Consider the following statements.

1. All CSE215 students are smart.
2. Some smart students have a GPA of less than 3.5.
3. There are students in CSE215 that have a GPA of 4.
4. All students with a GPA of 3.8 or more are enrolled in CSE215.

1.1
Convert each of the above statements to a quantified statement. Use $S$ as the set of all students, $s(x)$ to denote that $x$ is smart, and $c(x)$ to denote that $x$ is in CSE215. You may use the function $g(x)$ to represent $x$'s GPA.

1. $\forall x \in S, c(x) \rightarrow s(x)$
2. $\exists x \in S, s(x) \land (g(x) < 3.5)$
3. $\exists x \in S, c(x) \land (g(x) = 4)$
4. $\forall x \in S, (g(x) \geq 3.8) \rightarrow c(x)$

1.2
John is enrolled in CSE215. What can you say (if anything) about his GPA? Explain your answer briefly.

Proof. John is enrolled in CSE215. So, for $x = \text{John}$, $c(x)$. With statement 1, we can know that $s(x)$ is also true. However, we cannot get anything further regarding the GPA. So, we can say nothing about John’s GPA.  

1.3
Jane is not enrolled in CSE215. What can you say (if anything) about her GPA? Explain your answer briefly.

Proof. Jane is not enrolled in CSE215. So, for $x = \text{Jane}$, $\sim c(x)$. With statement 4, we can know that $\sim (g(x) \geq 3.8)$ is also true. Therefore, $g(x) < 3.8$, which means the GPA for Jane is less than 3.8.

\[\text{(Here, we should accept any brief explanation – even if it is not as formal as the solution.)}\]
2

Prove that $\sqrt{2} + \sqrt{3}$ is irrational. Hint: You may assume that $\sqrt{6}$ is irrational.

Proof. Assume that $\sqrt{2} + \sqrt{3}$ is rational, then we can get that $(\sqrt{2} + \sqrt{3})^2$ is also rational since rational number times another rational number is rational. Set $r = (\sqrt{2} + \sqrt{3})^2 = 5 + 2\sqrt{6}$ is rational, we can get $\sqrt{6} = \frac{r - 5}{2}$ is also rational since a rational number subtract by a rational number or divided by a rational nonzero number is also rational.

Here, $\sqrt{6}$ is irrational, contradicting to the original assumption and therefore $\sqrt{2} + \sqrt{3}$ is irrational.

3

Given any integer $n > 3$, prove that one of $n, n + 2$ or $n + 4$ must be composite. Hint: Use case analysis based on the quotient-remainder theorem.

Proof. For any integer $n > 3$, $n$ has a unique expression of $dq + r$ where $d, q, r$ are integers and $0 \leq r < d$. By setting $d = 3$, we can know that $n = 3q$ or $n = 3q + 1$ or $n = 3q + 2$.

1. When $n = 3q$, $n$ can be divisible by 3.
2. When $n = 3q + 1$, $n + 2 = 3q + 1 + 2 = 3(q + 1)$ can be divisible by 3.
3. When $n = 3q + 2$, $n + 4 = 3q + 2 + 4 = 3(q + 2)$ can be divisible by 3.

Since $n > 3$, $n + 2 > 3$ and $n + 4 > 3$, they must be a composite number if they are divisible by 3. So, in conclusion, for any number $n > 3$, one of $n, n + 2$ or $n + 4$ must be composite.

4

Prove or disprove the following two statements.

4.1

If $a$ is rational and $b$ is irrational, then $(a + b)$ is irrational.

Proof. We can prove the statement by contradiction. Suppose that when $a$ is rational and $b$ is irrational, then $(a + b)$ is rational.
Since \( a \) and \( a + b \) are rational, we can set \( a = \frac{p}{q}, a + b = \frac{r}{t} \) where \( p, q, r, t \) are integers and \( q \neq 0, t \neq 0 \).

\[
\begin{align*}
    b &= (a + b) - a \\
    &= \frac{r}{t} - \frac{p}{q} \\
    &= \frac{rq - pt}{tq} \\
\end{align*}
\]  

Clearly \( rq - pt \) and \( tq \) are both integers and \( tq \neq 0 \). Therefore, \( b \) is rational, contradict to the original assumption. So, If \( a \) is rational and \( b \) is irrational, then \( (a + b) \) is irrational. 

4.2

If \( a \) is rational and \( b \) is irrational, then \( ab \) is irrational.

Proof. In this statement, we can give a counterexample to show that it is false. When we set \( a = 0 \), which is a rational number, no matter what the value \( b \) is, \( ab = 0 \) is rational. So, the original statement is false. 

\[\text{2} \quad \text{We will give some credit to students who missed the “a=0” case, i.e., who proved the theorem assuming a \neq 0.}\]