Escher-Droste Effects

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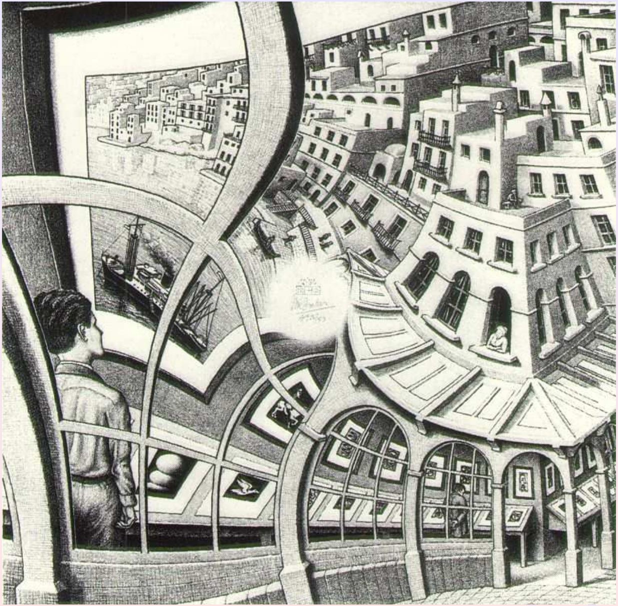
M.C.Escher's Print Gallery



Office Table

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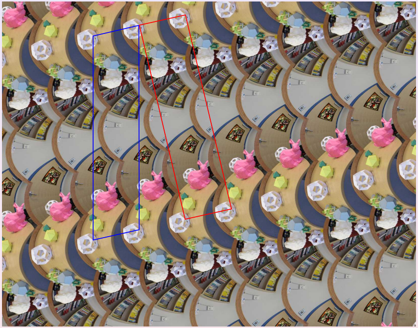
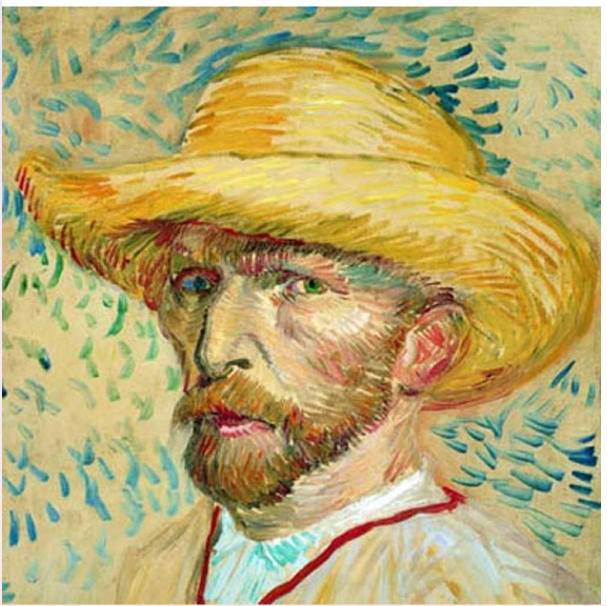


Self-portrait with straw hat, Van Gogh, Vincent 1887



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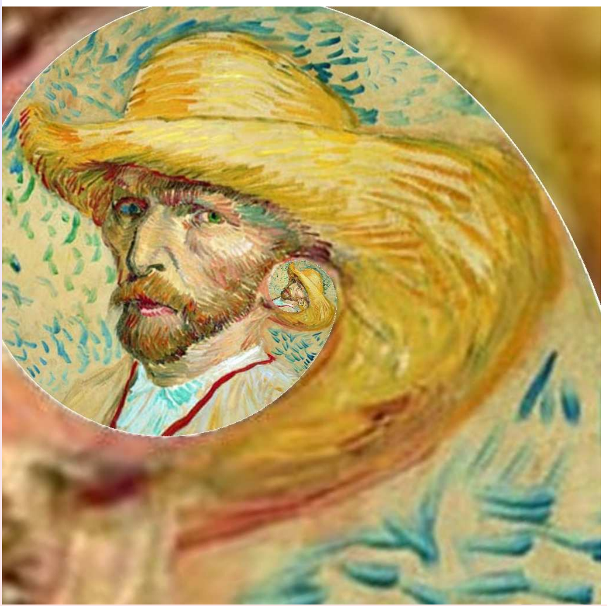

The Scream, Edvard Munch, 1893

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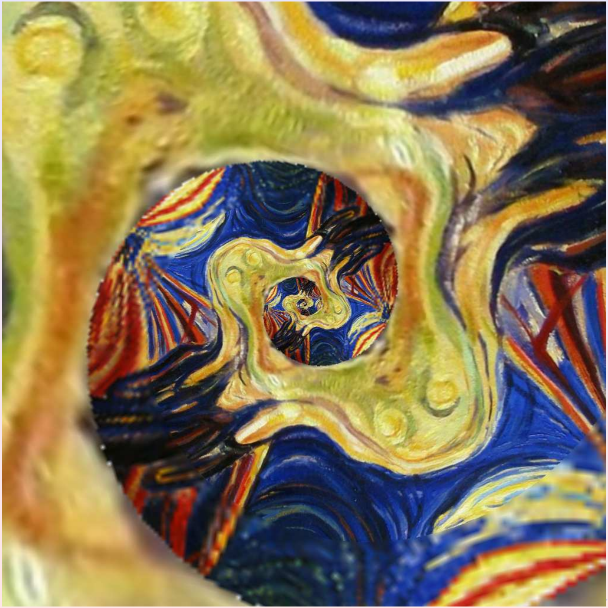


Nine Golden Fish

Nine Golden Fish

Mathematical Structure

- The image is invariant under scaling group, generated by

$$\phi(z) = sz, s \in \mathbb{C},$$

- The fixed point $\phi(z_0) = z_0$
- The quotient space

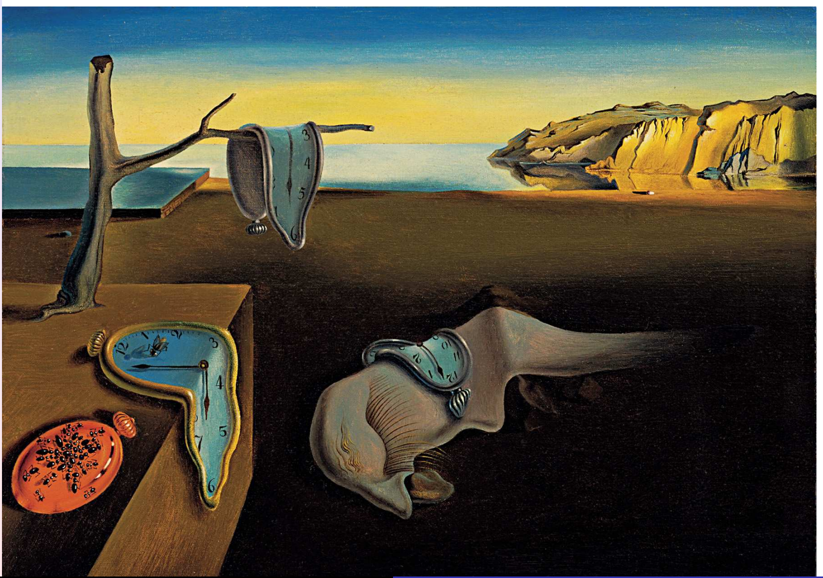
$$\{\mathbb{C} - z_0\}/\langle\phi\rangle$$

  is a torus
- The logrithm map lifts the torus to the universal covering space
- The homotopy class of the mapping from the torus to the torus is given by

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

  where $a_{11}a_{22} - a_{12}a_{21} = 1$.

Persistence of Memory



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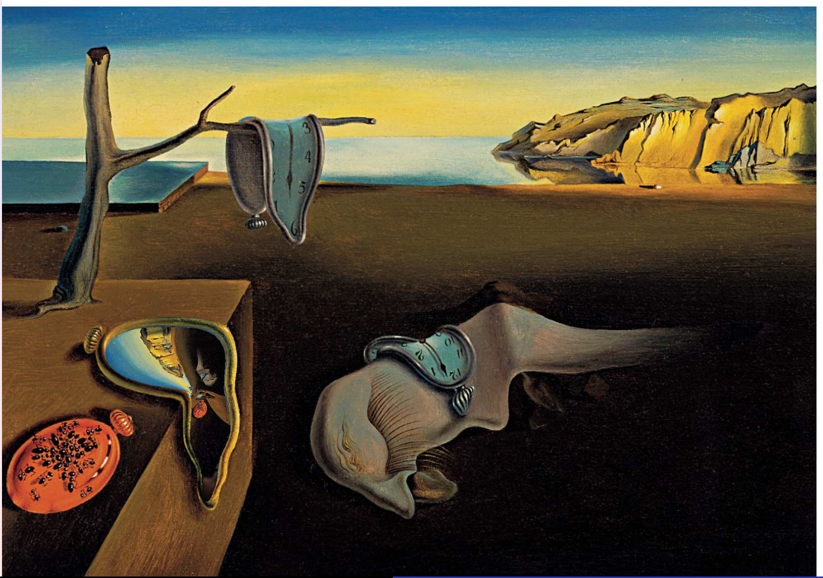
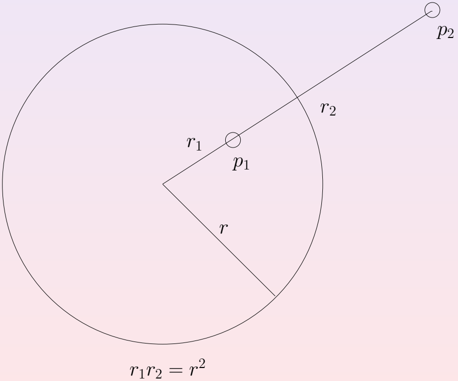


Persistence of Memory

Let $C(\mathbf{c}, r)$ represent a circle, with center $\mathbf{c}$ and radius r. The circular reflection is given by

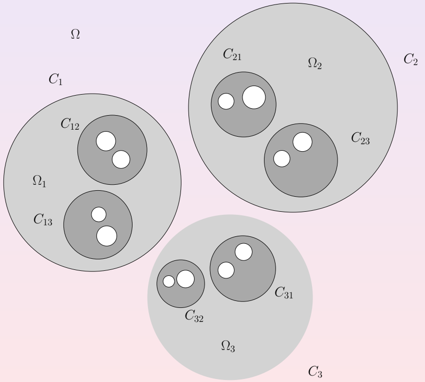
$$\rho_C(z) = \mathbf{c} + \frac{r^2}{\bar{z} - \bar{c}}$$



Circular Reflection

$$\partial\Omega = \{C_1, C_2, C_3\}, \Omega_k = \rho_k(\Omega), \rho_i(C_j) = C_{ij}, i \neq j$$

$$\partial\Omega_1 = \{C_1, C_{12}, C_{13}\}, \partial\Omega_2 = \{C_2, C_{21}, C_{23}\}, \partial\Omega_1 = \{C_3, C_{31}, C_{32}\}$$

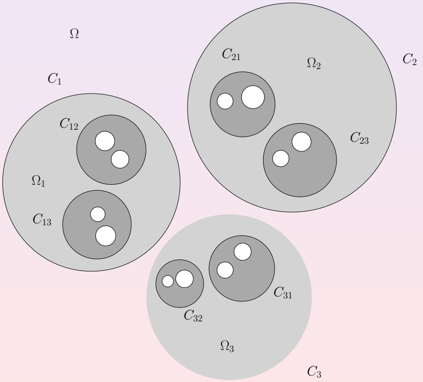


Naming convention

$$\Omega_{ij} = \rho_{ij}(\Omega_i), 1 \leq i,j \leq 3, i \neq j$$

$$C_{121} = \rho_{12}(C_1), C_{123} = \rho_{12}(C_3), \partial\Omega_{12} = \{C_{12}, C_{121}, C_{123}\}$$

$$C_{131} = \rho_{13}(C_1), C_{132} = \rho_{13}(C_2), \partial\Omega_{13} = \{C_{13}, C_{131}, C_{132}\}$$

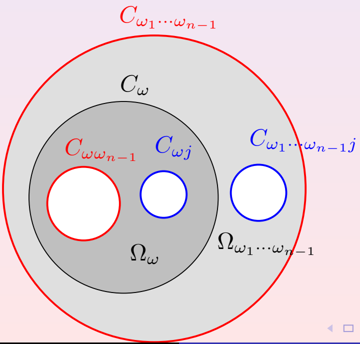


In general, for a circular domain with m interior holes, reflected regions and reflected circles are labeled with multiple indices

$$\omega = \omega_1\omega_2\cdots\omega_q, 1 \leq \omega_j \leq m, \omega_k \neq \omega_{k+1}, 1 \leq k \leq q-1.$$

$$\Omega_\omega = \rho_\omega(\Omega_{\omega_1\omega_2\cdots\omega_{q-1}}), C_{\omega\omega_{q-1}} = \rho_\omega(C_{\omega_1\omega_2\cdots\omega_{q-1}})$$

$$C_{\omega j} = \rho_\omega(C_{\omega_1\omega_2\cdots\omega_{q-1}j})$$



Suppose C_j is the circle $|z-c_j|=r_j^2$, C_0 is the unit circle $|z|=1$, denote $\rho_0(C_j)$ as C_j'. The Möbius map

$$\theta_j(z)=c_j+\frac{r_j^2}{1-\bar{c}_j z}$$

maps the exterior of C_j' to the interior of C_j, and C_j' to C_j.

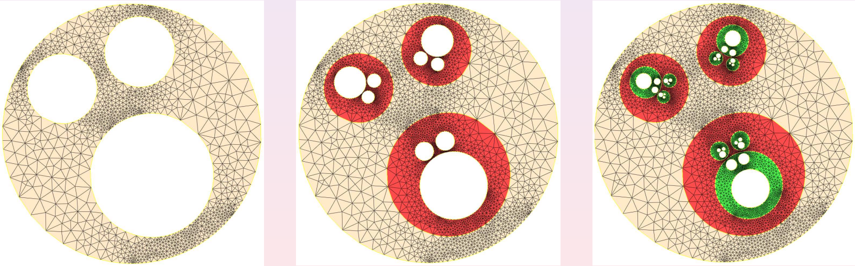
Definition (Schottky Group)

The infinite free group Θ of Mb̈oius mappings generated by composition of $2m$ basic Möbius mappings $\{\theta_j|j=1,2,\cdots,m\}$ and their inverses $\{\theta_j^{-1}|j=1,2,\cdots,m\}$.

Schottky Group

Consider the unbounded region Ω of the plane exterior to the $2m$ circles $\{C_j | j = 1, 2, \cdots, m\}$ and $\{C'_j | j = 1, 2, \cdots, m\}$. The union of copies of generated by the $\theta \in \Theta$ is denoted as

$$\Theta(\Omega) := \cup_{\theta \in \Theta} \theta(\Omega)$$



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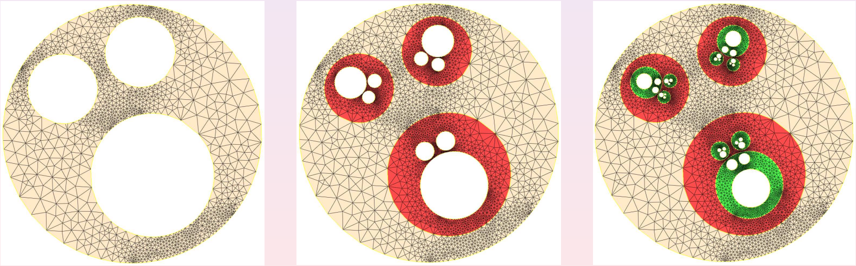


Schottky Group

Theorem

The complement set of $\Theta(\Omega)$

$$\Theta(\Omega)^c := \mathbb{C} - \Theta(\Omega)$$

is a Cantor set of zero measure.

Definition

The set of multi-indices of length n $n > 0$ is denoted as

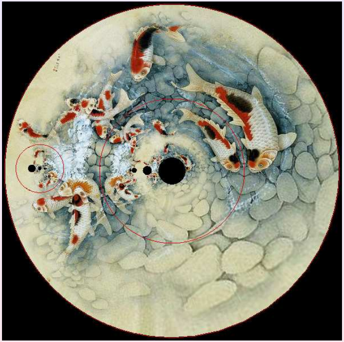
$$\sigma_n := \{\omega_1\omega_2\cdots\omega_n | 1 \leq \omega_j \leq m, \omega_k \neq \omega_{k+1}, 1 \leq k \leq n-1\}$$

Theorem

At level $n+1$*, the total area of holes*

$$\sum_{\omega\in\sigma_{n+1}} S(C_\omega) < \Delta^{4n} \sum_{i=1}^{m} S(C_i),$$

Schottky Group



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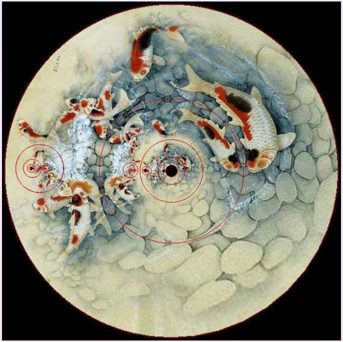

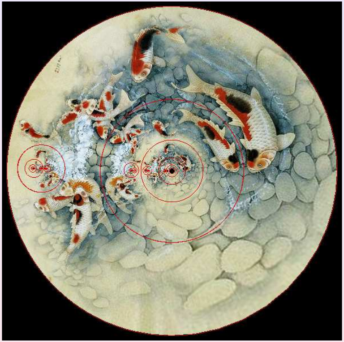

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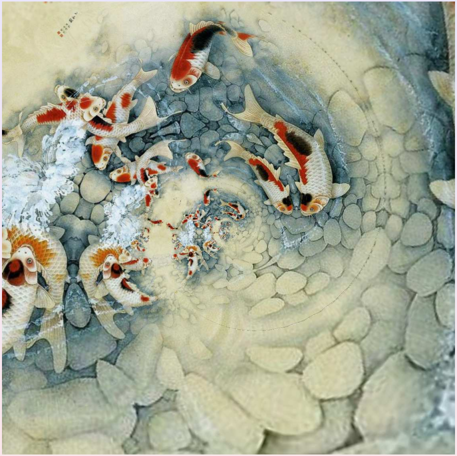

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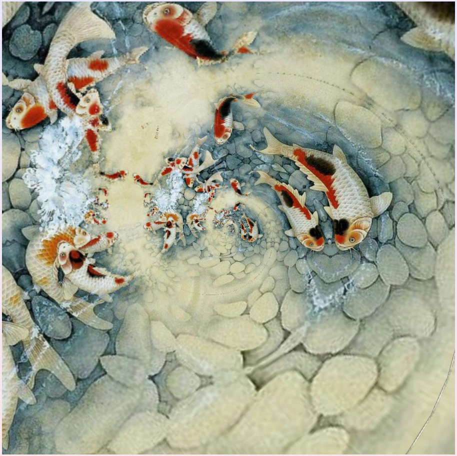

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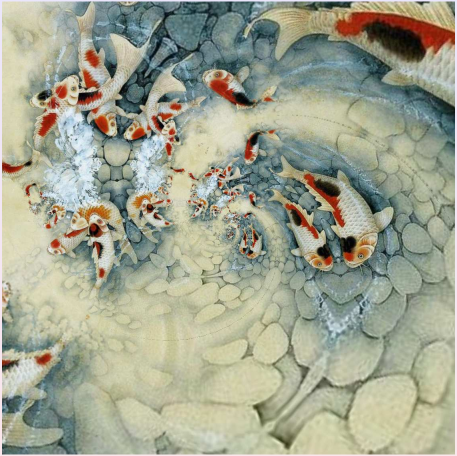

Schottky Group