Functions
Outline

• Functions
• One to one (injective) functions
• Onto (surjective) functions
• One-to-one correspondences
• Inverse functions
• Composition of functions
Functions Defined on General Sets

A function $f$ from a set $X$ to a set $Y$

$$f : X \rightarrow Y$$

$X$ is the domain.

$Y$ is the co-domain

Conditions for a Function

1. Every element in $X$ is related to some element in $Y$
2. No element in $X$ is related to multiple elements in $Y$

Range of $f = \{y \in Y \mid y = f(x), \ x \in X\}$
Arrow diagrams

An arrow diagram defines a function \( \text{iff} \)

1. Every element of \( X \) has exactly one arrow coming out of it
Arrow diagrams: Examples 1

\[ X = \{a, b, c\}, \quad Y = \{1, 2, 3, 4\} \]
Arrow diagrams: Examples 2.

\[ X = \{a, b, c\}, \quad Y = \{1, 2, 3, 4\} \]

\[ \text{domain of } f = \{a, b, c\}, \quad \text{co-domain of } f = \{1, 2, 3, 4\} \]

\[ \text{range of } f = \{2, 4\} \]

\[ \text{function representation as a set of pairs} = \{(a,2),(b,4),(c,2)\} \]
Function Equality

Two functions $F$ and $G$ are considered equal if:

- $F$ and $G$ have the same domain and co-domain, i.e.,
  $$F: X \rightarrow Y \text{ and } G: X \rightarrow Y$$

- They have the same set of arrows in the arrow-diagram, i.e.,
  $$F(x) = G(x) \text{ for all } x \in X,$$
  OR equivalently
  $$F \text{ and } G \text{ have the same set of related pairs.}$$
Function Equality

• Example: \( J_3 = \{0, 1, 2\} \)

\[ f : J_3 \rightarrow J_3 \quad \text{and} \quad g : J_3 \rightarrow J_3 \]

\[ f(x) = (x^2 + x + 1) \mod 3 \]

\[ g(x) = (x + 2)^2 \mod 3 \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( x^2 + x + 1 )</th>
<th>( f(x) = (x^2 + x + 1) \mod 3 )</th>
<th>( (x + 2)^2 )</th>
<th>( g(x) = (x + 2)^2 \mod 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>( 1 \mod 3 = 1 )</td>
<td>4</td>
<td>( 4 \mod 3 = 1 )</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>( 3 \mod 3 = 0 )</td>
<td>9</td>
<td>( 9 \mod 3 = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>( 7 \mod 3 = 1 )</td>
<td>16</td>
<td>( 16 \mod 3 = 1 )</td>
</tr>
</tbody>
</table>

\[ f(0) = g(0) = 1, \quad f(1) = g(1) = 0, \quad f(2) = g(2) = 1 \]

\[ f = g \]
Function: Example 1

Identity Function on a Set:
Given a set $X$, $I_X: X \rightarrow X$ is an identity function if

$$I_X(x) = x, \text{ for all } x \in X$$

Function for a sequence:

$1, -1/2, 1/3, -1/4, 1/5, \ldots, (-1)^n/(n + 1), \ldots$

$0 \rightarrow 1, \quad 1 \rightarrow -1/2, \quad 2 \rightarrow 1/3, \quad 3 \rightarrow -1/4, \quad 4 \rightarrow 1/5$

$n \rightarrow (-1)^n/(n + 1)$

$f: \mathbb{N} \rightarrow \mathbb{R}$, for each integer $n \geq 0$, $f(n) = (-1)^n/(n + 1)$

where ($\mathbb{N} = \mathbb{Z}^{\text{nonneg}}$) OR

$g: \mathbb{Z}^+ \rightarrow \mathbb{R}$, for each integer $n \geq 1$, $g(n) = (-1)^{n+1}/n$

where ($\mathbb{Z}^+ = \mathbb{Z}^{\text{nonneg}} - \{0\}$)
Functions: Example 2.

Cardinality of subsets

F : P(\{a, b, c\}) \rightarrow \mathbb{Z}_{\text{nonneg}}

For each X ∈ P(\{a, b, c\}), F(X) = \text{the number of elements in X (i.e., the cardinality of X)}
Functions: Example 3.

Logarithms and Logarithmic Functions.

- F: \( R^+ \rightarrow R \) with \( F(x) = \log_b x \)
- F: \( R \rightarrow R^+ \) with \( F(x) = b^x \)

The above are well-defined functions. Note that \( b^x \) is always positive, and \( \log_b x \) is defined only for positive \( x \)'s.

\[
\begin{align*}
\log_3 9 &= 2 & \text{because} & & 3^2 &= 9 \\
\log_{10}(1) &= 0 & \text{because} & & 10^0 &= 1 \\
\log_2 \frac{1}{2} &= -1 & \text{because} & & 2^{-1} &= \frac{1}{2} \\
\log_2 (2^m) &= m
\end{align*}
\]
The Hamming Distance Function

Let $S_n$ be the set of all strings of 0’s and 1’s of length $n$.

$$H: (S_n \times S_n) \rightarrow \mathbb{Z}^{\text{nonneg}}$$

For each pair of strings $(s, t) \in S_n \times S_n$

$H(s, t)$ = the number of positions in which $s$ and $t$ differ

For $n = 5$, $H(11111, 00000) = 5$

$H(10101, 00000) = 3$

$H(01010, 00000) = 2$
Function: Example 5

Boolean functions:

\[ f : \{0, 1\}^n \rightarrow \{0, 1\} \]

the domain = the set of all ordered n-tuples of 0’s and 1’s
the co-domain = the set \( \{0, 1\} \)

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P )</td>
<td>( Q )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
**Function: Example 5 contd**

\[ f : \{0, 1\}^3 \to \{0, 1\} \]

\[ f(x_1, x_2, x_3) = (x_1 + x_2 + x_3) \mod 2 \]

\[ f(0, 0, 0) = (0 + 0 + 0) \mod 2 = 0 \mod 2 = 0 \]

\[ f(0, 0, 1) = (0 + 0 + 1) \mod 2 = 1 \mod 2 = 1 \]

\[ f(0, 1, 0) = (0 + 1 + 0) \mod 2 = 1 \mod 2 = 1 \]

\[ f(0, 1, 1) = (0 + 1 + 1) \mod 2 = 2 \mod 2 = 0 \]

\[ f(1, 0, 0) = (1 + 0 + 0) \mod 2 = 1 \mod 2 = 1 \]

\[ f(1, 0, 1) = (1 + 0 + 1) \mod 2 = 2 \mod 2 = 0 \]

\[ f(1, 1, 0) = (1 + 1 + 0) \mod 2 = 2 \mod 2 = 0 \]

\[ f(1, 1, 1) = (1 + 1 + 1) \mod 2 = 3 \mod 2 = 1 \]
Examples of Non-Functions

1. \( f : \mathbb{R} \rightarrow \mathbb{R}, \ f(x) \) is the real number \( y \) such that \( x^2 + y^2 = 1 \)

2. \( f : \mathbb{Q} \rightarrow \mathbb{Z}, \ f(q) = m, \) if \( q = m/n \) for some integers \( m/n \) with \( n \neq 0 \)

Why?
Outline

• Functions
• **One to one (injective) functions**
• **Onto (surjective) functions**
• **One-to-one correspondences**
• **Inverse functions**
• **Composition of functions**
One-to-One Functions

F : X → Y is one-to-one (injective) if each element in Y has at most one arrow pointing to it.

Formally, F is one-to-one if:

∀ x₁, x₂ ∈ X, F(x₁) = F(x₂) → x₁ = x₂

NOT one-to-one:

Two distinct elements of X are sent to the same element of Y.
One-to-One Function: Example

- \( F \) and \( G: \{a, b, c, d\} \rightarrow \{u, v, w, x, y\} \)

\( F \) is one-to-one, \( G \) is not.

F is one-to-one, G is not.
Proving/Disproving One-to-Oneness

**To prove** $f$ is one-to-one:

Need to prove: $\forall x_1, x_2 \in X, F(x_1) = F(x_2) \Rightarrow x_1 = x_2$

Thus,

- Pick arbitrary elements $x_1$ and $x_2$ of $X$.
- Assume $f(x_1) = f(x_2)$.
- Show that $x_1 = x_2$.

**To show** $f$ is **not** one-to-one:

Find elements $x_1, x_2$ in $X$ s.t. $f(x_1) = f(x_2)$ but $x_1 \neq x_2$. 
1. $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 4x - 1$ for all $x \in \mathbb{R}$

2. $g : \mathbb{Z} \rightarrow \mathbb{Z}$, $g(n) = n^2$ for all $n \in \mathbb{Z}$
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Onto Functions

F: X → Y is **onto (surjective)** if each element y in Y has at least one arrow pointing to it.

Formally, F is onto if \( \forall y \in Y, \exists x \in X \text{ such that } F(x) = y \).

**Examples:**

![Diagram of onto function]

Each element y in Y equals F(x) for at least one x in X.
Not-Oneto Examples

\[ X = \text{domain of } F \quad \quad \quad \quad \quad \quad Y = \text{co-domain of } F \]

At least one element in \( Y \) does not equal \( F(x) \) for any \( x \) in \( X \).
Proving Onto

Recall \( F: X \rightarrow Y \) is onto if

\[ \forall y \in Y, \exists x \in X \text{ such that } F(x) = y. \]

To prove \( F \) is onto:

- Pick an arbitrary \( y \) in \( Y \).
- Show \( \exists x \in X \text{ such that } F(x) = y \)

To prove \( F \) is not onto:

- Find a \( y \) in \( Y \) s.t. \( y \neq F(x) \) for any \( x \) in \( X \).
Proving Ontoness: Exercises

1. \( f : \mathbb{R} \rightarrow \mathbb{R}, \ f(x) = 4x - 1 \) for all \( x \in \mathbb{R} \)
2. \( h : \mathbb{Z} \rightarrow \mathbb{Z}, \ h(n) = 4n - 1 \) for all \( n \in \mathbb{Z} \)
3. \( f : \mathbb{R} \rightarrow \mathbb{R}^+, \ f(x) = 2^x \)
4. \( f : \mathbb{R}^+ \rightarrow \mathbb{R}, \ f(x) = \log_2(x) \)

Does “2” matter in #3 and #4?
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One-to-One Correspondence

A *one-to-one correspondence* (or *bijection*)

= **both** one-to-one and onto.

![Diagram](image)
One-to-One Correspondence: Example

$$h : P(\{a, b\}) \rightarrow \{00, 01, 10, 11\}$$

If $a$ is in $A$, write a 1 in the 1\text{st} position of the string $h(A)$.
If $a$ is not in $A$, write a 0 in the 1\text{st} position of the string $h(A)$.
If $b$ is in $A$, write a 1 in the 2\text{nd} position of the string $h(A)$.
If $b$ is not in $A$, write a 0 in the 2\text{nd} position of the string $h(A)$.

<table>
<thead>
<tr>
<th>Subset of ${a, b}$</th>
<th>Status of $a$</th>
<th>Status of $b$</th>
<th>String in $S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>not in</td>
<td>not in</td>
<td>00</td>
</tr>
<tr>
<td>${a}$</td>
<td>in</td>
<td>not in</td>
<td>10</td>
</tr>
<tr>
<td>${b}$</td>
<td>not in</td>
<td>in</td>
<td>01</td>
</tr>
<tr>
<td>${a, b}$</td>
<td>in</td>
<td>in</td>
<td>11</td>
</tr>
</tbody>
</table>
One-to-One Correspondences

\[ F: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R} \]

\[ F(x, y) = (x + y, x - y), \text{ for all } (x, y) \in \mathbb{R} \times \mathbb{R} \]

Prove.
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For a one-to-one correspondence $F: X \rightarrow Y$, the inverse function for $F$ is defined as:

$$F^{-1}: Y \rightarrow X, \text{ s.t. } \forall y \in Y, F^{-1}(y) = x \in X \text{ s.t. } F(x) = y$$

Is $F^{-1}$ well-defined? One-to-one correspondence?
Inverse Function: Example

the inverse function for \( h \) is \( h^{-1} \):

\[
\begin{align*}
\mathcal{P}([a, b]) & \xrightarrow{h} S \\
\emptyset & \rightarrow 00 \\
\{a\} & \rightarrow 10 \\
\{b\} & \rightarrow 01 \\
\{a, b\} & \rightarrow 11 \\
\end{align*}
\]

\[
\begin{align*}
\mathcal{P}([a, b]) & \xrightarrow{h^{-1}} S \\
\emptyset & \rightarrow 00 & h^{-1}(00) = \emptyset \\
\{a\} & \rightarrow 10 & h^{-1}(10) = \{a\} \\
\{b\} & \rightarrow 01 & h^{-1}(01) = \{b\} \\
\{a, b\} & \rightarrow 11 & h^{-1}(11) = \{a, b\}
\end{align*}
\]
Inverse Functions: Exercise

\[ f : \mathbb{R} \rightarrow \mathbb{R}, \ f(x) = 4x - 1 \] for all real numbers \( x \).
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Composition of Functions

Let $f : X \rightarrow Y'$ and $g : Y \rightarrow Z$ be functions s.t. $Y' \subseteq Y$

The composition of $f$ and $g$ is a function $g \circ f : X \rightarrow Z : (g \circ f)(x) = g(f(x))$, for all $x \in X$
Composition of Functions: Example

Let $f : \{1,2,3\} \rightarrow \{a,b,c,d\}$ and $g : \{a,b,c,d,e\} \rightarrow \{x,y,z\}$
Composition of Functions: Exercise

Let $f: \mathbb{Z} \to \mathbb{Z}$ and $g: \mathbb{Z} \to \mathbb{Z}$

$f(n) = n + 1$, for all $n \in \mathbb{Z}$

$g(n) = n^2$, for all $n \in \mathbb{Z}$

$(g \circ f)(n)$

$(f \circ g)(n)$

$f \circ g \neq g \circ f$
Composition over Identity Function

Let $X = \{a, b, c, d\}$ and $Y = \{u, v, w\}$, $f : X \rightarrow Y$

$I_X : X \rightarrow X$ is an identity function
$I_X(x) = x$, for all $x \in X$

$(f \circ I_X)(x) = f(I_X(x)) = f(x)$, for all $x \in X$

$I_Y : Y \rightarrow Y$ is an identity function
$I_Y(y) = y$, for all $y \in Y$

$(I_Y \circ f)(x) = I_Y(f(x)) = f(x)$, for all $x \in X$
Composition over Inverse

Let \( f : \{a, b, c\} \rightarrow \{x, y, z\} \) be a one-to-one correspondence

\[
\begin{align*}
( f^{-1} \circ f)(a) &= f^{-1}( f(a)) = f^{-1}(z) = a \\
( f^{-1} \circ f)(b) &= f^{-1}( f(b)) = f^{-1}(x) = b \\
( f^{-1} \circ f)(c) &= f^{-1}( f(c)) = f^{-1}(y) = c
\end{align*}
\]

\[\Rightarrow f^{-1} \circ f = I_X \]

also \( f \circ f^{-1} = I_Y \)
Composition over Inverse: Proof

If $f: X \rightarrow Y$ is a one-to-one correspondence, then

(a) $f^{-1} \circ f = I_X$

(b) $f \circ f^{-1} = I_Y$

Prove.
Composition of 1-to-1 Functions

If \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) are both one-to-one functions, then \( g \circ f \) is also one-to-one.

Prove.
Composition of onto Functions

If \( f: X \rightarrow Y \) and \( g: Y \rightarrow Z \) are both onto functions, then \( g \circ f \) is onto.

Prove.
Image, and 2 Claims.
Image and inverse image

**Definition**

If \( f: X \rightarrow Y \) is a function and \( A \subseteq X \) and \( C \subseteq Y \), then

\[
f(A) = \{ y \in Y \mid y = f(x) \text{ for some } x \in A \}
\]

and

\[
f^{-1}(C) = \{ x \in X \mid f(x) \in C \}.
\]

\( f(A) \) is called the **image of** \( A \), and \( f^{-1}(C) \) is called the **inverse image of** \( C \).

Inverse sign over a SUBSET of domain signifies **image**
Example: Image and Inverse Image

Example: $X = \{1, 2, 3, 4\}$, $Y = \{a, b, c, d, e\}$, $f : X \rightarrow Y$

$$f(\{1,4\}) = \{b\} \quad f^{-1}(\{a,b\}) = \{1, 2, 4\}$$

$$f(X) = \{a, b, d\} \quad f^{-1}(\{c,e\}) = \emptyset$$
When does One-to-One $\iff$ Onto?

- If $f: X \to Y$ and $X$ and $Y$ have the same number of elements, then $f$ is one-to-one $\iff f$ is onto.

- If $f: X \to Y$ and $|X| > |Y|$, then $f$ can NOT be one-to-one.

[Proofs are in Chapter 9.4; we skip them for now]