Computing Surface Hyperbolic Structure and Real Projective Structures

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ACM Solid and Physical Modeling Symposium, 2006
M.C. Escher's art works: Angels and Devils

Regular division of the plane

Sphere with Angels and Devils

Circle limit IV Heaven and Hell
Geometries defined on surfaces

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Hyperbolic and Real Projective Structures
Motivation

Main Goals
- Define different geometries on surfaces.
- Systematically generalize planar algorithms to surfaces.

Example: Generalize planar spline schemes to surfaces.
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- Systematically generalize planar algorithms to surfaces.

Example: Generalize planar spline schemes to surfaces.
Planar Splines

Parametric Affine Invariant

The spline is invariant under the affine transformations of the knots and the parameters.
Idea: Geometry Structure
A mesh is covered by local coordinate charts. Geometric construction is invariant during the transition from one local coordinate to another.

Global Parameterization
Find atlas with special transition functions.
Manifold SPLines

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Global Parameterization
Find atlas with special transition functions.
Parameterizations as Finding Metrics

Idea: Flat Metrics

A planar parameterization of a mesh is equivalent to find the edge lengths such that the sum of surrounding angles for each vertex is $2\pi$. 

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Hyperbolic and Real Projective Structures
Parameterizations as Finding Metrics

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Hyperbolic and Real Projective Structures
Idea: Ricci Flow

Conformally adjust the edge lengths; the deformation of the edge lengths is driven by the current curvature.
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Main Ideas

Geometry Structure
A surface is covered by local coordinate charts. Geometric construction is invariant during the transition from one local coordinate to another.

Flat Metrics
Global parameterizations is formulated as finding flat metrics.

Ricci Flow
The conformal deformation of the edge lengths is driven by the curvature.
Erlangen Program - F. Klein 1872

Different geometries study the invariants under different transformation groups.

- **Euclidean Geometry**: Rigid motion on $\mathbb{R}^2$. Distances between arbitrary two points are the invariants.
- **Affine Geometry**: Affine transformations. Parallelism and **barry centric coordinates** are the invariants.
- **Real Projective Geometry**: Real projective transformations. Collinearity and **cross ratios** are the invariants.
Algorithms vs. Geometries

Delaunay Triangulation: Euclidean geometry.

Key concept used in Delaunay triangulation: distance.

Suppose \( \mathbf{P} \) is a planar point set, \( T(\mathbf{P}) \) is its Delaunay triangulation,

\[
\mathbf{P} = \{p_0, p_1, p_2, \cdots, p_n\}.
\]

Let \( g : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be a planar rigid motion, then,

\[
g(T(\mathbf{P})) = T(g(\mathbf{P})).
\]
Algorithms vs. Geometries

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Convex Hull: Real projective geometry.

Key concept used in convex hull: collinearity.

Suppose $P$ is a planar point set, $T(P)$ is its Delaunay triangulation,

$$P = \{p_0, p_1, p_2, \cdots, p_n\}.$$

Let $g : \mathbb{RP}^2 \rightarrow \mathbb{RP}^2$ be a real projective transformation, then,

$$g(C(P)) = C(g(P)).$$
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Let $g : \mathbb{RP}^2 \rightarrow \mathbb{RP}^2$ be a real projective transformation, then,

$$g(C(P)) = C(g(P)).$$
Definition ((X,G) invariant Algorithm)

Suppose $X$ is a topological space, $G$ is the transformation group on $X$. A geometric operator $\Omega$ defined on $X$ is $(X, G)$ invariant, if and only if

$$\Omega \circ g = g \circ \Omega, \forall g \in G.$$ 

Examples:

- Minkowski sum: Translation invariant.
- Voronoi Diagram: Rigid motion invariant.
- Polar form : Affine invariant.
Algorithms vs. Geometries

Central Problem

- Can different geometries be defined on general surfaces?
- Can different planar algorithms be generalized to surface domains directly?

The answers are yes and yes. The major theoretic tool is the Geometric Structure.
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The answers are yes and yes. The major theoretic tool is the *Geometric Structure.*
Manifold

\[ \phi_{\alpha}(U_{\alpha}) \quad \phi_{\beta}(U_{\beta}) \]

\[ U_{\alpha} \quad U_{\beta} \]

\[ S \]

\[ \phi_{\alpha} \quad \phi_{\beta} \]

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Hyperbolic and Real Projective Structures
Definition (Manifold)

A **manifold** is a topological space $\Sigma$ covered by a set of open sets $\{U_\alpha\}$. A homeomorphism $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ maps $U_\alpha$ to the Euclidean space $\mathbb{R}^n$. $(U_\alpha, \phi_\alpha)$ is called a chart of $\Sigma$, the set of all charts $\{(U_\alpha, \phi_\alpha)\}$ form the atlas of $\Sigma$. Suppose $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a transition map.

Transition maps satisfy cocycle condition, suppose $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, then

$$\phi_{\beta\gamma} \circ \phi_{\alpha\beta} = \phi_{\alpha\gamma}.$$
Definition ((X,G) Atlas)

Suppose $X$ is a topological space, $G$ is the transformation group of $X$. A manifold $\Sigma$ with an atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ is an $(X, G)$ atlas if

1. $\phi_\alpha(U_\alpha) \subset X$, for all charts $(U_\alpha, \phi_\alpha)$.
2. Transition maps $\phi_{\alpha\beta} \in G$. 
**Definition (Equivalent \((X, G)\) atlases)\)**

Two \((X, G)\) atlases \(\mathcal{A}_1\) and \(\mathcal{A}_2\) of \(\Sigma\) are **equivalent**, if their union is still an \((X, G)\) atlas of \(\Sigma\).

**Definition ((X,G) structure)\)**

An \((X, G)\) **structure** of a manifold \(\Sigma\) is an equivalent class of its \((X, G)\) atlases.
Definition (Equivalent $(X, G)$ atlases)

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Definition ($\langle X, G \rangle$ structure)

An $(X, G)$ structure of a manifold $\Sigma$ is an equivalent class of its $(X, G)$ atlases.
Common \((X, G)\) structure

**Spherical Structure**

- **X**: Unit sphere \(S^2\).
- **G**: Rotation group.
- **Surfaces**: Genus zero closed surfaces; any open surfaces.

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Common \((X,G)\) structure

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Hyperbolic and Real Projective Structures
Common \((X,G)\) structure

**Affine Structure**

- \(X\): Real plane \(\mathbb{R}^2\).
- \(G\): Affine transformation group.
- Surfaces: Genus one closed surface and open surfaces.
Common \((X,G)\) structure

\[
T(g(P))
\]

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- \(X\): Real plane \(\mathbb{R}^2\).
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- Surfaces: Genus one closed surface and open surfaces.
Common (X,G) structure

Hyperbolic Structure
- X: Hyperbolic plane $\mathbb{H}^2$.
- G: Möbius transformation group.
- Surfaces: with negative Euler number.

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Hyperbolic and Real Projective Structures
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\[ T(g(P)) \]

Real Projective Structure
- \(X\): Real projective plane \(\mathbb{RP}^2\).
- \(G\): Real projective transformation group.
- Surfaces: any surface.
Common \((X,G)\) structure

\[ T(g(P)) \]

Real Projective Structure

- \(X\): Real projective plane \(\mathbb{RP}^2\).
- \(G\): Real projective transformation group.
- Surfaces: any surface.
Conformal Structure

- $X$: Complex plane $\mathbb{C}$.
- $G$: Biholomorphic maps.
- Surfaces: any surface.
Pseudo \((X,G)\) structure

\[
P_t(g(P))
\]

Conformal Structure

- \(X\): Complex plane \(\mathbb{C}\).
- \(G\): Biholomorphic maps.
- Surfaces: any surface.
Global Tensor Product Structure
Relations Among Geometric Structures

Conformal structure (A holomorphic 1-form) induces affine structure.

Hyperbolic structure induces conformal structure.

Hyperbolic structure induces real projective structure.
Theorem

Suppose a manifold with an \((X, G)\) structure, then any \((X, G)\) invariant algorithms can be generalized on the manifold.

Corollary (Manifold Splines - Gu, He, Qin 2005)

Spline schemes based on polar forms can be defined on a manifold, if and only if the manifold has an affine structure.
Manifold Splines SPM2005

Hyperbolic and Real Projective Structures
Theorem (Benzecri 1959)

If a closed surface admits an affine structure, it has zero Euler class.

Real projective structure

- Real projective structure is general, it exists for all surfaces.
- Real projective structure is simple, all transitions are linear rational functions.
- Real projective structure is suitable for designing manifold spline schemes.
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There are too many related works, we only briefly review the most related ones. Detailed information can be found in the survey papers in each field.
Previous Works: Ricci Flow

Poincaré Conjecture

A closed three dimensional manifold is a topological three sphere, if all loops can be contracted to a point.

- Hamilton introduced surface Ricci flow in 1988, [Hamilton] and pointed out the direction of solving Poincaré conjecture.
- In 2003 to now, Perelman conquered the most difficult part of the proof. Many group of mathematicians are competing for the complete proof.
- In June 2006, complete proof (more than 320 pages) is published by Zhu and Cao. The proof covers Thurston geometrization conjecture, Poincaré conjecture is a corollary.
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Previous Works: Surface Conformal parameterization

Maps


Differential Forms

Previous Works: Surface Conformal parameterization

Maps

Differential Forms
Previous Works: Surface Conformal parameterization

Angles


Metrics

Previous Works: Surface Conformal parameterization

### Angles

### Metrics
Thurston developed the concept of geometric structures in 1976.

- Spherical structure

- Affine structure

- Hyperbolic structure and Real projective structure
Previously Works: Geometric Structures

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**Spherical structure**

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**Hyperbolic structure and Real projective structure**
Comparisons

Comparing with conventional methods

1. First work on hyperbolic surface parameterizations.
2. Formulate parameterization as finding special metrics.
3. Combinatorial vs. induced Euclidean metric.
5. Tangential relation vs. intersection.
6. Generalizable to 3-manifolds.
Conformal Metric

Definition

Suppose $\Sigma$ is a surface with a Riemannian metric,

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

Suppose $\lambda : \Sigma \to \mathbb{R}$ is a function defined on the surface, then $e^{2\lambda}g$ is also a Riemannian metric on $\Sigma$ and called a conformal metric. $e^{2\lambda}$ is called the conformal factor.

Angles are invariant measured by conformal metrics.
Given a surface $\Sigma$ with a Riemannian metric $g$, find a function $\lambda : \Sigma \rightarrow \mathbb{R}$, such that $e^{2\lambda}g$ is one of the followings:

- **Uniform flat metric**
  
  \[ \bar{K} \equiv 0, \]

  for interior points

  \[ \bar{k}_g \equiv \text{const} \]

  for boundary points. The constant values are determined by the conformal structure of $\Sigma$.

- **Uniformization metric**
  
  \[ \bar{K} \equiv \text{const}, \]

  for interior points

  \[ \bar{k}_g \equiv 0. \]

The tool to calculate the above metrics is **Ricci flow**.
Definition (Surface Ricci Flow)

A closed surface with a Riemannian metric $g$, the Ricci flow on it is defined as

$$\frac{dg_{ij}}{dt} = (\bar{K} - K)g_{ij}, \quad \bar{K} = \frac{2\pi \chi(\Sigma)}{S(\Sigma)}$$

$\chi(\Sigma)$ is the Euler number, $S(\Sigma)$ is the total area of $\Sigma$. 

Surface Ricci Flow
Theorem (Hamilton 1982)

For a closed surface of non-positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) everywhere.

Theorem (Chow)

For a closed surface of positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) everywhere.
Surfaces are represented as polyhedron triangular meshes.

- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, S^2$. 

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Hyperbolic and Real Projective Structures
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Surfaces are represented as polyhedron triangular meshes.

- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$. 
Curvature

- Discrete curvature: $K : V = \{\text{vertices}\} \to \mathbb{R}^1$.

\[
K(v) = 2\pi - \sum_i \alpha_i, \ v \notin \partial M; \ K(v) = \pi - \sum_i \alpha_i, \ v \in \partial M
\]

- Discrete Gauss-Bonnet theorem

\[
\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi \chi(M).
\]
Discrete Metrics

- **Discrete Metric**: $l : E = \{\text{all edges}\} \to \mathbb{R}^1$, satisfies triangular inequality.
- **Metrics determine curvatures by cosine law.**

\[
\cos \theta_i = \frac{l_j^2 + l_k^2 - l_i^2}{2l_jl_k}, \quad l \neq j \neq k \neq i
\]
## Metrics vs. Curvatures

- All metrics for a mesh $L(\Sigma)$ form a convex polytope.
- All admissible curvature configurations for a mesh $K(\Sigma)$ also form a convex polytope.
- The mapping from the metrics to the curvatures

$$\Phi : L(\Sigma) \rightarrow K(\Sigma),$$

is not one to one.
- The mapping from a conformal class of metrics to the curvatures is a homeomorphism.
Conformal maps Properties

- transform infinitesimal circles to infinitesimal circles.
- preserve the intersection angles among circles.

Idea - Approximate conformal metric deformation

Replace infinitesimal circles by circles with finite radii.
Circle Packing Metric

We associate each vertex \( v_i \) with a circle with radius \( \gamma_i \). On edge \( e_{ij} \), the two circles intersect at the angle of \( \Phi_{ij} \). The edge lengths are

\[
l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j \cos \Phi_{ij}
\]

CP Metric \((\Sigma, \Gamma, \Phi)\), \(\Sigma\) triangulation,

\[
\Gamma = \{\gamma_i|\forall v_i\}, \Phi = \{\phi_{ij}|\forall e_{ij}\}
\]
Discrete Conformal Equivalent Metric

Definition (Conformal Circle Packing Metrics)

Two circle packing metrics \( \{ \Sigma, \Phi_1, \Gamma_1 \} \) and \( \{ \Sigma, \Phi_2, \Gamma_2 \} \) are conformal equivalent, if

- The radii of circles are different, \( \Gamma_1 \neq \Gamma_2 \).
- The intersection angles are same, \( \Phi_1 \equiv \Phi_2 \).

In practice, the circle radii and intersection angles are optimized to approximate the induced Euclidean metric of the mesh as close as possible.
Definition (Discrete Ricci flow)

A mesh $\Sigma$ with a circle packing metric $\{\Sigma, \Gamma, \Phi\}$, where $\Gamma = \{\gamma_i, v_i \in V\}$ are the vertex radii, $\Phi = \{\Phi_{ij}, e_{ij} \in E\}$ are the angles associated with each edge, the discrete Ricci flow on $\Sigma$ is defined as

$$\frac{d\gamma_i}{dt} = (\bar{K}_i - K_i)\gamma_i,$$

where $\bar{K}_i$ are the target curvatures on vertices. If $\bar{K}_i \equiv 0$, the flow with normalized total area leads to a metric with constant Gaussian curvature.

Idea

Metric deformation is driven by curvature.
Theorem (Chow and Luo 2002)

A discrete Euclidean Ricci flow \( \{ \Sigma, \Gamma, \Phi \} \rightarrow \{ M, \bar{\Gamma}, \Phi \} \) converges.

\[
|K_i(t) - \bar{K}_i| < c_1 e^{-c_2 t},
\]

and

\[
|\gamma_i(t) - \bar{\gamma}_i| < c_1 e^{-c_2 t},
\]

where \( c_1, c_2 \) are positive numbers.
Definition

Let $u_i = \ln \gamma_i$, the Ricci energy is defined as

$$f(u) = \int_{u_0}^{u} \sum_{i=1}^{n} (K_i - \bar{K}_i) du_i,$$

where $u = (u_1, u_2, \ldots, u_n)$, $u_0 = (0, 0, \ldots, 0)$. 

Variational Euclidean Ricci flow
Theorem (Ricci Energy)

*Euclidean Ricci energy is well defined and convex, namely, there exists a unique global minimum.*

Proof.

In an Euclidean triangle, with angles \((\theta_1, \theta_2, \theta_3)\) and radius \((\gamma_1, \gamma_2, \gamma_3)\), let \(u_i = \ln \gamma_i\), according to Euclidean cosine law,

\[
\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i}.
\]

Therefore \(\omega = \sum \theta_i du_i\) is a closed 1-form. The Euclidean Ricci energy is well defined. Direct computation verifies that Hessian matrix is positive definite.
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Newton's method for Euclidean Ricci energy

**Gradient descent Method**

Ricci flow is the gradient descent method for minimizing Ricci energy,

\[ \nabla f = (K_1 - \bar{K}_1, K_2 - \bar{K}_2, \ldots, K_n - \bar{K}_n). \]

**Newton's method**

The Hessian matrix of Ricci energy is

\[ \frac{\partial^2 f}{\partial u_i \partial u_j} = \frac{\partial K_i}{\partial u_j}. \]

Newton’s method can be applied directly.
Algorithm: uniform flat metric for closed surfaces

**Ricci Flow for Uniform Flat Metric**

Suppose $\Sigma$ is a closed genus one mesh,

1. Compute the circle packing metric $(\Gamma, \Phi)$.
2. Set the target curvature to be zero for each vertex

$$\bar{K}_i \equiv 0, \forall v_i \in V$$

3. Minimize the Euclidean Ricci energy using Newton’s method to get the target radii $\bar{\Gamma}$.
4. Compute the target flat metric.
Algorithm: uniform flat metric for open surfaces

Given a surface $\Sigma$ with genus $g$ and $b$ boundaries, then its Euler number is

$$\chi(\Sigma) = 2 - 2g - b.$$ 

Suppose the boundary of $\Sigma$ is a set of closed curves

$$\partial \Sigma = C_1 \cup C_2 \cup C_3 \cdots C_b.$$ 

The total curvature for each $C_i$ is denoted as $2m_i \pi$, $m_i \in \mathbb{Z}$, and

$$\sum_{i=1}^{b} m_i = \chi(\Sigma).$$ 

The target curvature for interior vertices are zeros.
Algorithm: uniform flat metric for open surfaces

Euclidean Ricci flow for open surfaces

- Use Newton’s method to minimize the Ricci energy to update the metric.
- Adjust the boundary vertex curvature to be proportional to the ratio between the current lengths of the adjacent edges and the current total length of the boundary component.
- Repeat until the process converges.
Algorithm: Flatten a mesh with a uniform flat metric

1. Determine the planar shape of each triangle using 3 edge lengths.

2. Glue all triangles on the plane along their common edges by rigid motions. Because the metric is flat, the gluing process is coherent and results in a planar embedding.
Euclidean Uniform Flat Metric

original surface

- genus 1, 3 boundaries

universal cover

- embedded in $\mathbb{R}^2$

texture mapping
Euclidean Uniform Flat Metric

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Hyperbolic and Real Projective Structures
Different boundaries are mapped to straight lines.
Euclidean Uniform Flat Metric

original surface  fundamental domain  universal cover
Conformal Model: Poincaré Disk

A unit disk $|z| < 1$ with the Riemannian metric

$$ds^2 = \frac{4dzd\bar{z}}{(1 - \bar{z}z)^2}.$$
Conformal Model: Poincaré Disk

**Poincaré disk**

The **rigid motion** is the Möbius transformation

\[ e^{i\theta} \frac{Z - Z_0}{1 - \overline{Z}_0 Z}. \]
Poincaré disk

The **hyperbolic line** through two point $z_0, z_1$ is the circular arc through $z_0, z_1$ and perpendicular to the boundary circle $|z| = 1$. 
A hyperbolic circle \((c, \gamma)\) on the Poincaré disk is also an Euclidean circle \((C, R)\) on the plane, such that \(C = \frac{2 - 2\mu^2}{1 - \mu^2|c|^2}\),
\[ R^2 = |C|^2 - \frac{|c|^2 - \mu^2}{1 - \mu^2|c|^2}, \mu = \frac{e^r - 1}{e^r + 1}. \]
Discrete Hyperbolic Ricci Flow

Definition (Discrete Hyperbolic Ricci Flow)

Let

\[ u_i = \ln \tanh \frac{\gamma_i}{2}, \]

Discrete hyperbolic Ricci flow for a mesh \( \Sigma \) is

\[ \frac{du_i}{dt} = \bar{K}_i - K_i, \bar{K}_i \equiv 0, \]

the Euler number of \( \Sigma \) is negative, \( \chi(\Sigma) < 0. \)
A hyperbolic discrete Ricci flow \((M, \Gamma, \Phi) \rightarrow (\bar{M}, \bar{\Gamma}, \Phi)\) converges,

\[ |K_i(t) - \bar{K}_i| < c_1 e^{-c_2 t}, \]

and

\[ |\gamma_i(t) - \bar{\gamma}_i| < c_1 e^{-c_2 t}, \]

where \(c_1, c_2\) are positive numbers.
The discrete Hyperbolic Ricci energy is defined as

$$f(u) = \int_{u_0}^{u} \sum_{i=1}^{n} (\bar{K}_i - K_i) du_i.$$ 

Discrete hyperbolic Ricci flow is the gradient descendent method to minimize the discrete hyperbolic ricci energy.
**Theorem (Hyperbolic Discrete Ricci Energy)**

Discrete hyperbolic Ricci energy is well defined and convex, namely, there exists a unique global minimum.

**Proof.**

In a hyperbolic triangle, with angles \((\theta_1, \theta_2, \theta_3)\) and radius \((\gamma_1, \gamma_2, \gamma_3)\), \(u_i = \ln \tanh \frac{\gamma_i}{2}\), according to hyperbolic cosine law,

\[
\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i}.
\]

Therefore \(\omega = \sum \theta_i du_i\) is a closed 1-form. The hyperbolic Ricci energy is convex. Direct computation verifies the Hessian matrix is positive definite.
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Algorithm: Computing Hyperbolic uniformization metric

Hyperbolic Ricci Energy Optimization

1. Set target curvature $K(v_i) \equiv 0$.
2. Optimize the hyperbolic Ricci energy using Newton’s method, with the constraint the total area is preserved.

Flattening Mesh in Hyperbolic Space

1. Determine the shape of each triangle.
2. Glue the hyperbolic triangles coherently by Möbius transformation.

Key: all computations use hyperbolic geometry.
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Genus 0 surface with 3 boundaries. The double covered surface is of genus 2. The boundaries are mapped to hyperbolic lines.
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Hyperbolic Uniformization Metric

Genus 0 surface with 3 boundaries. The double covered surface is of genus 2. The boundaries are mapped to hyperbolic lines.
Embedding in the upper half plane hyperbolic space model. Different period embedded in the hyperbolic space. The boundaries are mapped to hyperbolic lines.
Universal Covering Space and Deck Transformation

**Universal Cover**

A pair \((\tilde{\Sigma}, \pi)\) is a universal cover of a surface \(\Sigma\), if

- Surface \(\tilde{\Sigma}\) is simply connected.
- Projection \(\pi : \tilde{\Sigma} \rightarrow \Sigma\) is a local homeomorphism.

**Deck Transformation**

A transformation \(\phi : \tilde{\Sigma} \rightarrow \tilde{\Sigma}\) is a deck transformation, if

\[
\pi = \pi \circ \phi.
\]

A deck transformation maps one period to another.
Definition (Fuchsian Group)

Suppose $\Sigma$ is a surface, $g$ is its uniformization metric, $(\tilde{\Sigma}, \pi)$ is the universal cover of $\Sigma$. $g$ is also the uniformization metric of $\tilde{\Sigma}$. A deck transformation of $(\tilde{\Sigma}, g)$ is a Möbius transformation. All deck transformations form the Fuchsian group of $\Sigma$.

Fuchsian group indicates the intrinsic symmetry of the surface.
The Fuchsian group is isomorphic to the fundamental group

<table>
<thead>
<tr>
<th></th>
<th>$e^{i\theta}$</th>
<th>$z_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$-0.631374 + i0.775478$</td>
<td>$+0.730593 + i0.574094$</td>
</tr>
<tr>
<td>$b_1$</td>
<td>$+0.035487 - i0.999370$</td>
<td>$+0.185274 - i0.945890$</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$-0.473156 + i0.880978$</td>
<td>$-0.798610 - i0.411091$</td>
</tr>
<tr>
<td>$b_2$</td>
<td>$-0.044416 - i0.999013$</td>
<td>$+0.035502 + i0.964858$</td>
</tr>
</tbody>
</table>
Klein Model

Another Hyperbolic space model is Klein Model, suppose \( s, t \) are two points on the unit disk, the distance is

\[
d(s, t) = \text{arccosh} \frac{1 - s \cdot t}{\sqrt{(1 - s \cdot s)(1 - t \cdot t)}}
\]

Poincaré vs. Klein Model

From Poincaré model to Klein model is straightforward

\[
\beta(z) = \frac{2z}{1 + \bar{z}z}, \quad \beta^{-1}(z) = \frac{1 - \sqrt{1 - \bar{z}z}}{\bar{z}z},
\]

Assume \( \phi \) is a Möbius transformation, then transition maps \( \beta \circ \phi \circ \beta^{-1} \) are real projective.
Real projective structure

The embedding of the universal cover in the Poincaré disk is converted to the embedding in the Klein model, which induces a real projective atlas of the surface.
Surface  Hyperbolic Structure  Projective Structure
Hyperbolic and Real Projective Structure

Surface, courtesy of Cindy Grimm

Hyperbolic Structure

Projective Structure
Hyperbolic and Real Projective Structure

Surface

Hyperbolic Structure

Projective Structure
Hyperbolic and Real Projective Structure

Surface

Hyperbolic Structure

Projective Structure
Hyperbolic Uniformization Metric

Miao Jin, Feng Luo, Xianfeng (David) Gu

Hyperbolic and Real Projective Structures
Performance

- Based on OpenMesh library on Windows platform, [Sovakar and Kobbelt 2005].
- Eight model (7k faces), $10^2$ seconds on 1.7G CPU with 1G RAM laptop.

Curvature error vs. running time. Red curve Newton’s method; Blue curve: gradient decent method.
Challenges

- Intrinsically nonlinear method.
- Intrinsically the conformal factor may be exponential.
- Determine the optimal initial circle packing metric.
- Embed universal cover in the Poincaré disk.
Summary

Contributions

- Introduce general geometric structures
  - Different geometries can be defined on surfaces.
  - Planar algorithms can be systematically generalized to surfaces.
- Ricci flow method to compute special metrics.
  - Uniform flat metric.
  - Uniformization metric.
- Algorithms to compute geometric structures
  - hyperbolic structure
  - real projective structure
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Future Works

- Design spline schemes based on real projective geometry.
- Hierarchical approach for Ricci energy optimization.
- Surface classification using Fuchsian group.
- Generalize planar geometric algorithms to surface domains using geometric structures.
- Ricci flow on 3-manifolds.
For more information, please email to gu@cs.sunysb.edu.

Thank you!
W. Thurston. 
*Geometry and Topology of 3-manifolds.*

K. Stephenson.
*Introduction to Circle Packing, the theory of discrete analytic functions.*

R. S. Hamilton.
*The Ricci flow on surfaces.*

Colin de Verdiere.
*Yves Un principe variationnel pour les empilements de cercles.*
References II

B. Chow, F. Luo.
*Combinatorial Ricci Flows on Surfaces.*

A. Bobenko, B. Springborn.
*Variational principles for circle patterns and Koebe's theorem.*

L. Kharevych, B. Springborn, P. Schröder
*Discrete Conformal Mapping via Circle Patterns.*
SGP2005.

*Least squares conformal maps for automatic texture atlas generation.*
M. Desbrun, M. Meyer, P. Alliez.  
*Intrinsic parameterizations of surface meshes.*  

A. Sheffer, E. de Sturler.  
*Parameterization of faced surfaces for meshing using angle based flattening.*  

A. Sheffer, B. Levy, M. Mogilnitsky, A. Bogomyakov.  
*ABF++: Fast and robust angle based flattening.*  

X. Gu, S.-T. Yau.  
*Computing Conformal Structures of Surfaces.*  
Communications in Information and Systems, Vol2, No 2.
*Quasi-conformally flat mapping the human cerebellum*, MICCAI 1999.

Algorithm: Computing Circle Packing Metric

- Input: A triangular mesh $\Sigma$.
- Output: A circle packing metric $(\Gamma, \Phi)$.

1. On face $f_{ijk}$, compute

$$\gamma_{jk} = \frac{l_j + l_k - l_i}{2}, \quad \gamma_{ki} = \frac{l_k + l_i - l_j}{2}, \quad \gamma_{ij} = \frac{l_i + l_j - l_k}{2},$$

2. For each vertex $v_i$, computes $\gamma_i$,

$$\gamma_i = \frac{1}{m} \sum_{f_{ijk} \in F} \gamma_{jk},$$

$m$ is the number of faces adjacent to vertex $v_i$. 

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Hyperbolic and Real Projective Structures
3. for each edge $e_{ij}$ with edge length $l_{ij}$, compute

$$\nu_{ij} = \frac{\gamma_i^2 + \gamma_j^2 - l_{ij}^2}{2\gamma_i\gamma_j}$$

set $\Phi_{ij}$,

$$\Phi_{ij} = \begin{cases} 
0 & \nu_{ij} > 1 \\
\cos^{-1} \nu_{ij} & 0 \leq \nu_{ij} \leq 1 \\
\frac{\pi}{2} & \nu_{ij} < 0
\end{cases}$$
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