, \text{Blue}(x) \rightarrow \text{Triangle}(x)\).
  - FALSE

- \(\exists y \text{ such that Square}(y) \land \text{RightOf}(d, y)\).
  - TRUE

- \(\exists z \text{ such that Square}(z) \land \text{Gray}(z)\).
  - FALSE

Statements with Multiple Quantifiers in Tarski’s World

\(\forall \exists\)

- For all triangles \(x\), there is a square \(y\) such that \(x\) and \(y\) have the same color

  TRUE


<table>
<thead>
<tr>
<th>Given (x)</th>
<th>choose (y)</th>
<th>and check that (y) is the same color as (x).</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d)</td>
<td>(e)</td>
<td>yes ✓</td>
</tr>
<tr>
<td>(f) or (i)</td>
<td>(h) or (g)</td>
<td>yes ✓</td>
</tr>
</tbody>
</table>
Statements with Multiple Quantifiers in Tarski’s World

- There is a triangle $x$ such that for all circles $y$, $x$ is to the right of $y$
  TRUE

<table>
<thead>
<tr>
<th>Choose $x =$</th>
<th>Then, given $y =$</th>
<th>check that $x$ is to the right of $y$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d$ or $i$</td>
<td>$a$</td>
<td>yes $\checkmark$</td>
</tr>
<tr>
<td>$b$</td>
<td>$c$</td>
<td>yes $\checkmark$</td>
</tr>
<tr>
<td>$c$</td>
<td>$d$</td>
<td>yes $\checkmark$</td>
</tr>
</tbody>
</table>

Interpreting Statements with Two Different Quantifiers

- Quantifiers are performed in the order in which the quantifiers occur:
  - $\forall x$ in $D$, $\exists y$ in $E$ such that $P(x, y)$
    - for whatever element $x$ in $D$ you must find an element $y$ in $E$ that “works” for that particular $x$
  - $\exists x$ in $D$ such that $\forall y$ in $E$, $P(x, y)$
    - find one particular $x$ in $D$ that will “work” no matter what $y$ in $E$ anyone might choose
Interpreting Statements with Two Different Quantifiers

- $\exists$ an item I such that $\forall$ students S, S chose I.
  TRUE

- $\exists$ a student S such that $\forall$ stations Z, $\exists$ an item I in Z such that S chose I
  TRUE

- $\forall$ students S and $\forall$ stations Z, $\exists$ an item I in Z such that S chose I
  FALSE

Negations of Multiply-Quantified Statements

- Apply negation to quantified statements from left to right:

  $\sim(\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } P(x, y))$
  $\equiv \exists x \text{ in } D \text{ such that } \sim(\exists y \text{ in } E \text{ such that } P(x, y))$
  $\equiv \exists x \text{ in } D \text{ such that } \forall y \text{ in } E, \sim P(x, y)$.

  $\sim(\exists x \text{ in } D \text{ such that } \forall y \text{ in } E, P(x, y))$
  $\equiv \forall x \text{ in } D, \sim(\forall y \text{ in } E, P(x, y))$
  $\equiv \forall x \text{ in } D, \exists y \text{ in } E \text{ such that } \sim P(x, y)$.
Negating Statements in Tarski’s World

- For all squares \( x \), there is a circle \( y \) such that \( x \) and \( y \) have the same color

- **Negation**
  \[
  \exists \text{ a square } x \text{ such that } \neg (\exists \text{ a circle } y \text{ such that } x \text{ and } y \text{ have the same color})
  \]
  \[
  \equiv \exists \text{ a square } x \text{ such that } \forall \text{ circles } y, x \text{ and } y \text{ do not have the same color}
  \]
  TRUE: Square \( e \) is black and no circle is black.

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Negating Statements in Tarski’s World

- There is a triangle \( x \) such that for all squares \( y \), \( x \) is to the right of \( y \)

- **Negation**
  \[
  \forall \text{ triangles } x, \neg (\forall \text{ squares } y, x \text{ is to the right of } y)
  \]
  \[
  \equiv \forall \text{ triangles } x, \exists \text{ a square } y \text{ such that } x \text{ is not to the right of } y
  \]
  TRUE: Square \( g \) or \( j \) is at the rightmost side.
Quantifier Order in Tarski’s World

- For every square $x$ there is a triangle $y$ such that $x$ and $y$ have different colors
  TRUE
- There exists a triangle $y$ such that for every square $x$, $x$ and $y$ have different colors
  FALSE

Formalizing Statements in Tarski’s World

- $\text{Triangle}(x)$ means “$x$ is a triangle”
- $\text{Circle}(x)$ means “$x$ is a circle”
- $\text{Square}(x)$ means “$x$ is a square”
- $\text{Blue}(x)$ means “$x$ is blue”
- $\text{Gray}(x)$ means “$x$ is gray”
- $\text{Black}(x)$ means “$x$ is black”
- $\text{RightOf}(x, y)$ means “$x$ is to the right of $y$”
- $\text{Above}(x, y)$ means “$x$ is above $y$”
- $\text{SameColorAs}(x, y)$ means “$x$ has the same color as $y$”
- $x = y$ denotes the predicate “$x$ is equal to $y$”
Formalizing Statements in Tarski’s World

• For all circles $x$, $x$ is above $f$
  $\forall x (\text{Circle}(x) \rightarrow \text{Above}(x, f))$

• Negation:
  $\sim (\forall x (\text{Circle}(x) \rightarrow \text{Above}(x, f)))$
  $\equiv \exists x \sim (\text{Circle}(x) \rightarrow \text{Above}(x, f))$
  $\equiv \exists x (\text{Circle}(x) \land \sim \text{Above}(x, f))$

Formalizing Statements in Tarski’s World

• There is a square $x$ such that $x$ is black
  $\exists x (\text{Square}(x) \land \text{Black}(x))$

• Negation:
  $\sim (\exists x (\text{Square}(x) \land \text{Black}(x)))$
  $\equiv \forall x \sim (\text{Square}(x) \land \text{Black}(x))$
  $\equiv \forall x (\sim \text{Square}(x) \lor \sim \text{Black}(x))$
Formalizing Statements in Tarski’s World

- There is a square x such that for all triangles y, x is to right of y
  \[ \exists x (\text{Square}(x) \land \forall y (\text{Triangle}(y) \rightarrow \text{RightOf}(x, y))) \]

- Negation:
  \[ \neg (\exists x (\text{Square}(x) \land \forall y (\text{Triangle}(y) \rightarrow \text{RightOf}(x, y)))) \equiv \forall x (\neg \text{Square}(x) \lor \exists y (\neg (\text{Triangle}(y) \rightarrow \text{RightOf}(x, y)))) \]

Prolog (Programming in logic)

- Prolog statements:
  
isabove(g, b₁). color(g, gray). color(b₁, blue). isabove(b₁, w₁).
  color(b₂, blue). color(w₁, white). isabove(w₂, b₂).
  color(b₃, blue). color(w₂, white). isabove(b₂, b₃).

  \[ \text{isabove}(X, Z) :- \text{isabove}(X, Y), \text{isabove}(Y, Z). \]

  ?- color(b₁, blue).
  \[ \text{TRUE} \]

  ?- isabove(X, w₁).
  \[ X = b₁; \quad X = g \]
Prolog (Programming in logic)

- Prolog statements:
  
isabove(g, b₁). color(g, gray). color(b₃, blue). isabove(b₁, w₁).
color(b₁, blue). color(w₁, white). isabove(w₁, b₂).
color(b₂, blue). color(w₂, white). isabove(b₂, b₃).

\[
\text{isabove}(X, Z) :\text{- isabove}(X, Y), \text{isabove}(Y, Z).
\]

?- isabove(b₂, w₁).
No.

?- color(w₁, X).
X = white