Abstract— We exploit the Laplace-Beltrami (LB) operator to represent shapes, which in turn is used to visualize certain standard datasets specifically for biomedical applications. LB spectral measures are isometry invariant and are one of the most powerful ways to represent shape, also called “Shape-DNA”. We define a shape signature by finding the eigenvalues and eigenvectors of the LB matrix for the given model. For computational simplicity we use the isosurfaces instead of using the volume data directly. Isosurfaces are extracted from the volume data and the LB matrix is evaluated on these extracted isosurfaces. The eigenvalues and eigenvectors of the LB matrix for the extracted isosurfaces are used to represent shapes. We call these eigenvectors shape eigenvectors as they successfully define various shapes in a model. We use an interface which looks like a modified form of parallel coordinates where the shape eigenvectors are represented by parallel lines. This interface is called parallel segmentation and it can be used for visualizing selected regions in the 3D model by just varying the values using the cursor. We demonstrate the results of our method on several standard datasets and show the effectiveness of our method for biomedical visualization.

Keywords- Laplace-Beltrami; Shape-DNA; shape eigenvectors; shape signature; parallel segmentation

I. INTRODUCTION

In recent years work has been done in volume visualization based on shape for medical and biological applications. Visualization based on size, texture, density and other such parameters have already been developed. However, development of good visualization based on shape still remains a challenge. It is indeed a complex task to understand shape. Most of the methods that are used to represent shape provide a characterization of the geometric and structural properties for the object boundary. They do not provide any explicit way to describe the semantic relevance of the shape. Spectral methods have gained much interest in various applications of computer graphics [1]. The major applications include the shape correspondence [2], matching and retrieval [3, 4] and segmentation [5]. Spectral methods are also used for specific applications like mesh compression [6] and some imaging applications [7]. We use one such powerful spectral operator, called the Laplace-Beltrami (LB) operator to represent shapes and in turn use this shape information to design a transfer function for volume rendering. In particular, the eigenfunctions of the LB operator yield a family of real-valued functions that provide interesting insights into the morphology and structure of the shapes. The family of eigenvalues of the LB matrix defines the spectrum of the shape. These LB spectral values are one of the most powerful ways to represent shapes and hence this spectra has been termed as Shape-DNA.

Isosurfaces are extracted from the given volume data and the LB operator is applied on these extracted isosurfaces. We use the eigenvalues and eigenvectors of the LB operator for representing shape. As these eigenvectors are used to represent shape, we term these eigenvectors as shape eigenvectors. Different shape eigenvectors essentially help us to segment the model based on the various shapes present. An interactive user interface is developed by representing the segmentation of different shape eigenvectors as parallel lines. The color and opacity values are assigned and varied by moving the sliders over these lines. The interface looks similar to that of parallel coordinates and thus we call this interface parallel segmentation as it can be used to segment and understand different shapes in the given model.

II. LAPLACE-BELTRAMI OPERATOR

Let \( \zeta \) be a Riemannian manifold defined by a function \( f: \zeta \rightarrow \mathbb{R} \) and with a metric \( g \). Then, the LB operator \( \Delta \) of that function \( f \) is defined as

\[
\Delta f = \text{div}(\text{grad } f)
\]

where \( \text{grad } f \) is the gradient of \( f \) and \( \text{div} \) on the manifold. The same operator when defined elaborately is given by the following formula:

\[
\Delta f = |\text{det}[g_{ij}]|^{-\frac{1}{2}} \frac{\partial}{\partial x^k} \left( g^{kl} |\text{det}[g_{ij}]|^{-\frac{1}{2}} \frac{\partial f}{\partial x^l} \right)
\]

where \( \delta_{ij} \) are the elements of the matrix representing the metric \( g \) in the coordinate system given by local coordinates \( x^i \).

In this paper, we use the LB operator on mesh structures. Thus, essentially we compute the LB of a function that is sampled at the vertices of the mesh. The only information known is the value of the function at these vertices and moreover the function is not smooth. As the smooth LB requires a smooth function and also a metric, we cannot use the smooth LB and need a discrete LB operator. A discrete LB operator \( D \) can be computed by using the following equation:
\[ \Delta f_i = \frac{1}{2A} \sum_{j \in N_i(i)} (\cot \alpha_{ij} + \cot \beta_{ij})(f_i - f_j) \]

where \( \alpha_{ij} \) and \( \beta_{ij} \) are the angles opposite to the edge \( ij \) where \( j \) is the 1-ring neighborhood of \( i \), \( A \) is the Voronoi region area of \( i \), \( N_i(i) \) has all the vertices \( j \) which lie at 1-ring neighborhood of \( i \), \( f \) is a given function sampled at the vertices of the mesh.

Using this equation, given any triangular mesh, a LB matrix \( L \) can be constructed as following:

\[
L_{ij} = \begin{cases} 
-\frac{\cot \alpha_{ij} + \cot \beta_{ij}}{2A_i} & \text{if } i,j \text{ are adjacent} \\
\sum_{k} \frac{\cot \alpha_{ij} + \cot \beta_{ij}}{2A_i} & \text{if } i=j \\
0 & \text{otherwise}
\end{cases}
\]

where all the terms are same as defined previously for some \( i \) and \( j \). Then the spectrum is obtained by solving the following equation:

\[ L \mathbf{v} = \lambda \mathbf{v} \]

where \( \mathbf{v} \) is a vector of dimension \( n \) (number of vertices in the mesh). The value of \( \mathbf{v} \) is a function value at each of the \( n \) vertices on the mesh. The solution to this problem is the family of eigenvalues which form the spectrum.

Hence, the LB spectrum is the family of eigenvalues \( \{0, \lambda_1, \lambda_2, \ldots, \lambda_n \} \) obtained by solving the above eigenvalue problem. Each eigenvalue has an eigenvector and it is this eigenvector that has relevant shape information.

A. Accuracy of Discrete Laplace-Beltrami operator

For the sake of simpler numerical computation, instead of using continuous LB operator, we use discrete LB on triangular meshes. The accuracy of the discrete operator has been studied in [8]. The results of using the discrete LB have been compared with the standard results already known from theory. The exact eigenfunctions are known for rectangles and spheres from theory. The approximate eigenfunctions are computed by using different discretizations of the LB operator and the correctness is compared by evaluating the errors with respect to the exactly known eigenfunctions. These comparisons of the discrete LB with the standard results prove the accuracy of the LB operator. The robustness of the cotangent operator has also been studied and discussed [9]. It justifies the usage of the cotangent operator and also verifies the results by evaluating the LB eigenfunctions for meshes with different mesh structures but same geometry.

B. Properties

Isometry Two geometric objects are isometric if a homeomorphism from one to the other exists preserving (geodesic) distances, i.e., mapping curves to curves with equal arc length. This homeomorphism is then called an isometry. Similarly two geometric objects are congruent if they can be transformed into each other by rigid motions (translations and rotations) as well as reflections. LB operator has the isometry property. Irrespective of the objects representation and location, the LB operator generates the same matrix and same eigenvalues and eigenvectors. This is due to the fact that LB operator considers geodesic distance instead of Euclidean distance. The LB operator depends on the gradient and divergence. These in turn are dependent only on the intrinsic geometry. Thus the LB operator preserves the shape information irrespective of the deformations.

Completeness The computed eigenvalues and eigenvectors of the LB matrix give a complete characterization of the shape, thus representing the shape uniquely. Further these eigenvectors can be used to reconstruct the solid. However, there are very rare exceptions possible where two shapes have same LB spectra which can be ignored. Thus, the LB spectrum of any shape is unique.

Scaling The LB spectrum is independent of the objects size and hence is scale invariant. The scale information is taken care by normalization of the LB matrix. After obtaining the LB matrix, it is normalized, before the eigenvalues and eigenvectors are computed. Scaling a manifold by a factor \( k \) results in the eigenvalues scaled by a factor \( 1/k^2 \). Hence, normalizing the eigenvalues will ensure the LB spectrum to be invariant to scaling.

Shape-DNA The eigenvalues can be calculated for different object representations in different dimensions and can even be calculated for grayscale or color images. Another advantage of the LB spectra is that it can even be applied to solids containing cavities (solids bounded by several not connected surfaces), for example an ice-cube containing fully enclosed bubbles. Most techniques that work on boundary representations and not on the solid itself have difficulties with several boundary components. But, LB spectra can be applied on solids itself. Irrespective of the position, posture of the object the LB spectra captures the shape information. Because the spectrum of the LB operator contains intrinsic shape information it has been called the Shape-DNA. Thus, this Shape-DNA can be used to represent shapes in objects.

Just like the DNA in real world, the LB shape-DNA also contains intrinsic information unique to the object and so the LB spectrum is a very good choice for shape representation. Two surfaces with same shape-DNA and same geometry, have same eigenfunctions. Such is the efficiency of the LB spectrum in describing the shape information. We showed the power of LB shape eigenvectors to identify and classify shapes by testing it on a phantom dataset. Figure 1 shows the results of LB shape eigenvectors on a phantom dataset we created. The phantom data include various standard shapes like the sphere, ring and two drop like structures. The shape eigenvectors are presented successively to see that different shapes have been successfully identified by different shape eigenvectors. In Fig. 1 (a), one drop like structure has been identified and similarly in Figures 1 (b), (c) and (d), ring,
second drop like structure and the sphere have been identified respectively. Fig. 1 (e) shows all the eigenvectors identifying all the shapes. Thus, the LB operator works successfully for the phantom dataset in classifying shapes.

**Fig. 1.** The LB shape eigenvectors for a phantom dataset. The phantom dataset has different shapes of sphere, two drop like structures and a ring. (a) One drop like structure (in blue) has been identified by the first lowest frequency eigenvector. (b) The ring (in light pink) has been identified by the second lowest frequency eigenvector. (c) The second drop like structure (in light pink) has been identified by the third lowest frequency eigenvector. (d) The sphere (in light pink) has been identified by the fourth lowest frequency eigenvector. (e) All the four eigenvectors in different colors

### III. LAPLACE-BELTRAMI SHAPE EIGENVECTORS

Though LB operator is a generalized operator which can be applied on any Riemannian manifold, most of the published literature to the best of our knowledge, only discuss the discrete LB operator for mesh surfaces only (in particular triangular mesh surfaces) and not for volumes. In other words, the discrete LB operator has been applied on triangular meshes and not on tetrahedral meshes. The development of discrete LB operator for the tetrahedral mesh is itself a challenge of its own and forms a part of our future work. For now we use the isosurfaces and apply the discrete LB operator on them. Isosurfaces are extracted from the volume data using the classic marching cubes algorithm [10] by considering different isovalues. LB operator is applied on these isosurfaces to obtain a LB matrix. This LB matrix as mentioned previously is a sparse matrix, as we obtain it by solving a generalized eigenvalue problem. The corresponding eigenvalues and eigenfunctions are evaluated on this matrix by using the equations mentioned in the previous section. The family of eigenvalues forms the spectrum and the corresponding eigenvectors contain intrinsic shape information which can be used to define shapes in that isosurface. Each eigenvector inherently possesses certain amount of shape information and represent a particular shape of the isosurface. Highlighting a specific eigenvector highlights a specific shape of the isosurface. As the eigenvectors of the LB matrix is used to represent shapes we term these eigenvectors as shape eigenvectors. Different eigenvectors of the LB matrix essentially depict different shapes in the model. Higher frequency eigenvectors represent global shapes whereas the lower frequency eigenvectors help to classify the global model into smaller shapes.

**Fig. 2.** (a) Combined results of using all the shape eigenvectors on the engine dataset (isosurface 200). The results assign different parts of the engine to different colors for clear distinction; (b) All the shape eigenvectors on the fuel dataset (isosurface 100)

We demonstrate the results of our method on several standard datasets like the engine and fuel datasets. Different parts of the engine dataset are clearly shown by using the LB operator in Fig. 2 (a). Clear distinction of different segments of the engine dataset can be observed in the result. The isosurface used in the case of this engine dataset is 200. We chose this isosurface deliberately so as to check the efficiency of our technique in classifying shapes. Fig. 2 (b) shows the highest four eigenvectors (in terms of magnitude) of the LB matrix of the fuel dataset (isosurface 100). The isosurface 100 of the fuel dataset shows different parts with different shapes. It has a body which is almost a hollow cylinder, a crown which
resembles a cup and four jewel like structures of different
shapes.

IV. USER INTERFACE

Given any volume data, we extract the isosurfaces for different
isovalues and apply the LB operator to obtain the
corresponding eigenvalues and eigenvectors. The
eigenvectors, called the shape eigenvectors, have to be
presented to the user so that different parameters like the color
and opacity values can be varied in an interactive way. Here,
we propose an interface similar to that of parallel coordinates.
A series of vertical parallel lines are presented before the user
and every line is used to decide the segmentation of each
eigenvector. Thus, the segmentation of the shape-eigenvectors
are represented by a series of parallel lines. A horizontal line
called as “zero-shape line” is initially presented. By moving
this zero-shape line over the vertical parallel lines different
color and opacity values are assigned as well as varied to
obtain any region of interest by segmenting the shape-
eigenvectors. At the same time, weights can also be assigned
to the eigenvectors using the interface. Right clicking on the
zero shape line and moving the slider will assign weights to
different shape eigenvectors. The sliders help the user to
segment the shape-eigenvectors by assigning color and opacity
values. Assigning weights, further help to focus on any region
of interest and classify various shapes of the model, defined by
the shape-eigenvectors.

Fig. 3. Parallel segmentation user interface, showing the parallel lines, the
zero-shape line and the sliders to vary the parameters.

By providing many parameters to the user, we provide as
much flexibility as possible so that the user can accurately
classify shapes of the model and choose a desired region of
interest. As we are using parallel lines to assign colors and to
segment the given model into different shapes we call this
interface as parallel segmentation. Fig. 3 shows the user
interface with the different lines and sliders on the zero shape
line. The initial position of the “zero-shape line” as the name
suggests is 0. All the lines are normalized between [-1,1].
Moving the slider will help to choose a particular element of
the n-dimensional eigenvector and thus segment it. Weights
are also assigned to different eigenvectors so that the users can
explore and navigate different segments of the given model.
However, this parallel segmentation is not analogous to the
parallel coordinates as there is no reduction of dimension here.

V. MEDICAL AND BIOLOGICAL APPLICATIONS

This volume visualization and segmentation based on shape
has specific application in medical and biomedical fields. The
application also extends to areas like physics and chemistry to
study the motion of electrons and atoms. Fig. 4 shows the
result of applying our method by using the different
eigenvectors of the neghip dataset (for the isosurfaces 67 and
120). Both the number of segments and the shape of the
segments varied between the two isosurfaces. Yet it can be
seen that our methods produced similar results albeit the
change in the number of segments, shape and size of the
segments. The neghip protein structure has various
components with different shapes. Thus, segmentation based
on only shape will ensure to understand the protein structure
better. Our results help to visualize the neghip protein dataset
profundly which help to better explore the protein structure.
The isosurface values of 67 and 120 are shown by Figures 4
(a) and (b) respectively.

Fig. 4. Contrasting the results of our method on different isosurfaces of neghip
protein dataset (isosurface 67 and isosurface 120). (a) shows the isosurface 67
and (b) shows the isosurface 120 of the neghip dataset. Albeit the change in
the shape and size of the segments, various segments are clearly evident, in
both the isosurfaces.

Figures 5 and 6 show the results on still more complicated
data set of a foot (isosurface 140) consisting of many shapes.
Very minute parts of the foot have been identified by our
method. Figures 6 (a) and (b) shows different parts of the foot
being highlighted and in each of the figures classification is
performed by setting various parameters in the parallel
segmentation. From the Figures 6 (a) and (b), the color of the
second finger from right has changed by moving the slider. Thus, the user can focus on that finger, if needed, by just changing the slider of the highlighted eigenvector. These results show the flexibility of the user interface to detect different shapes and to focus on any region of interest. Finally, in Figure 5 we show the results by removing the lighting so that the segmentation results based on shapes are evident. The left side shows the results without the lighting whereas the right side shows the corresponding results with lighting. All the results assert that our method using LB operator is very useful in volume visualization for exploration based on shapes.

![Fig. 5. Results of applying our method on CT foot dataset. On the left side, we have the results without lighting while on the right side, we have the corresponding results with lighting. The results on the left, though not smooth, clearly show the segmentation of the foot based on different shapes.](image)

### VI. CONCLUSION

We have exploited the LB operator to represent shapes which in turn is used for visualizing biomedical and medical data. LB spectral measures are isometry invariant and are one of the most powerful ways to represent shape, also called Shape-DNA. LB operator is applied on different isosurfaces of the volume data and the corresponding eigenvalues and eigenvectors are computed. The eigenvectors obtained provide intrinsic shape information. As these eigenvectors represent different shapes, these are called shape eigenvectors. A user interface is designed which resembles parallel segmentation on the basis of its design. The user can precisely explore the desired regions of interest by using our interface. We have shown the results on datasets like fuel and hydrogen atom and also show its application on medical and biomedical datasets like neghip and foot datasets. It can be observed that our method based on shapes using the LB operator yields very good results.

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![Fig. 6. (a) and (b) compare the results obtained by just changing the position of the slider on the highlighted eigenvector, keeping all the other parameters the same. When we change the slider position, we can observe a change in the color of the second finger from right (from earthen color to light purple). This proves that the highlighted eigenvector in fact corresponds to that particular shape and so varying the parameter resulted in change of color.](image)

### REFERENCES