Logic
Outline

• Basics
• Truth Tables of Operators
• Logic Formulae
  – Evaluation; Truth Tables
• Equivalence
  – Laws; Proving
• Inferences
Logic

• All about what is TRUE and what is FALSE.

• Proposition/Statements:
  – “Sentences” that are either T or F, and never both.
  – E.g., (a) Today is Sunday. (b) If today is Sunday, then it will rain today.
  – NOT propositions: (a) “x + y = 5” since x and y are unknown, (b) “How far is the next town?”
Logic Variables and Operators

Logic Variables:
- Of unknown value (but still T or F).
- p, q, r, t are commonly used as logic variables.

Logic Operators
1. NOT (\(\sim\))
   - E.g., \(\sim p\)
2. AND (\(\land\))
   - E.g., \(p \land q\)
3. OR (\(\lor\))
   - E.g., \(p \lor q\)
4. IMPLIES (\(\rightarrow\))
   - E.g., \(p \rightarrow q\)
Compound Statements

Formed of variables and logic operators.
E.g.,

• \( \sim p \)
• \( \sim(p \lor \sim q) \)
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Truth Table for $\sim$ $p$

When is ($\sim p$) TRUE?

When $p$ is FALSE.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\sim p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>
Truth Table for $\wedge$

When is $(p \wedge q)$ TRUE?

When both $p$ and $q$ are true.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \wedge q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
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<tr>
<td>T</td>
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<td>F</td>
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<td>F</td>
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<td>F</td>
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<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
Truth Table for $\lor$

When is $(p \lor q)$ true?

When either one is true.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \lor q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
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<tr>
<td>F</td>
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</tbody>
</table>
Truth Table for “→”

When is (p→q) true?
= “if p then q”
Always, unless p is true and q is false.
\[ p \rightarrow q \]

- Why is \((p \rightarrow q)\) always true when \(p\) is \(F\).
- Let \(p\) = “Sky is blue”
- Let \(q\) = “its daytime”.

\(p \rightarrow q\) is “If (sky is blue) then (its daytime).

- Consider a night. Here, \(p\) is \(F\) and \(q\) is \(F\).
- Or, consider a cloudy day. Here, \(p\) is \(F\), \(q\) is \(T\).
- Should \(p \rightarrow q\) hold, at nights or cloudy days?
- Yes, \(p \rightarrow q\) is a valid claim (even at night or cloudy days).
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Logic Formulae

\[(\neg(\neg p \land \neg q)) \land (\neg(p \land q))\]

• When is the above true?
• Can the above be simplified?

Construct its truth table.

Parentheses used to avoid ambiguity.
Truth Table for Logic Formulae

\((\sim(\sim p \land \sim q)) \land (\sim(p \land q))\)

• Evaluate the above for all combination of values of variables (in this case, p and q).

• Thus,

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>((\sim(\sim p \land \sim q)) \land (\sim(p \land q)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>?</td>
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<tr>
<td>T</td>
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<td>?</td>
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<tr>
<td>F</td>
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<td>?</td>
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<tr>
<td>F</td>
<td>F</td>
<td>?</td>
</tr>
</tbody>
</table>
(1) Enumerating all combinations

- Consider a formula composed of N variables.
- How do you enumerate all combination of values?
  - Binary method:
  - For loop:
  - Recursive:

  There are $2^N$ combinations.
(2) Evaluating a Formula

\[(\neg(\neg p \land \neg q)) \land (\neg(p \land q))\]

- What is its value, when p is TRUE and q is FALSE?

\[\implies (\neg(\neg T \land \neg F)) \land (\neg(T \land F))\]

\[= (\neg(F \land T)) \land (\neg F)\]

\[= (\neg F) \land (T)\]

\[= T.\]
Truth Table for Formula

\((\sim(\sim p \land \sim q)) \land (\sim(p \land q))\)

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>((\sim(\sim p \land \sim q)) \land (\sim(p \land q)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
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</tbody>
</table>
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Equivalence

• Two formulae are equivalent, if they have the same truth table.
• E.g., \((p \rightarrow q)\) and \((\sim p \lor q)\) are equivalent.
• Notation: \(X \equiv Y\), where X and Y are formulae.
Equivalence Laws – I

1. $\land$ and $\lor$ are **commutative**.
   - $p \land q \equiv q \land p$
   - $p \lor q \equiv q \lor p$

2. $\land$ and $\lor$ are **associative**.
   - $p \land (q \land r) \equiv (p \land q) \land r$
   - $p \lor (q \lor r) \equiv (p \lor q) \lor r$

3. $\land$ and $\lor$ are **distributive**.
   - $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$
   - $p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$
Equivalence Laws – II

1. De Morgan’s Laws.
   - $\neg(p \land q) \equiv \neg p \lor \neg q$
   - $\neg(p \lor q) \equiv \neg p \land \neg q$
Proving Equivalence

1. Show the truth tables are same. (Obvious)
2. Prove using laws. E.g., \((p \lor \neg q) \land q \equiv p \land q\)

Proof:
\[
(p \lor \neg q) \land q \equiv q \land (p \lor \neg q) \\
\equiv (q \land p) \lor (q \land \neg q) \\
\equiv (q \land p) \lor F \\
\equiv (q \land p) \\
\equiv p \land q
\]
More Facts about \( (p \rightarrow q) \)

\((p \rightarrow q)\) is equivalent to all of the below:

- \((\sim q \rightarrow \sim p)\)
- \((\sim p \lor q)\)
- \((\sim (p \land \sim q))\)
- \((\sim (p \land \sim q))\)

Easy to confirm using truth tables.

To see intuitively, try:

\( p = \text{Alex is a father} \)

\( q = \text{Alex is a male} \)

Here, \((p \rightarrow q)\) holds. Do the others mean the same?
Other Concepts

- **Tautology**: A formula that is always true.
  - E.g., \((p \lor \neg p), (p \land q) \lor (\neg p \lor \neg q)\).

- **Contradiction**: A formula that is always false.

- **Bi-conditional** \((p \leftrightarrow q)\)
  - Equivalent to: \((p \rightarrow q) \land (q \rightarrow p)\).
  - \((p \leftrightarrow q)\) is true if and only if \(p\) and \(q\) have the same values (check).
Necessary and Sufficient

1. “p is **sufficient** for q” means (p→q).
2. “p is **necessary** for q” means (¬p→¬q). Right?
   above also implies that:
   “p is necessary for q” means (q→p). Why?

Thus, p↔q means p is necessary and sufficient for q.
If and only if

• \((p \rightarrow q)\) means “if \(p\) then \(q\)” OR “\(q\) is true if \(p\) is true”

• What does “\(q\) is true only if \(p\) is true” mean?
  – \(p\) is necessary for \(q\).
  – \((\neg p \rightarrow \neg q)\)
  – \((q \rightarrow p)\)
Interesting Implication Example

• Prosecutor:
  – "If the defendant is guilty, then he had an accomplice."

• Defense Attorney:
  – "That's not true!!"

• What did the defense attorney just claim??
  \[ \sim(p \rightarrow q) \equiv \sim(\sim p \lor q) \equiv \sim(\sim(p \land \sim q)) \equiv p \land \sim q \]
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Arguments; Making Inferences

• We don’t know the values of \( p \) and \( q \).
• But, we know \((p \lor q)\) and \( \neg p \) are true.
• Can we say anything about \( p, q \)?
• Yes.
  – \( p \) is false.
  – \( q \) is true.

• How?
Inferences

Given (as true) formulae $X$ and $Y$. Can we infer (deduce as true) $Z$? Written as:

$$X, Y$$

$$\therefore Z$$

When:

• We can show that $(X \land Y) \rightarrow Z$ is always true (i.e., is a Tautology). How? Truth table!
Inference Example

Given \((p \rightarrow (q \lor \sim r))\) and \((q \rightarrow (p \land r))\)

Can we infer \((p \rightarrow r)\)?

<table>
<thead>
<tr>
<th>(p)</th>
<th>(q)</th>
<th>(r)</th>
<th>(\sim r)</th>
<th>(q \lor \sim r)</th>
<th>(p \land r)</th>
<th>(p \rightarrow q \lor \sim r)</th>
<th>(q \rightarrow p \land r)</th>
<th>(p \rightarrow r)</th>
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<tbody>
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</table>

This row shows it is possible for an argument of this form to have true premises and a false conclusion. Hence this form of argument is invalid.

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# Common Inference Rules

<table>
<thead>
<tr>
<th>Rule</th>
<th>Expression 1</th>
<th>Expression 2</th>
<th>Expression 3</th>
<th>Expression 4</th>
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<tbody>
<tr>
<td><strong>Modus Ponens</strong></td>
<td>$p \rightarrow q$</td>
<td>$p$</td>
<td>$q$</td>
<td><strong>Elimination</strong></td>
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<td>$a. p \lor q$</td>
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<td>$\sim q$</td>
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<td>$p$</td>
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<td><strong>Modus Tollens</strong></td>
<td>$p \rightarrow q$</td>
<td>$\sim q$</td>
<td>$\sim p$</td>
<td><strong>Transitivity</strong></td>
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<td>$p \rightarrow q$</td>
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<td>$p$</td>
</tr>
<tr>
<td><strong>Generalization</strong></td>
<td>a. $p$</td>
<td>b. $q$</td>
<td></td>
<td><strong>Proof by</strong></td>
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<td><strong>Division into Cases</strong></td>
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<td>$p \lor q$</td>
<td>$p \lor q$</td>
<td></td>
<td>$p \lor q$</td>
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<tr>
<td></td>
<td></td>
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<td></td>
<td>$q \rightarrow r$</td>
</tr>
<tr>
<td><strong>Specialization</strong></td>
<td>a. $p \land q$</td>
<td>b. $p \land q$</td>
<td>$p$</td>
<td>$q$</td>
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<tr>
<td><strong>Conjunction</strong></td>
<td>$p$</td>
<td>$q$</td>
<td>$p \land q$</td>
<td></td>
</tr>
</tbody>
</table>
Inference Example – I

1. If I was reading in the kitchen, then my glasses would be on the kitchen table.
2. If glasses are on the kitchen table, then I saw them at breakfast.
3. I did not see my glasses at breakfast.
4. I was reading the newspaper in the living room or the kitchen.
5. If I was reading the newspaper in the living room, then my glasses are on the coffee table.

Where are the glasses?

6. Glasses not on kitchen table [From 2, 3, Modus Tollens].
7. I was not reading in the kitchen [From 1, 6, Modus Tollens]
8. Reading in the living room [From 4, 7, Elimination]
9. Glasses on the coffee table. [From 5, 8, Modus Ponens]
Knights and Knaves: knights always tell the truth and knaves always lie

A says: B is a knight.
B says: A and I are of opposite type.

Suppose A is a knight.
∴ What A says is true. by definition of knight
∴ B is also a knight. That’s what A said.
∴ What B says is true. by definition of knight
∴ A and B are of opposite types. That’s what B said.
∴ We have arrived at the following contradiction: A and B are both knights and A and B are of opposite type.
∴ The supposition is false. by the contradiction rule
∴ A is not a knight. negation of supposition
∴ A is a knave. since A is not a knight.
∴ What A says is false. by definition of knave
∴ B is not a knight. ~(what A said).
∴ B is also a knave. by elimination
Use of contradiction Rule

- Given some facts.
- P (supposition)
- Q (inferred/given from above).
- ~Q (inferred/given from above).
- False (from Q ∧ ~Q).
- Thus, we have shown (Facts ∧ P) → False.
- Thus, ~(Facts ∧ P).
- Thus, ~p.
Use of Transitivity

• Given X and Y, we infer Z.
• Now, from Y and Z, we infer W.
• Can we say that W can be inferred from X and Y?

• \((X \land Y) \rightarrow Z\).
• \((X \land Y) \rightarrow Y\). (trivial)
• Thus, \((X \land Y) \rightarrow (Y \land Z)\)
• Also, \((Y \land Z) \rightarrow W\).
• Thus, \((X \land Y) \rightarrow W\) (by transitivity)