Computing Surface Hyperbolic Structure and Real Projective Structures

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M.C. Escher’s art works: Angels and Devils

Regular division of the plane

Sphere with Angels and Devils

Circle limit IV Heaven and Hell
Motivation

Main Goals

- Define different geometries on surfaces.
- Systematically generalize planar algorithms to surfaces.

Example: Generalize planar spline schemes to surfaces.
Main Goals

- Define different geometries on surfaces.
- Systematically generalize planar algorithms to surfaces.

Example: Generalize planar spline schemes to surfaces.
Planar Splines

Parametric Affine Invariant

The spline is invariants under the affine transformations of the knots and the parameters.
Idea: Geometry Structure

A mesh is covered by local coordinate charts. Geometric construction is invariant during the transition from one local coordinate to another.

Global Parameterization

Find atlas with special transition functions.
Manifold SPLines

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Global Parameterization

Find atlas with special transition functions.

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Hyperbolic and Real Projective Structures
Idea: Flat Metrics

A planar parameterization of a mesh is equivalent to find the edge lengths such that the sum of surrounding angles for each vertex is $2\pi$. 
Parameterizations as Finding Metrics

Idea: Flat Metrics

A planar parameterization of a mesh is equivalent to find the edge lengths such that the sum of surrounding angles for each vertex is $2\pi$. 
Idea: Ricci Flow
Conformally adjust the edge lengths; the deformation of the edge lengths is driven by the current curvature.
Ricci Flow

Idea: Ricci Flow
Conformally adjust the edge lengths; the deformation of the edge lengths is driven by the current curvature.
Main Ideas

Geometry Structure
A surface is covered by local coordinate charts. Geometric construction is invariant during the transition from one local coordinate to another.

Flat Metrics
Global parameterizations is formulated as finding flat metrics.

Ricci Flow
The conformal deformation of the edge lengths is driven by the curvature.
Erlangen Program - F. Klein 1872

Different geometries study the invariants under different transformation groups.

- Euclidean Geometry: Rigid motion on $\mathbb{R}^2$. Distances between arbitrary two points are the invariants.
- Affine Geometry: Affine transformations. Parallelism and barry centric coordinates are the invariants.
- Real Projective Geometry: Real projective transformations. Collinearity and cross ratios are the invariants.
Suppose $P$ is a planar point set, $T(P)$ is its Delaunay triangulation,

$$P = \{p_0, p_1, p_2, \cdots, p_n\}.$$ 

Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be a planar rigid motion, then,

$$g(T(P)) = T(g(P)).$$
Delaunay Triangulation: Euclidean geometry.

Key concept used in Delaunay triangulation: distance.

Suppose $\mathcal{P}$ is a planar point set, $\mathbf{T}(\mathcal{P})$ is its Delaunay triangulation,

$$\mathcal{P} = \{p_0, p_1, p_2, \ldots, p_n\}.$$

Let $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a planar rigid motion, then,

$$g(\mathbf{T}(\mathcal{P})) = \mathbf{T}(g(\mathcal{P})).$$
Convex Hull: Real projective geometry.

Key concept used in convex hull: collinearity.

Suppose $P$ is a planar point set, $T(P)$ is its Delaunay triangulation,

$$P = \{p_0, p_1, p_2, \cdots, p_n\}.$$

Let $g : \mathbb{RP}^2 \rightarrow \mathbb{RP}^2$ be a real projective transformation, then,

$$g(C(P)) = C(g(P)).$$
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Let $g : \mathbb{RP}^2 \to \mathbb{RP}^2$ be a real projective transformation, then,

$$g(C(P)) = C(g(P)).$$
(X,G) Invariant Algorithms

Definition ((X,G) invariant Algorithm)
Suppose \( X \) is a topological space, \( G \) is the transformation group on \( X \). A geometric operator \( \Omega \) defined on \( X \) is \((X,G)\) invariant, if and only if

\[
\Omega \circ g = g \circ \Omega, \quad \forall g \in G.
\]

Examples:
- Minkowski sum: Translation invariant.
- Voronoi Diagram: Rigid motion invariant.
- Polar form: Affine invariant.
Central Problem

- Can different geometries be defined on general surfaces?
- Can different planar algorithms be generalized to surface domains directly?

The answers are yes and yes. The major theoretic tool is the Geometric Structure.
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- Can different planar algorithms be generalized to surface domains directly?

The answers are yes and yes. The major theoretic tool is the *Geometric Structure*.
Manifold

\[ U_\alpha \quad U_\beta \]

\[ S \]

\[ \phi_\alpha \quad \phi_\beta \]

\[ \phi_\alpha(U_\alpha) \quad \phi_\beta(U_\beta) \]

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Hyperbolic and Real Projective Structures
Definition (Manifold)

A **manifold** is a topological space $\Sigma$ covered by a set of open sets $\{ U_\alpha \}$. A homeomorphism $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ maps $U_\alpha$ to the Euclidean space $\mathbb{R}^n$. $(U_\alpha, \phi_\alpha)$ is called a chart of $\Sigma$, the set of all charts $\{(U_\alpha, \phi_\alpha)\}$ form the atlas of $\Sigma$. Suppose $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a transition map.

Transition maps satisfy cocycle condition, suppose $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, then

$$\phi_{\beta\gamma} \circ \phi_{\alpha\beta} = \phi_{\alpha\gamma}.$$
\((X, G)\) structure

**Definition ((X,G) Atlas)**

Suppose \(X\) is a topological space, \(G\) is the transformation group of \(X\). A manifold \(\Sigma\) with an atlas \(\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}\) is an \((X, G)\) atlas if

1. \(\phi_\alpha(U_\alpha) \subset X\), for all charts \((U_\alpha, \phi_\alpha)\).
2. Transition maps \(\phi_{\alpha\beta} \in G\).
Definition (Equivalent $(X, G)$ atlases)

Two $(X, G)$ atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\Sigma$ are equivalent, if their union is still an $(X, G)$ atlas of $\Sigma$.

Definition ($(X, G)$ structure)

An $(X, G)$ structure of a manifold $\Sigma$ is an equivalent class of its $(X, G)$ atlases.
**Definition (Equivalent \((X, G)\) atlases)**

Two \((X, G)\) atlases \(\mathcal{A}_1\) and \(\mathcal{A}_2\) of \(\Sigma\) are *equivalent*, if their union is still an \((X, G)\) atlas of \(\Sigma\).

**Definition \(((X, G)\) structure)**

An \((X, G)\) *structure* of a manifold \(\Sigma\) is an equivalent class of its \((X, G)\) atlases.
Common \((X,G)\) structure

- **X**: Unit sphere \(S^2\).
- **G**: Rotation group.
- **Surfaces**: Genus zero closed surfaces; any open surfaces.

Spherical Structure
Common \((X,G)\) structure

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**Spherical Structure**
Common \((X, G)\) structure

**Affine Structure**

- \(X\): Real plane \(\mathbb{R}^2\).
- \(G\): Affine transformation group.
- Surfaces: Genus one closed surface and open surfaces.
Common \((X,G)\) structure

**Affine Structure**

- \(X\): Real plane \(\mathbb{R}^2\).
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Common \((X,G)\) structure

- **\(X\):** Hyperbolic plane \(\mathbb{H}^2\).
- **\(G\):** Möbius transformation group.
- **Surfaces:** with negative Euler number.

### Hyperbolic Structure

- \(X\): Hyperbolic plane \(\mathbb{H}^2\).
- \(G\): Möbius transformation group.
- Surfaces: with negative Euler number.
Common \((X,G)\) structure

\begin{itemize}
    \item \(X\): Hyperbolic plane \(\mathbb{H}^2\).
    \item \(G\): Möbius transformation group.
    \item Surfaces: with negative Euler number.
\end{itemize}
Common \((X, G)\) structure

Real Projective Structure

- \(X\): Real projective plane \(\mathbb{RP}^2\).
- \(G\): Real projective transformation group.
- Surfaces: any surface.
Common \((X,G)\) structure

- **\(X\)**: Real projective plane \(\mathbb{RP}^2\).
- **\(G\)**: Real projective transformation group.
- **Surfaces**: any surface.
Pseudo \( (X,G) \) structure

Conformal Structure

- \( X \): Complex plane \( \mathbb{C} \).
- \( G \): Biholomorphic maps.
- Surfaces: any surface.
Pseudo \((X,G)\) structure

\(X\): Complex plane \(\mathbb{C}\).

\(G\): Biholomorphic maps.

Surfaces: any surface.

Conformal Structure
Conformal Structure

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Hyperbolic and Real Projective Structures
Conformal Structure

Global Tensor Product Structure
Relations Among Geometric Structures

- Conformal structure (A holomorphic 1-form) induces affine structure.
- Hyperbolic structure induces conformal structure.
- Hyperbolic structure induces real projective structure.
Theorem

Suppose a manifold with an \((X, G)\) structure, then any \((X, G)\) invariant algorithms can be generalized on the manifold.

Corollary (Manifold Splines - Gu, He, Qin 2005)

Spline schemes based on polar forms can be defined on a manifold, if and only if the manifold has an affine structure.
Manifold Splines SPM2005

Hyperbolic and Real Projective Structures
Topological Obstructions

Theorem (Benzécri 1959)

If a closed surface admits an affine structure, it has zero Euler class.

Real projective structure

- Real projective structure is general, it exists for all surfaces.
- Real projective structure is simple, all transitions are linear rational functions.
- Real projective structure is suitable for designing manifold spline schemes.
Theorem (Benzécri 1959)

*If a closed surface admits an affine structure, it has zero Euler class.*

Real projective structure

- Real projective structure is *general*, it exists for all surfaces.
- Real projective structure is *simple*, all transitions are linear rational functions.
- Real projective structure is *suitable* for designing manifold spline schemes.
Previous Works

Related Works

1. Geometric Structures
2. Circle packing
3. Ricci flow
4. Conformal surface parameterizations

There are too many related works, we only briefly review the most related ones. Detailed information can be found in the survey papers in each field.
Conformal Metric

**Definition**

Suppose $\Sigma$ is a surface with a Riemannian metric,

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

Suppose $\lambda : \Sigma \to \mathbb{R}$ is a function defined on the surface, then $e^{2\lambda}g$ is also a Riemannian metric on $\Sigma$ and called a conformal metric. $e^{2\lambda}$ is called the conformal factor.

Angles are invariant measured by conformal metrics.
Given a surface $\Sigma$ with a Riemannian metric $g$, find a function $\lambda : \Sigma \to \mathbb{R}$, such that $e^{2\lambda}g$ is one of the followings:

- **Uniform flat metric**
  \[ \bar{K} \equiv 0, \]
  for interior points
  \[ \bar{k}_g \equiv \text{const} \]
  for boundary points. The constant values are determined by the conformal structure of $\Sigma$.

- **Uniformization metric**
  \[ \bar{K} \equiv \text{const}, \]
  for interior points
  \[ \bar{k}_g \equiv 0. \]

The tool to calculate the above metrics is **Ricci flow**.
Definition (Surface Ricci Flow)

A closed surface with a Riemannian metric $g$, the Ricci flow on it is defined as

$$\frac{dg_{ij}}{dt} = (\bar{K} - K)g_{ij}, \quad \bar{K} = \frac{2\pi \chi(\Sigma)}{S(\Sigma)}$$

$\chi(\Sigma)$ is the Euler number, $S(\Sigma)$ is the total area of $\Sigma$. 
Theorem (Hamilton 1982)

For a closed surface of non-positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) everywhere.

Theorem (Chow)

For a closed surface of positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) everywhere.
Surfaces are represented as polyhedron triangular meshes.

- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2$, $S^2$. 

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Hyperbolic and Real Projective Structures
Surfaces are represented as polyhedron triangular meshes.

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Surfaces are represented as polyhedron triangular meshes.

- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$. 
Curvature

- Discrete curvature: $K : V = \{\text{vertices}\} \rightarrow \mathbb{R}^1$.

\[
K(v) = 2\pi - \sum_i \alpha_i, \ v \notin \partial M; K(v) = \pi - \sum_i \alpha_i, \ v \in \partial M
\]

- Discrete Gauss-Bonnet theorem

\[
\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi \chi(M).
\]
Discrete Metrics

Metric

- **Discrete Metric**: $l : E = \{\text{all edges}\} \to \mathbb{R}^1$, satisfies triangular inequality.
- Metrics determine curvatures by cosine law.

\[
\cos \theta_i = \frac{l_j^2 + l_k^2 - l_i^2}{2l_jl_k}, \quad l \neq j \neq k \neq i
\]
Relation between Metrics and Curvatures

Metrics vs. Curvatures

- All metrics for a mesh $L(\Sigma)$ form a convex polytope.
- All admissible curvature configurations for a mesh $K(\Sigma)$ also form a convex polytope.
- The mapping from the metrics to the curvatures

$$\Phi : L(\Sigma) \rightarrow K(\Sigma),$$

is not one to one.
- The mapping from a conformal class of metrics to the curvatures is a homeomorphism.
Conformal maps Properties

- transform infinitesimal circles to infinitesimal circles.
- preserve the intersection angles among circles.

Idea - Approximate conformal metric deformation

Replace infinitesimal circles by circles with finite radii.
Circle Packing Metric

**CP Metric**

We associate each vertex $v_i$ with a circle with radius $\gamma_i$. On edge $e_{ij}$, the two circles intersect at the angle of $\Phi_{ij}$. The edge lengths are

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j\cos\Phi_{ij}$$

CP Metric ($\Sigma, \Gamma, \Phi$), $\Sigma$ triangulation,

$$\Gamma = \{\gamma_i|\forall v_i\}, \Phi = \{\phi_{ij}|\forall e_{ij}\}$$
Definition (Conformal Circle Packing Metrics)

Two circle packing metrics \( \{\Sigma, \Phi_1, \Gamma_1\} \) and \( \{\Sigma, \Phi_2, \Gamma_2\} \) are conformal equivalent, if

- The radii of circles are different, \( \Gamma_1 \neq \Gamma_2 \).
- The intersection angles are same, \( \Phi_1 \equiv \Phi_2 \).

In practice, the circle radii and intersection angles are optimized to approximate the induced Euclidean metric of the mesh as close as possible.
Definition (Discrete Ricci flow)

A mesh $\Sigma$ with a circle packing metric $\{\Sigma, \Gamma, \Phi\}$, where $\Gamma = \{\gamma_i, v_i \in V\}$ are the vertex radii, $\Phi = \{\Phi_{ij}, e_{ij} \in E\}$ are the angles associated with each edge, the discrete Ricci flow on $\Sigma$ is defined as

$$\frac{d\gamma_i}{dt} = (\bar{K}_i - K_i)\gamma_i,$$

where $\bar{K}_i$ are the target curvatures on vertices. If $\bar{K}_i \equiv 0$, the flow with normalized total area leads to a metric with constant Gaussian curvature.

Idea

Metric deformation is driven by curvature.
Theorem (Chow and Luo 2002)

A discrete Euclidean Ricci flow \( \{\Sigma, \Gamma, \Phi\} \to \{M, \bar{\Gamma}, \Phi\} \) converges.

\[
|K_i(t) - \bar{K}_i| < c_1 e^{-c_2 t},
\]

and

\[
|\gamma_i(t) - \bar{\gamma}_i| < c_1 e^{-c_2 t},
\]

where \( c_1, c_2 \) are positive numbers.
Definition

Let $u_i = \ln \gamma_i$, the **Ricci energy** is defined as

$$f(u) = \int_{u_0}^{u} \sum_{i=1}^{n} (K_i - \bar{K}_i) du_i,$$

where $u = (u_1, u_2, \cdots, u_n)$, $u_0 = (0, 0, \cdots, 0)$. 
Theorem (Ricci Energy)

Euclidean Ricci energy is well defined and convex, namely, there exists a unique global minimum.

Proof.

In an Euclidean triangle, with angles \((\theta_1, \theta_2, \theta_3)\) and radius \((\gamma_1, \gamma_2, \gamma_3)\), let \(u_i = \ln \gamma_i\), according to Euclidean cosine law,

\[
\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i}.
\]

Therefore \(\omega = \sum \theta_i du_i\) is a closed 1-form. The Euclidean Ricci energy is well defined. Direct computation verifies that Hessian matrix is positive definite.
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Newton’s method for Euclidean Ricci energy

Gradient descent Method

Ricci flow is the gradient descent method for minimizing Ricci energy,
\[ \nabla f = (K_1 - \bar{K}_1, K_2 - \bar{K}_2, \ldots, K_n - \bar{K}_n). \]

Newton’s method

The Hessian matrix of Ricci energy is
\[ \frac{\partial^2 f}{\partial u_i \partial u_j} = \frac{\partial K_i}{\partial u_j}. \]

Newton’s method can be applied directly.
Algorithm: uniform flat metric for closed surfaces

Ricci Flow for Uniform Flat Metric

Suppose \( \Sigma \) is a closed genus one mesh,

1. Compute the circle packing metric \((\Gamma, \Phi)\).
2. Set the target curvature to be zero for each vertex
   \[ \bar{K}_i \equiv 0, \forall v_i \in V \]
3. Minimize the Euclidean Ricci energy using Newton’s method to get the target radii \(\bar{\Gamma}\).
4. Compute the target flat metric.
Algorithm: uniform flat metric for open surfaces

Given a surface $\Sigma$ with genus $g$ and $b$ boundaries, then its Euler number is

$$\chi(\Sigma) = 2 - 2g - b.$$  

Suppose the boundary of $\Sigma$ is a set of closed curves

$$\partial \Sigma = C_1 \cup C_2 \cup C_3 \cdots C_b.$$  

The total curvature for each $C_i$ is denoted as $2m_i \pi$, $m_i \in \mathbb{Z}$, and

$$\sum_{i=1}^{b} m_i = \chi(\Sigma).$$  

The target curvature for interior vertices are zeros.

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Hyperbolic and Real Projective Structures
Algorithm: uniform flat metric for open surfaces

Euclidean Ricci flow for open surfaces

- Use Newton’s method to minimize the Ricci energy to update the metric.
- Adjust the boundary vertex curvature to be proportional to the ratio between the current lengths of the adjacent edges and the current total length of the boundary component.
- Repeat until the process converges.
Algorithm: Flatten a mesh with a uniform flat metric

Embedding

1. Determine the planar shape of each triangle using 3 edge lengths.

2. Glue all triangles on the plane along their common edges by rigid motions. Because the metric is flat, the gluing process is coherent and results in a planar embedding.
Euclidean Uniform Flat Metric

original surface

genus 1, 3 boundaries

universal cover
embedded in \( \mathbb{R}^2 \)

texture mapping
Euclidean Uniform Flat Metric
Different boundaries are mapped to straight lines.
Euclidean Uniform Flat Metric

original surface  fundamental domain  universal cover
Conformal Model: Poincaré Disk

A unit disk $|z| < 1$ with the Riemannian metric

$$ds^2 = \frac{4dzd\bar{z}}{(1 - \bar{z}z)^2}.$$
The **rigid motion** is the Möbius transformation

\[ e^{i\theta} \frac{Z - Z_0}{1 - \bar{Z}_0 Z}. \]
Conformal Model: Poincaré Disk

Poincaré disk

The **hyperbolic line** through two point $z_0, z_1$ is the circular arc through $z_0, z_1$ and perpendicular to the boundary circle $|z| = 1$. 
A hyperbolic circle \((c, \gamma)\) on Poincaré disk is also an Euclidean circle \((C, R)\) on the plane, such that

\[
C = \frac{2 - 2\mu^2}{1 - \mu^2|c|^2},
\]

\[
R^2 = |C|^2 - \frac{|c|^2 - \mu^2}{1 - \mu^2|c|^2}, \quad \mu = \frac{e^r - 1}{e^r + 1}.
\]
Definition (Discrete Hyperbolic Ricci Flow)

Let

\[ u_i = \ln \tanh \frac{\gamma_i}{2}, \]

Discrete hyperbolic Ricci flow for a mesh \( \Sigma \) is

\[ \frac{du_i}{dt} = \bar{K}_i - K_i, \bar{K}_i \equiv 0, \]

the Euler number of \( \Sigma \) is negative, \( \chi(\Sigma) < 0 \).
Theorem (Discrete Hyperbolic Ricci flow, Chow and Luo 2002)

A hyperbolic discrete Ricci flow \((M, \Gamma, \Phi) \rightarrow (M, \bar{\Gamma}, \Phi)\) converges,

\[
|K_i(t) - \bar{K}_i| < c_1 e^{-c_2 t},
\]

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\[
|\gamma_i(t) - \bar{\gamma}_i| < c_1 e^{-c_2 t},
\]

where \(c_1, c_2\) are positive numbers.
The discrete Hyperbolic Ricci energy is defined as

\[ f(u) = \int_{u_0}^{u} \sum_{i=1}^{n} (\bar{K}_i - K_i) du_i. \]

Discrete hyperbolic Ricci flow is the gradient descent method to minimize the discrete hyperbolic Ricci energy.
Theorem (Hyperbolic Discrete Ricci Energy)

Discrete hyperbolic Ricci energy is well defined and convex, namely, there exists a unique global minimum.

Proof.

In a hyperbolic triangle, with angles \((\theta_1, \theta_2, \theta_3)\) and radius \((\gamma_1, \gamma_2, \gamma_3)\), \(u_i = \ln \tanh \frac{\gamma_i}{2}\), according to hyperbolic cosine law,

\[
\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i}.
\]

Therefore \(\omega = \sum \theta_i du_i\) is a closed 1-form. The hyperbolic Ricci energy is convex. Direct computation verifies the Hessian matrix is positive definite.
**Theorem (Hyperbolic Discrete Ricci Energy)**

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Therefore \( \omega = \sum \theta_i du_i \) is a closed 1-form. The hyperbolic Ricci energy is convex. Direct computation verifies the Hessian matrix is positive definite.
Algorithm: Computing Hyperbolic uniformization metric

Hyperbolic Ricci Energy Optimization

1. Set target curvature $K(v_i) \equiv 0$.
2. Optimize the hyperbolic Ricci energy using Newton’s method, with the constraint the total area is preserved.

Flattening Mesh in Hyperbolic Space

1. Determine the shape of each triangle.
2. Glue the hyperbolic triangles coherently by Möbius transformation.

Key: all computations use hyperbolic geometry.
Algorithm: Computing Hyperbolic uniformization metric

### Hyperbolic Ricci Energy Optimization
1. Set target curvature $K(v_i) \equiv 0$.
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### Flattening Mesh in Hyperbolic Space
1. Determine the shape of each triangle.
2. Glue the hyperbolic triangles coherently by Möbius transformation.

Key: all computations use hyperbolic geometry.
Genus 0 surface with 3 boundaries. The double covered surface is of genus 2. The boundaries are mapped to hyperbolic lines.
Hyperbolic Uniformization Metric

Genus 0 surface with 3 boundaries. The double covered surface is of genus 2. The boundaries are mapped to hyperbolic lines.
Genus 0 surface with 3 boundaries. The double covered surface is of genus 2. The boundaries are mapped to hyperbolic lines.
Embedding in the upper half plane hyperbolic space model. Different period embedded in the hyperbolic space. The boundaries are mapped to hyperbolic lines.
Universal Cover

A pair \((\tilde{\Sigma}, \pi)\) is a universal cover of a surface \(\Sigma\), if

- Surface \(\tilde{\Sigma}\) is simply connected.
- Projection \(\pi : \tilde{\Sigma} \rightarrow \Sigma\) is a local homeomorphism.

Deck Transformation

A transformation \(\phi : \tilde{\Sigma} \rightarrow \tilde{\Sigma}\) is a deck transformation, if

\[
\pi = \pi \circ \phi.
\]

A deck transformation maps one period to another.
Fuchsian Group

Definition (Fuchsian Group)

Suppose $\Sigma$ is a surface, $g$ is its uniformization metric, $(\tilde{\Sigma}, \pi)$ is the universal cover of $\Sigma$. $g$ is also the uniformization metric of $\tilde{\Sigma}$. A deck transformation of $(\tilde{\Sigma}, g)$ is a Möbius transformation. All deck transformations form the Fuchsian group of $\Sigma$.

Fuchsian group indicates the intrinsic symmetry of the surface.
The Fuchsian group is isomorphic to the fundamental group.

<table>
<thead>
<tr>
<th></th>
<th>$e^{i\theta}$</th>
<th>$z_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$-0.631374 + i0.775478$</td>
<td>$+0.730593 + i0.574094$</td>
</tr>
<tr>
<td>$b_1$</td>
<td>$+0.035487 - i0.999370$</td>
<td>$+0.185274 - i0.945890$</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$-0.473156 + i0.880978$</td>
<td>$-0.798610 - i0.411091$</td>
</tr>
<tr>
<td>$b_2$</td>
<td>$-0.044416 - i0.999013$</td>
<td>$+0.035502 + i0.964858$</td>
</tr>
</tbody>
</table>
Another Hyperbolic space model is Klein Model, suppose $s, t$ are two points on the unit disk, the distance is

$$d(s, t) = \text{arccosh} \frac{1 - s \cdot t}{\sqrt{(1 - s \cdot s)(1 - t \cdot t)}}$$

From Poincaré model to Klein model is straightforward:

$$\beta(z) = \frac{2z}{1 + \bar{z}z}, \quad \beta^{-1}(z) = \frac{1 - \sqrt{1 - \bar{z}z}}{\bar{z}z}$$

Assume $\phi$ is a Möbius transformation, then transition maps $\beta \circ \phi \circ \beta^{-1}$ are real projective.
The embedding of the universal cover in the Poincaré disk is converted to the embedding in the Klein model, which induces a real projective atlas of the surface.
Hyperbolic and Real Projective Structure

Surface  Hyperbolic Structure  Projective Structure
Surface, courtesy of Cindy Grimm

Hyperbolic Structure

Projective Structure
Hyperbolic and Real Projective Structure

Surface

Hyperbolic Structure

Projective Structure
Performance

Based on OpenMesh library on Windows platform, [Sovakar and Kobbelt 2005].

Eight model (7k faces), $10^2$ seconds on 1.7G CPU with 1G RAM laptop.

Curvature error vs. running time. Red curve Newton’s method; Blue curve : gradient decent method.
Challenges

- Intrinsically nonlinear method.
- Intrinsically the conformal factor may be exponential.
- Determine the optimal initial circle packing metric.
- Embed universal cover in the Poincaré disk.
Contributions

- Introduce general geometric structures
  - Different geometries can be defined on surfaces.
  - Planar algorithms can be systematically generalized to surfaces.
- Ricci flow method to compute special metrics.
  - Uniform flat metric.
  - Uniformization metric.
- Algorithms to compute geometric structures
  - hyperbolic structure
  - real projective structure
Summary

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Future Works

- Design spline schemes based on real projective geometry.
- Hierarchical approach for Ricci energy optimization.
- Surface classification using Fuchsian group.
- Generalize planar geometric algorithms to surface domains using geometric structures.
- Ricci flow on 3-manifolds.
For more information, please email to gu@cs.sunysb.edu.

Thank you!