Inversive Distance Euclidean Curvature Flows

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Cosine law

\[ A = l_j l_k \sin \theta_i \]

\[
2l_j l_k \cos \theta_i = l_j^2 + l_k^2 - l_i^2
\]

\[
-2l_j l_k \sin \theta_i \frac{d\theta_i}{dl_i} = -2l_i
\]

\[
\frac{d\theta_i}{dl_i} = \frac{l_i}{A}
\]
Cosine law

\[ l_j = l_i \cos \theta_k + l_k \cos \theta_i \]

\[ 2l_jl_k \cos \theta_i = l_j^2 + l_k^2 - l_i^2 \]

\[ 2l_j = 2l_k \cos \theta_i - 2l_jl_k \sin \theta_i \]

\[ \frac{d\theta_i}{dl_j} = \frac{l_k \cos \theta_i - l_j}{A} \]

\[ = -\frac{l_j \cos \theta_k}{A} \]

\[ = -\frac{d\theta_i}{dl_i} \cos \theta_k \]
Cosine law

\[
\begin{align*}
\odot \quad v_k \quad \odot \quad v_j \quad \odot \quad v_i \\
\theta_k \quad \theta_i \quad \theta_j \\
l_j \quad l_i \quad l_k \\
\end{align*}
\]

\[l_k^2 = r_i^2 + r_j^2 + 2r_i r_j l_{ij}\]

\[
\begin{align*}
\frac{d l_i}{dr_j} &= \frac{2r_j + 2r_k l_{jk}}{2l_i r_j} \\
2l_i \frac{d l_i}{dr_j} &= 2r_j + 2r_k l_{jk} \\
l_i^2 &= r_j^2 + r_k^2 + 2r_j r_k l_{jk} + \frac{r_j^2 - r_k^2}{2l_i r_j}
\end{align*}
\]
Cosine law

Let $u_i = \log r_i$, then

$$
\begin{pmatrix}
\frac{d\theta_1}{du_1} \\
\frac{d\theta_2}{du_2} \\
\frac{d\theta_3}{du_3}
\end{pmatrix}
= \frac{-1}{A}
\begin{pmatrix}
l_1 & 0 & 0 \\
0 & l_2 & 0 \\
0 & 0 & l_3
\end{pmatrix}
\begin{pmatrix}
-1 & \cos \theta_3 & \cos \theta_2 \\
\cos \theta_3 & -1 & \cos \theta_1 \\
\cos \theta_2 & \cos \theta_1 & -1
\end{pmatrix}
\begin{pmatrix}
l_1 & 0 & 0 \\
0 & l_2 & 0 \\
0 & 0 & l_3
\end{pmatrix}
\begin{pmatrix}
\frac{l_1^2 + r_2^2 - r_3^2}{2l_1 r_2} & \frac{l_1^2 + r_3^2 - r_2^2}{2l_1 r_3} & \frac{l_1^2 + r_3^2 - r_2^2}{2l_1 r_3} \\
\frac{l_2^2 + r_1^2 - r_3^2}{2l_2 r_1} & 0 & \frac{l_2^2 + r_3^2 - r_1^2}{2l_2 r_3} \\
\frac{l_3^2 + r_1^2 - r_2^2}{2l_3 r_1} & \frac{l_3^2 + r_2^2 - r_1^2}{2l_3 r_2} & 0
\end{pmatrix}
\begin{pmatrix}
r_1 & 0 & 0 \\
0 & r_2 & 0 \\
0 & 0 & r_3
\end{pmatrix}
\begin{pmatrix}
0 \\
r_2 \\
r_3
\end{pmatrix}
$$
Suppose a point $p$ is not coincident of the center of a circle $c = (c, r)$ on the plane, the line through $c$ and $p$ intersects the circle at $q_1$ and $q_2$, $T$ is the tangent point, then the power of $p$ with respect to the circle is

$$pow(p, c) = |pq_1||pq_2| = |pT|^2 = |pc|^2 - r^2.$$
Suppose there are two circles $c_1 = (c_1, r_1), c_2 = (c_2, r_2)$, the equi-power line is the locus

$$\text{pow}(p, c_1) = \text{pow}(p, c_2).$$

The equation of $p$ is

$$|p - c_1|^2 - r_1^2 = |p - c_2|^2 - r_2^2.$$

If two circles intersect at $p_1$ and $p_2$, then the line through them is the equi-power line.
Suppose there are two circles $c_1 = (c_1, r_1)$, the line through $c_1$ and $c_2$ intersects the equi-power line at the point $p$. Assume the length between $c_1$ and $c_2$ is $l$. The distance from $p$ to $c_2$ is denoted as $d_{21}$, then

\[ d_{12} = \frac{l^2 + r_1^2 - r_2^2}{2l}, \]

\[ d_{21} = \frac{l^2 + r_2^2 - r_1^2}{2l}, \]

obviously, $d_{12} + d_{21} = l$. 

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Discrete Ricci Flow
compute the power of $p$ with respect to two circles

\[
\text{pow}(p, c_1) = d_{12}^2 - r_1^2
\]

\[
\text{pow}(p, c_2) = d_{21}^2 - r_2^2
\]

\[
d_{12}^2 - d_{21}^2 = (d_{12} + d_{21})(d_{12} - d_{21})
\]

\[
= l \frac{r_1^2 - r_2^2}{l} = r_1^2 - r_2^2.
\]
Lemma

The equi-power line is orthogonal to the line connecting the centers.

Proof.

Define a function \( \phi(p) = \text{pow}(p, c_1) - \text{pow}(p, c_2) \),

\[
\phi(p) = \langle p - c_1, p - c_1 \rangle - r_1^2 - \langle p - c_2, p - c_2 \rangle + r_2^2
\]

\[
d\phi(p) = \langle dp, c_2 - c_1 \rangle
\]
Given three circles $c_k, k = 1, 2, 3$, then three equi-power lines intersect at one point $o$, which is called the **power center**.

The equi-power lines of $c_1, c_2$ and $c_1, c_3$ intersects at the point $o$. Then

$$\text{pow}(o, c_1) = \text{pow}(o, c_2) = \text{pow}(o, c_3)$$

so $o$ is also on the equi-power line of $c_2, c_3$. 
There are 3 circles $c_k = (c_k, r_k)$, the power center $o$ is also the center of the unique circle $(p, r)$, which is orthogonal to all 3 circles.

$$\text{pow}(p, c_k) = \langle p - c_k, p - c_k \rangle - r_k^2 = r^2$$

so the power center is the center of the circle which is orthogonal to the 3 circles.

$$\frac{d\theta_i}{du_j} = \frac{h_k}{l_k}$$
Cosine law

\[ l_k^2 = r_i^2 + r_j^2 + 2l_{ij}r_ir_j \]

\[ 2l_k \frac{dl_k}{dr_j} = 2r_j + 2r_i l_{ij} \]

\[ r_j \frac{dl_k}{dr_j} = \frac{2r_j^2 + 2r_ir_j l_{ij}}{2l_k} \]

\[ = \frac{l_k^2 + r_j^2 - r_i^2}{2l_k} \]

Therefore

\[ \frac{dl_k}{du_j} = d_{ji}. \]
The distance from $o$ to edge $[v_i, v_j]$ is $h_k$. 

\[
\begin{align*}
\frac{d\theta_i}{du_j} &= \frac{d\theta_j}{du_i} = \frac{h_k}{l_k} \\
\frac{d\theta_j}{du_k} &= \frac{d\theta_k}{du_j} = \frac{h_i}{l_i} \\
\frac{d\theta_k}{du_i} &= \frac{d\theta_i}{du_k} = \frac{h_j}{l_j}
\end{align*}
\]
Cosine law

\[
\cos \theta = \frac{d_{jk} - d_{ji} \cos \theta_j}{l_i} \cdot \frac{dl_i}{l_i l_k \sin \theta_j} = \frac{h_k}{l_k} \cdot \frac{h_k \sin \theta_j}{l_k \sin \theta_j}
\]

Proof.

\[
\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_i}{\partial l_i} \frac{\partial l_i}{\partial u_j} + \frac{\partial \theta_i}{\partial l_k} \frac{\partial l_k}{\partial u_j} = \frac{\partial \theta_i}{\partial l_i} \left( \frac{\partial l_i}{\partial u_j} - \frac{\partial l_k}{\partial u_j} \cos \theta_j \right)
\]

\[
= \frac{l_i}{A} (d_{jk} - d_{ji} \cos \theta_j)
\]

\[
= \frac{dl_i}{l_i l_k \sin \theta_j}
\]

\[
= \frac{h_k \sin \theta_j}{l_k \sin \theta_j}
\]

\[
= \frac{h_k}{l_k}
\]
The Discrete Ricci energy of Inversive distance CP metric is convex, but the metric space is non-convex. Therefore it has local rigidity, not global rigidity.
Shrink three circles to vertices, then the power center $o$ becomes the circum-center.

\[
\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_i}{\partial l_i} \left( \frac{\partial l_i}{\partial u_j} - \frac{\partial l_k}{\partial u_j} \cos \theta_j \right)
\]

\[
= \frac{l_i}{A} (l_i - l_k \cos \theta_j)
\]

\[
= \frac{2l_i d}{l_i l_k \sin \theta_j}
\]

\[
= \frac{2h_k}{l_k}
\]

\[
= \cot \theta_k
\]

$l_{ij} \leftarrow e^{u_i} l_{ij} e^{u_j}$
The Discrete Ricci energy of discrete Yamabe flow is convex, but the metric space is non-convex. Therefore it has local rigidity, not global rigidity.