Surface and Volume Based Techniques for Shape Modeling and Analysis

G. Patané[1], X. Li[2], David Gu[3]

[1]CNR-IMATI, Italy
[2]Louisiana State University, USA
[3]Stony Brook University, USA

SIGGRAPH Asia 2013 Course

Discrete Optimal Mass Transportation

Minkowski Problem

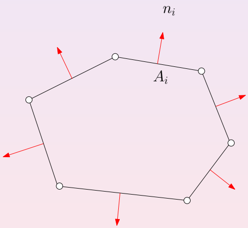
Minkowski problem - 2D Case

Example

A convex polygon P in $\mathbb{R}^2$ is determined by its edge lengths A_i and the unit normal vectors $\mathbf{n}_i$.

Take any $\mathbf{u} \in \mathbb{R}^2$ and project P to $\mathbf{u}$, then $\langle \sum_i A_i \mathbf{n}_i, \mathbf{u} \rangle = \mathbf{0}$, therefore

$$\sum_i A_i \mathbf{n}_i = \mathbf{0}.$$



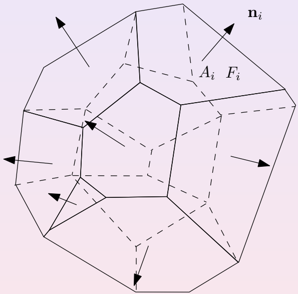
Minkowski Problem

Given k unit vectors $\mathbf{n}_1, \cdots, \mathbf{n}_k$ not contained in a half-space in $\mathbb{R}^n$ and $A_1, \cdots, A_k > 0$, such that

$$\sum_i A_i \mathbf{n}_i = \mathbf{0},$$

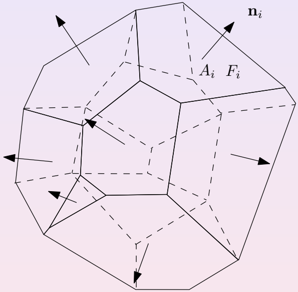
find a compact convex polytope P with exactly k codimension-1 faces $F_1, \cdots, F_k$, such that

1. $area(F_i) = A_i$,
2. $\mathbf{n}_i \perp F_i$.



Theorem (Minkowski)

P exists and is unique up to translations.



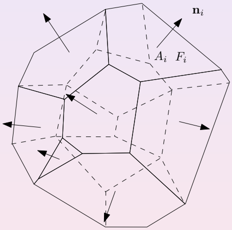
Minkowski's Proof

Given $\mathbf{h} = (h_1, \cdots, h_k)$, $h_i > 0$, define compact convex polytope

$$P(\mathbf{h}) = \{\mathbf{x} | \langle \mathbf{x}, \mathbf{n}_i \rangle \le h_i, \forall i\}$$

Let $Vol : \mathbb{R}_+^k \to \mathbb{R}_+$ be the volume $Vol(\mathbf{h}) = vol(P(\mathbf{h}))$, then

$$\frac{\partial Vol(\mathbf{h})}{\partial h_i} = area(F_i)$$



using Lagrangian multiplier, the solution (up to scaling) to MP is the critical point of Vol on $\{\mathbf{h} | h_i \ge 0, \sum h_i A_i = 1\}$. Uniqueness part is proved using Brunn-Minkowski inequality, which implies $(Vol(\mathbf{h}))^{\frac{1}{n}}$ is concave in $\mathbf{h}$.

Piecewise Linear Convex Function

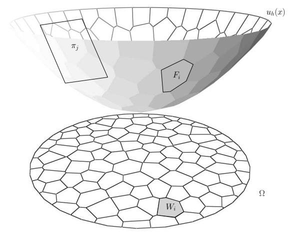
A Piecewise Linear convex function

$$f(\mathbf{x}) := \max\{\langle \mathbf{x}, \mathbf{p}_i \rangle + h_i | i = 1, \cdots, k\}$$

produces a convex cell decomposition W_i of $\mathbb{R}^n$:

$$W_i = \{\mathbf{x} | \langle \mathbf{x}, \mathbf{p}_i \rangle + h_i \geq \langle \mathbf{x}, \mathbf{p}_j \rangle + h_j, \forall j\}$$

Namely, $W_i = \{\mathbf{x} | \nabla f(\mathbf{x}) = \mathbf{p}_i\}$.



Alexandrov Theorem

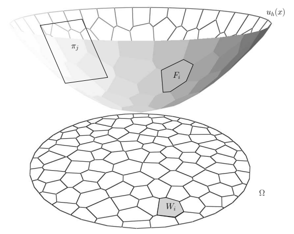
Theorem (Alexandrov 1950)

Given Ω compact convex domain in $\mathbb{R}^n$, $p_1, \cdots, p_k$ distinct in $\mathbb{R}^n$, $A_1, \cdots, A_k > 0$, such that $\sum A_i = Vol(\Omega)$, there exists PL convex function

$$f(\mathbf{x}) := \max\{\langle \mathbf{x}, \mathbf{p}_i \rangle + h_i | i = 1, \cdots, k\}$$

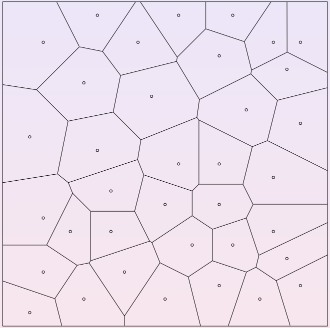
unique up to translation such that

$$Vol(W_i) = Vol(\{\mathbf{x} | \nabla f(\mathbf{x}) = \mathbf{p}_i\}) = A_i.$$



Alexandrov's proof is topological, not variational.

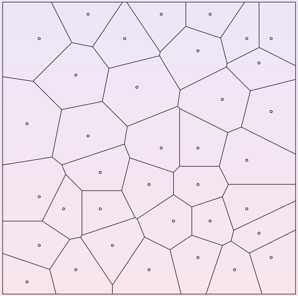
Voronoi Decomposition



Voronoi Diagram

Given $p_1, \cdots, p_k$ in $\mathbb{R}^n$, the Voronoi cell W_i at p_i is

$$W_i = \{\mathbf{x} | |\mathbf{x} - p_i|^2 \leq |\mathbf{x} - p_j|^2, \forall j\}.$$

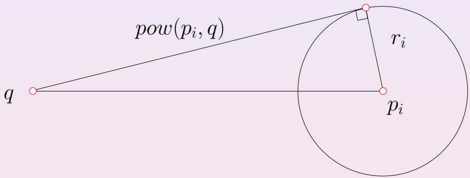


Power Distance

Power Distance

Given $\mathbf{p}_i$ associated with a sphere $(\mathbf{p}_i, r_i)$ the power distance from $\mathbf{q} \in \mathbb{R}^n$ to $\mathbf{p}_i$ is

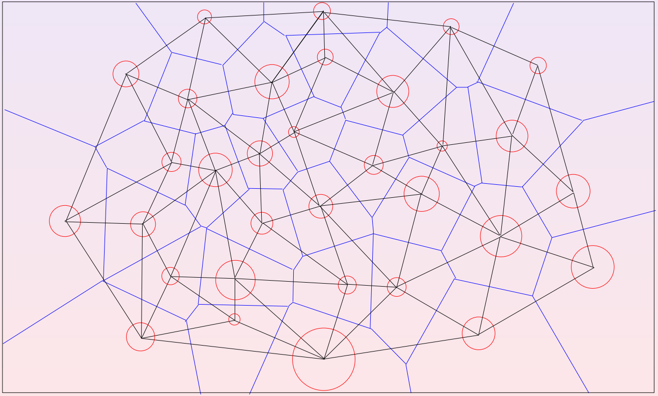
$$pow(\mathbf{p}_i, \mathbf{q}) = |\mathbf{p}_i - \mathbf{q}|^2 - r_i^2.$$



Power Diagram

Given $p_1, \cdots, p_k$ in $\mathbb{R}^n$ and power weights $h_1, \cdots, h_k$, the power Voronoi cell W_i at p_i is

$$W_i = \{\mathbf{x} | |\mathbf{x} - p_i|^2 + h_i \leq |\mathbf{x} - p_j|^2 + h_j, \forall j\}.$$



PL convex function vs. Power diagram

Lemma

Suppose $f(x) = \max\{\langle \mathbf{x}, \mathbf{p}_i \rangle + h_i\}$ *is a piecewise linear convex function, then its gradient map induces a power diagram,*

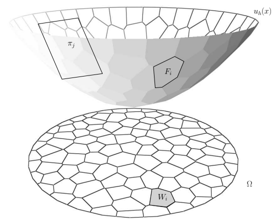
$$W_i = \{\mathbf{x} | \nabla f = \mathbf{p}_i\}.$$

Proof.

$\langle \mathbf{x}, \mathbf{p}_i \rangle + h_i \geq \langle \mathbf{x}, \mathbf{p}_j \rangle + h_j$ is equivalent to

$$|\mathbf{x} - p_i|^2 - 2h_i - |p_i|^2 \leq |x - p_j|^2 - 2h_j - |p_j|^2.$$

□



Theorem (Gu-Luo-Sun-Yau 2012)

Ω is a compact convex domain in $\mathbb{R}^n$, $p_1,\cdots,p_k$ distinct in $\mathbb{R}^n$, $s:\Omega\to\mathbb{R}$ is a positive continuous function. For any $A_1,\cdots,A_k>0$ with $\sum A_i=\int_\Omega s(\mathbf{x})d\mathbf{x}$, there exists a vector $(h_1,\cdots,h_k)$ so that

$$f(\mathbf{x})=\max\{\langle \mathbf{x},\mathbf{p}_i\rangle+h_i\}$$

satisfies $\int_{W_i\cap\Omega}s(\mathbf{x})d\mathbf{x}=A_i$, where $W_i=\{\mathbf{x}|\nabla f(\mathbf{x})=\mathbf{p}_i\}$. Furthermore, $\mathbf{h}$ is the minimum point of the convex function

$$E(\mathbf{h})=\int_{\mathbf{0}}^{\mathbf{h}}\sum_{i=1}^{k}w_i(\eta)d\eta_i-\sum_{i=1}^{k}A_ih_i,$$

where $w_i(\eta)=\int_{W_i(\eta)\cap\Omega}s(\mathbf{x})d\mathbf{x}$ is the volume of the cell.

Variational Proof

X. Gu, F. Luo, J. Sun and S.-T. Yau, “Variational Principles for Minkowski Type Problems, Discrete Optimal Transport, and Discrete Monge-Ampere Equations”, arXiv:1302.5472

Variational Proof

Proof.

For $\mathbf{h} = (h_1, \cdots, h_k)$ in $\mathbb{R}^k$, define the PL convex function f as above and let $W_i(\mathbf{h}) = \{\mathbf{x} | \nabla f(\mathbf{x}) = \mathbf{p}_i\}$ and $w_i(\mathbf{h}) = vol(W_i(\mathbf{h}))$,

1. $H = \{\mathbf{h} \in \mathbb{R}^k | w_i(\mathbf{h}) > 0, \forall i\}$ is non-empty open convex set in $\mathbb{R}^k$.
2. $\frac{\partial w_i}{\partial h_j} = \frac{\partial w_j}{\partial h_i} \leq 0$ for $i \neq j$. Thus the differential 1-form $\sum w_i(\mathbf{h}) dh_i$ is closed in H. Therefore $\exists$ a smooth $F : H \to \mathbb{R}$ so that $\frac{\partial F}{\partial h_i} = w_i(h)$
3. $\sum \frac{\partial w_i(\mathbf{h})}{\partial h_i} = 0$, due to $\sum w_i(\mathbf{h}) = vol(\Omega)$. Therefore the Hessian of F is diagonally dominated, $F(\mathbf{h})$ is convex in H.
4. F is strictly convex in $H_0 = \{\mathbf{h} \in H | \sum h_i = 0\}$ so that $\nabla F = (w_1, \cdots, w_k)$.

If F strictly convex on an open convex set Ω in $\mathbb{R}^k$ then $\nabla F : \Omega \to R^k$ is one-one. This shows the uniqueness part of Alexandrov's theorem. □

Variational Proof

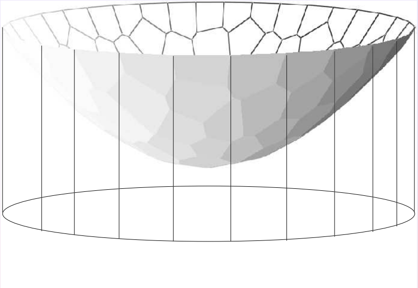
Proof.

It can be shown that the convex function

$$G(\mathbf{h}) = F(\mathbf{h}) - \sum A_i h_i$$

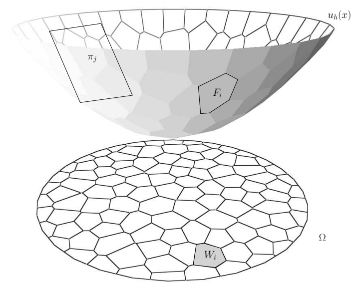
has a minimum point in H_0, which is the solution to Alexandrov's theorem. □

Geometric Interpretation



One can define a cylinder through $\partial\Omega$, the cylinder is truncated by the xy-plane and the convex polyhedron. The energy term $\int^{\mathbf{h}} \sum w_i(\eta) d\eta_i$ equals to the volume of the truncated cylinder.

Computational Algorithm

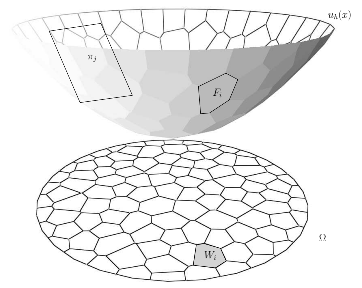


The convex energy is

$$E(h_1, h_2, \cdots, h_k) = \sum_{i=1}^{k} A_i h_i - \int_{\mathbf{0}}^{\mathbf{h}} \sum_{j=1}^{k} W_j dh_j,$$

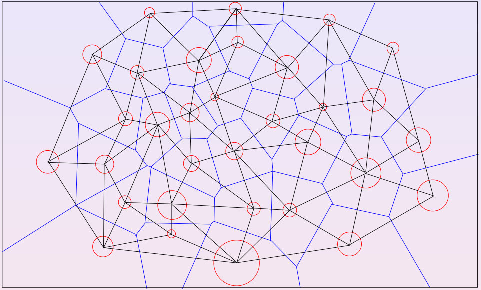
Geometrically, the energy is the volume beneath the parabola.

Computational Algorithm



The gradient of the energy is the areas of the cells

$$\nabla E(h_1, h_2, \cdots, h_k) = (A_1 - w_1, A_2 - w_2, \cdots, A_k - w_k)$$



The Hessian of the energy is the length ratios of edge and dual edges,

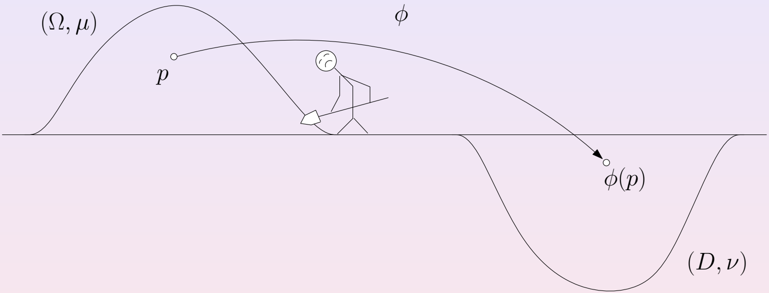
$$\frac{\partial w_i}{\partial h_j} = \frac{|e_{ij}|}{|\bar{e}_{ij}|}$$

Computational Algorithm

1. Initialize $\mathbf{h} = \mathbf{0}$
2. Compute the Power Voronoi diagram, and the dual Power Delaunay Triangulation
3. Compute the cell areas, which gives the gradient ∇E
4. Compute the edge lengths and the dual edge lengths, which gives the Hessian matrix of E, $Hess(E)$
5. Solve linear system
$$\nabla E = Hess(E) d\mathbf{h}$$
6. Update the height vector
$$(h) \leftarrow \mathbf{h} - \lambda d\mathbf{h},$$
where λ is a constant to ensure that no cell disappears
7. Repeat step 2 through 6, until $\|d\mathbf{h}\| < \varepsilon$.

Optimal Mass Transport Mapping

Optimal Transport Problem



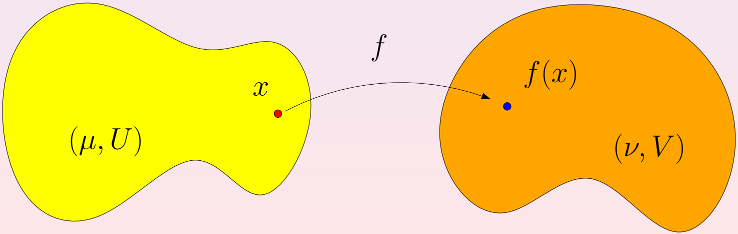
Earth movement cost.

Problem Setting

Find the best scheme of transporting one mass distribution (μ, U) to another one (ν, V) such that the total cost is minimized, where U, V are two bounded domains in $\mathbb{R}^n$, such that

$$\int_U \mu(x)dx = \int_V \nu(y)dy,$$

$0 \leq \mu \in L^1(U)$ and $0 \leq \nu \in L^1(V)$ are density functions.



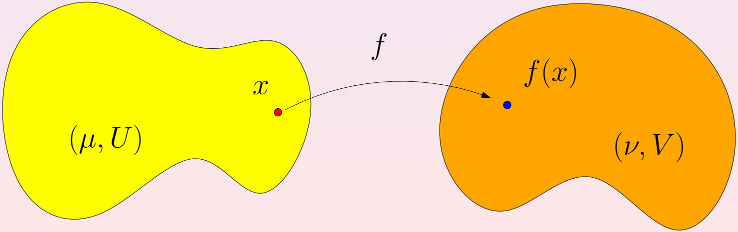
For a transport scheme s (a mapping from U to V)

$$s : \mathbf{x} \in U \to \mathbf{y} \in V,$$

the total cost is

$$C(s) = \int_U \mu(\mathbf{x}) c(\mathbf{x}, s(\mathbf{x})) d\mathbf{x}$$

where $c(\mathbf{x}, \mathbf{y})$ is the cost function.



Cost Function $c(x,y)$

The cost of moving a unit mass from point x to point y.

$$Monge(1781) : c(x,y) = |x-y|.$$

This is the natural cost function. Other cost functions include

$$\begin{aligned} c(x,y) &= |x-y|^p, p \neq 0 \\ c(x,y) &= -\log|x-y| \\ c(x,y) &= \sqrt{\varepsilon + |x-y|^2}, \varepsilon > 0 \end{aligned}$$

Any function can be cost function. It can be negative.

Optimal Transportation Map

Problem

Is there an optima mapping $T : U \to V$ such that the total cost $\mathscr{C}$ is minimized,

$$\mathscr{C}(T) = \inf\{\mathscr{C}(s) : s \in \mathscr{S}\}$$

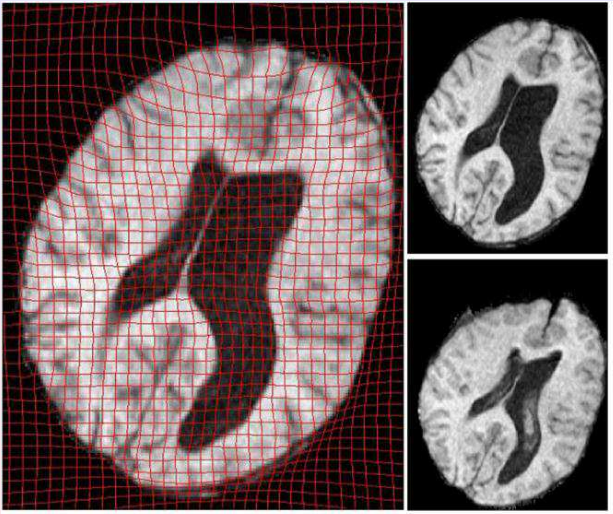
where $\mathscr{S}$ is the set of all measure preserving mappings, namely $s : U \to V$ satisfies

$$\int_{s^{-1}(E)} \mu(x)dx = \int_E \nu(y)dy, \forall \, Borel \ set \ E \subset V$$

Applications

- Economy: producer-consumer problem, gas station with capacity constraint,
- Probability: Wasserstein distance
- Image processing: image registration
- Digital geometry processing: surface registration

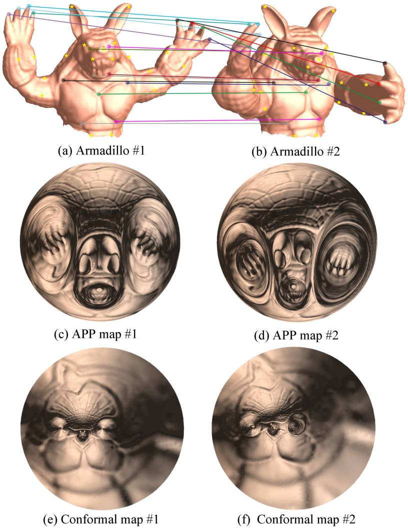
Image Registration



A. Tannenbaum: Medical image registration

Determine the locations of gas stations $\{p_1, p_2, \cdots, p_k\}$ with capacities $\{c_1, c_2, \cdots, c_k\}$ in a city with gasoline consumption density μ, such that the total square of distances from each family to the corresponding gas station is minimized.

Surface Registration



(a) Armadillo #1 (b) Armadillo #2

(c) APP map #1 (d) APP map #2

(e) Conformal map #1 (f) Conformal map #2

Z. Su, W. Zeng, R. Shi, Y. Wang, J. Sun, J. Gao, X. Gu, "Area Preserving Brain Mapping", CVPR, June, 2013.

Solutions

Three categories:

1. Discrete category: both (μ, U) and (ν, V) are discrete,
2. Semi-continuous category: (μ, U) is continuous, (ν, V) is discrete,
3. Continuous category: both (μ, U) and (ν, V) are continuous.

Kantorovich's Approach

Both (μ, U) and (ν, V) are discrete. μ and ν are Dirac measures. (μ, U) is represented as

$$\{(\mu_1, \mathbf{p}_1), (\mu_2, \mathbf{p}_2), \cdots, (\mu_m, \mathbf{p}_m)\},$$

(ν, V) is

$$\{(\nu_1, \mathbf{q}_1), (\nu_2, \mathbf{q}_2), \cdots, (\nu_n, \mathbf{q}_n)\}.$$

A transportation plan $f : \{\mathbf{p}_i\} \to \{\mathbf{q}_j\}$, $f = \{f_{ij}\}$, f_{ij} means how much mass is moved from $(\mu_i, \mathbf{p}_i)$ to $(\nu_j, \mathbf{q}_j)$, $i \leq m, j \leq n$. The optimal mass transportation plan is:

$$\min_f f_{ij} c(\mathbf{p}_i, \mathbf{q}_j)$$

with constraints:

$$\sum_{j=1}^{n} f_{ij} = \mu_i, \sum_{i=1}^{m} f_{ij} = \nu_j.$$

Optimizing a linear energy on a convex set, solvable by linear programming method.

Kantorovich's Approach

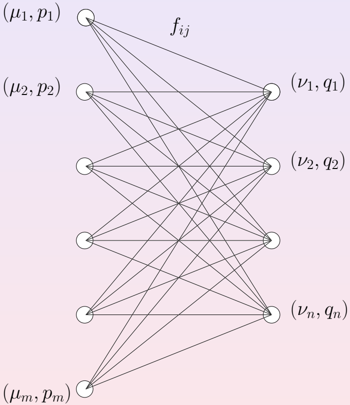
Kantorovich won Nobel's prize in economics.

$$\min_f \sum_{ij} f_{ij} c(\mathbf{p}_i, \mathbf{p}_j),$$

such that

$$\sum_j f_{ij} = \mu_i, \sum_i f_{ij} = \nu_j.$$

mn unknowns in total. The complexity is quite high.



Brenier's Approach

Theorem (Brenier)

If $\mu,\nu > 0$ and U is convex, and the cost function is quadratic distance,

$$c(\mathbf{x},\mathbf{y}) = |\mathbf{x}-\mathbf{y}|^2$$

then there exists a convex function $f : U \to \mathbb{R}$ unique upto a constant, such that the unique optimal transportation map is given by the gradient map

$$T : \mathbf{x} \to \nabla f(\mathbf{x}).$$

Brenier's Approach

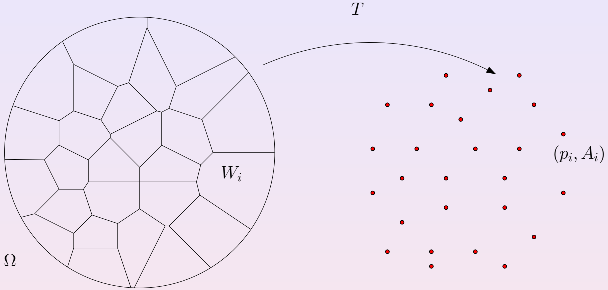
Continuous Category: In smooth case, the Brenier potential $f : U \to \mathbb{R}$ statisfies the Monge-Ampere equation

$$det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = \frac{\mu(\mathbf{x})}{\nu(\nabla f(\mathbf{x}))},$$

and $\nabla f : U \to V$ minimizes the quadratic cost

$$\min_f \int_U |\mathbf{x} - \nabla f(\mathbf{x})|^2 d\mathbf{x}.$$

Semi-Continuous Category: Discrete Optimal Transportation Problem



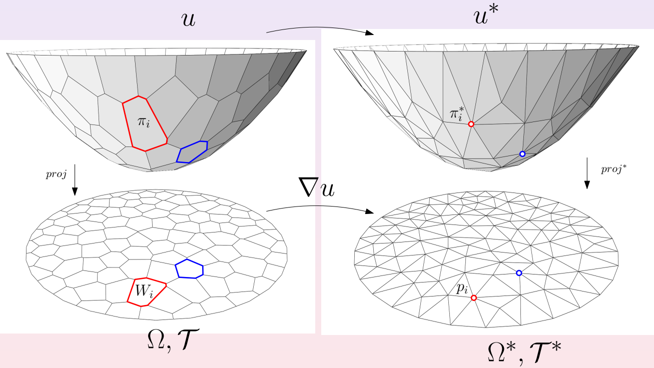
Given a compact convex domain U in $\mathbb{R}^n$ and $p_1, \cdots, p_k$ in $\mathbb{R}^n$ and $A_1, \cdots, A_k > 0$, find a transport map $T : \Omega \to \{p_1, \cdots, p_k\}$ with $vol(T^{-1}(p_i)) = A_i$, so that T minimizes the transport cost

$$\int_U |\mathbf{x} - T(\mathbf{x})|^2 d\mathbf{x}.$$

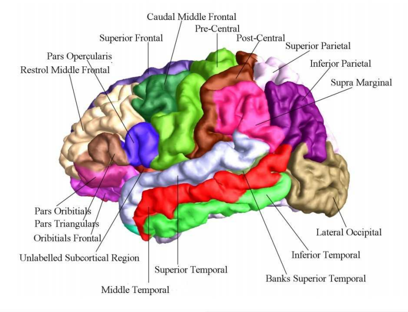
Alexandrov Map vs Optimal Transport Map

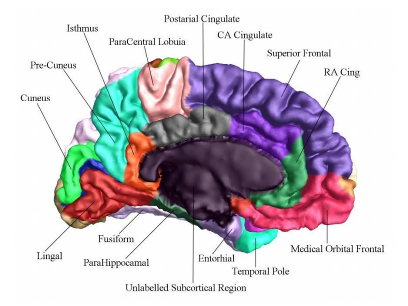
Theorem (Aurenhammer-Hoffmann-Aronov 1998)

Alexandrov map ∇f is the optimal transport map.

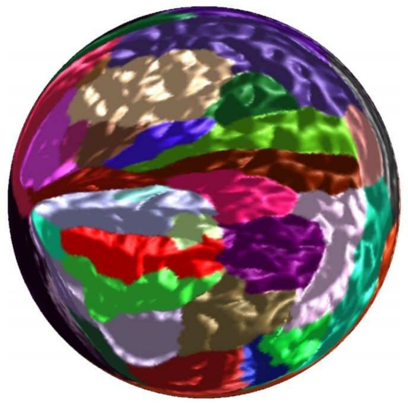


Optimal Transport Map Examples

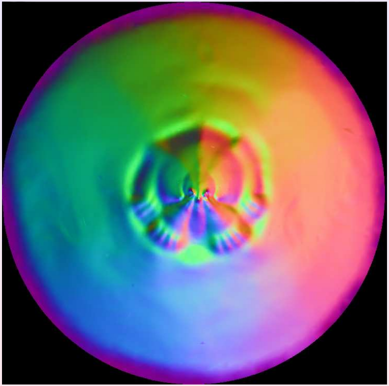




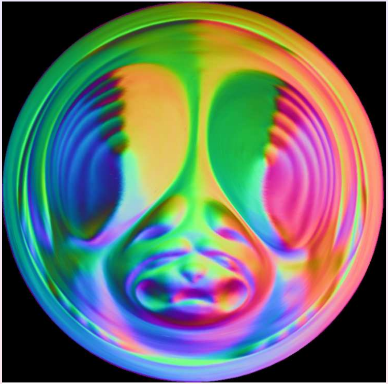
Optimal Transport Map Examples



Normal Map

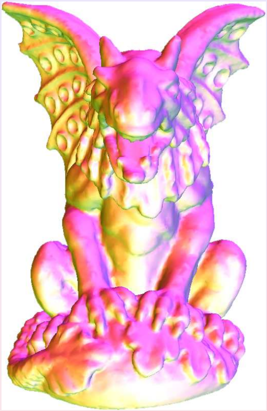



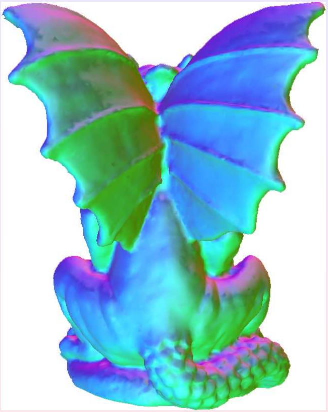
Conformal mapping



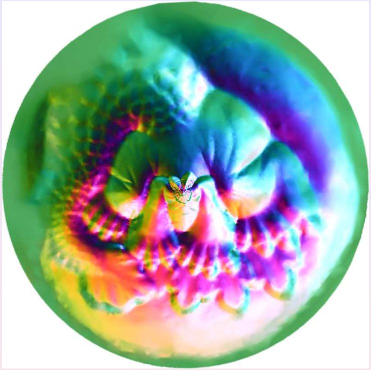
Area-preserving mapping

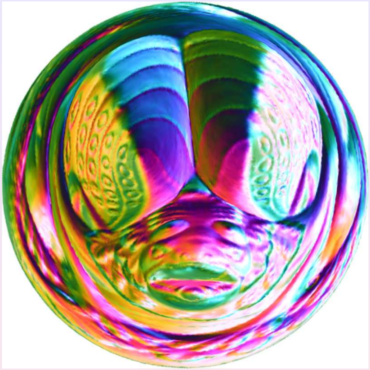
Visualization





Visualization





X. Zhao, Z. Su, X. Gu, A. Kaufman, J. Sun, J. Gao, F. Luo, “Area-preservation Mapping using Optimal Mass Transport”, IEEE TVCG, 2013.

Visualization

(a) Front view *(b) Angle-preserving* *(c) Area-preserving* *(d) Back view*

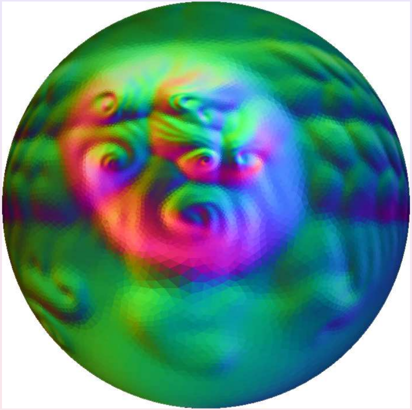
Angle-perserving parameterization vs. area-preserving parameterization

(a) 2x *(b) 3x* *(c) 4x* *(d) 6x*

Importance driven parameterization. The Buddha's head region is magnified by different factors

Visualization

Visualization



Visualization