1 Equational Laws for Quantifiers

Theorem 1 (Propositional Substitutions) *If a formula A is a propositional tautology, then by substituting for propositional variables in A any formula of the first order language we obtain a formula which is a first order tautology.*

Example 1 *Consider the following propositional tautology:* $((a \Rightarrow b) \Rightarrow (\neg a \cup b))$. *Substituting $\exists x P(x,z)$ for a, and $\forall y R(y,z)$ for b, we obtain the formula*

$$((\exists x P(x,z) \Rightarrow \forall y R(y,z)) \Rightarrow (\neg \exists x P(x,z) \cup \forall y R(y,z))).$$

The theorem 1 guarantee that this formula is a predicate language (first order) tautology.

We will consider here only tautologies which have a form of a **logical equivalence** and write $A \equiv B$ to denote that formulas A and B are logically equivalent.
Definition Reminder: $A \equiv B$ if and only of the formula $((A \Rightarrow B) \cap (B \Rightarrow A))$ is a tautology. Directly from the theorem 1 we get that the following is true.

Fact 1 *If $A \equiv B$ is a propositional equivalence, A', B' are first order formulas obtained by a substitution of any formula of the first order language for propositional variables in A and B, respectively, then $A' \equiv B'$ also holds in the first order logic.*

Example 2 *Consider the following propositional logical equivalence:* $(a \Rightarrow b) \equiv (\neg a \cup b)$. *Substituting $\exists x P(x,z)$ for a, and $\forall y R(y,z)$ for b, we get from the fact 1 that the following equivalence holds:*

$$(\exists x P(x,z) \Rightarrow \forall y R(y,z)) \equiv (\neg \exists x P(x,z) \cup \forall y R(y,z)).$$

We will prove also the following, intuitively quite obvious theorem which helps to build new logical equivalences from the old, known ones.

Theorem 2 *For any formulas $A(x), B(x) \in \mathcal{F}$, if $A(x)$ and $B(x)$ are logically equivalent, so are the formulas $\forall x A(x)$ and $\forall x B(x)$, and $\exists x A(x)$ and $\exists x B(x)$,respectively. I.e., the following holds.*

$$If \quad A(x) \equiv B(x), \text{ then } \forall x A(x) \equiv \forall x B(x) \text{ and } \exists x A(x) \equiv \exists x B(x).$$

Example 3 *We know from the example 2 that the formulas $(\exists x P(x,z) \Rightarrow \forall y R(y,z))$ and $(\neg \exists x P(x,z) \cup \forall y R(y,z))$ are logically equivalent. We get, as the direct consequence of the theorem 2 the following equivalences:*

$$\forall z(\exists x P(x,z) \Rightarrow \forall y R(y,z)) \equiv \forall z(\neg \exists x P(x,z) \cup \forall y R(y,z)),$$

$$\exists z(\exists x P(x,z) \Rightarrow \forall y R(y,z)) \equiv \exists z(\neg \exists x P(x,z) \cup \forall y R(y,z)).$$

The **Theorem 1** and **Theorem 2** show us how to use the propositional tautologies and simple use of quantifiers to build first order tautologies. The substitution technique is valid not for logical equivalences only, but we will concentrate here on them. We will show now that we can use only the propositional tautologies and theorems 1, 2 to prove the formulas $\neg\forall x\neg(A(x) \cup B)$ and $\neg\forall x(\neg A(x) \cap \neg B)$ are logically equivalent.

Example 4 *By the substituting* $A(x)$ *for* a*, and any formula* B *for* b*, in the propositional de Morgan law:* $\neg(a \cup b) \equiv (\neg a \cap \neg b)$*, we get that*

$$\neg(A(x) \cup B) \equiv (\neg A(x) \cap \neg B).$$

Applying the theorem 2 to the above we obtain that

$$\forall x\neg(A(x) \cup B) \equiv \forall x(\neg A(x) \cap \neg B).$$

We know, from the propositional logic, that for any variables a, b*,* $a \equiv b$ *if and only if* $\neg a \equiv \neg b$*. Substituting* $\forall x\neg(A(x) \cup B)$ *and* $\forall x(\neg A(x) \cap \neg B)$ *for* a *and* b*, respectively, we get that* $\forall x\neg(A(x) \cup B) \equiv \forall x(\neg A(x) \cap \neg B)$ *if and only if* $\neg\forall x\neg(A(x) \cup B) \equiv \neg\forall x(\neg A(x) \cap \neg B)$*. But we know, that* $\forall x\neg(A(x) \cup B) \equiv \forall x(\neg A(x) \cap \neg B)$ *holds, so we conclude that the equivalence*

$$\neg\forall x\neg(A(x) \cup B) \equiv \neg\forall x(\neg A(x) \cap \neg B)$$

is true.

As we can see, it is possible to obtain a fair amount of laws of quantifiers from the propositional tautologies and theorems 1 and 2, but obviously, we will for example never obtain a following, intuitively true law:

$$(\forall x A(x) \Rightarrow \exists x A(x)).$$

Some more examples of important first order tautologies (laws of quantifiers) which cannot be obtained by application of the above theorems 1, 2 to the propositional tautologies will be given later. We will concentrate now only on those laws which have a form of a logical equivalence. One of the most important are the following De Morgan's Laws.

De Morgan's Laws

$$\neg\forall x A(x) \equiv \exists x\neg A(x) \tag{1}$$

$$\neg\exists x A(x) \equiv \forall x\neg A(x) \tag{2}$$

We will prove the De Morgan's Laws later and now we will apply them to show that the quantifiers can be defined one by the other i.e. that the following Definability Laws hold.

Definability Laws

$$\forall x A(x) \equiv \neg\exists x \neg A(x) \tag{3}$$

$$\exists x A(x) \equiv \neg\forall x \neg A(x) \tag{4}$$

The law (3) is often used as a definition of the universal quantifier in terms of the existential one (and negation), the law (4) as a **definition** of the existential quantifier in terms of the universal one (and negation).

Proof of (3) Substituting any formula $A(x)$ for a variable a in the propositional tautology $((a \Rightarrow \neg\neg a) \cap (\neg\neg a \Rightarrow a))$ we get the following first order logical equivalence: $A(x) \equiv \neg\neg A(x)$. Applying the theorem 2 to the above we obtain $\exists x A(x) \equiv \exists\neg\neg A(x)$. By the de Morgan Law 1 $\exists\neg\neg A(x) \equiv \neg\forall\neg A(x)$ and hence $\exists x A(x) \equiv \neg\forall\neg A(x)$, what ends the proof.

Proof of (3) We obtain $\forall x A(x) \equiv \forall\neg\neg A(x)$ in a similar way as above. By the de Morgan Law 2, $\forall\neg\neg A(x) \equiv \neg\exists\neg A(x)$ and hence $\forall x A(x) \equiv \neg\exists\neg A(x)$, what ends the proof.

Other important equational laws are the following Introduction and Elimination Laws. We will prove later the first two of them. We will show here that the laws (7) - (12) can be deduced from laws (5) and (6), the de Morgan Laws, Definability Laws, propositional tautologies and theorems1, 2.

Introduction and Elimination Laws

If B is a formula such that B **does not contain any free occurrence** of x, then the following logical equivalences hold.

$$\forall x(A(x) \cup B) \equiv (\forall x A(x) \cup B) \tag{5}$$

$$\forall x(A(x) \cap B) \equiv (\forall x A(x) \cap B) \tag{6}$$

$$\exists x(A(x) \cup B) \equiv (\exists x A(x) \cup B) \tag{7}$$

$$\exists x(A(x) \cap B) \equiv (\exists x A(x) \cap B) \tag{8}$$

$$\forall x(A(x) \Rightarrow B) \equiv (\exists x A(x) \Rightarrow B) \tag{9}$$

$$\exists x(A(x) \Rightarrow B) \equiv (\forall x A(x) \Rightarrow B) \tag{10}$$

$$\forall x(B \Rightarrow A(x)) \equiv (B \Rightarrow \forall x A(x)) \tag{11}$$

$$\exists x(B \Rightarrow A(x)) \equiv (B \Rightarrow \exists x A(x)) \tag{12}$$

The equivalences (5)- (8) make it possible to introduce a quantifier that precedes a disjunction or a conjunction into one component on the condition that the other component does not contain any free occurrence of a variable which is bound by that quantifier. These equivalences also make possible to eliminate a quantifier from a component of a disjunction or a conjunction and to place it before that disjunction or conjunction as a whole, on the condition that the other component does not contain any free occurrence of a variable which that quantifier would then bind.

The equivalences (9) - (12)make it possible to introduce a quantifier preceding an implication into the consequent of that implication, on the condition that that antecedent does not contain any free occurrence of a variable which is bound by that quantifier; they also make it possible to introduce a universal quantifier preceding an implication into the consequent of that implication while changing it into an existential quantifier in the process,on the condition that the consequent of that implication does not contain any free occurrence of a variable bound by that quantifier. Equivalences (9) - (12) further enable us to eliminate quantifiers from the antecedent of an implication to the position preceding the whole implication, while changing a universal quantifier into an existential one, and vice versa, in the process, and also to eliminate quantifiers from the consequent of an implication to the position preceding the whole implication; the conditions that the other component of the implication in question does not contain any free occurrence of a variable which that quantifier would then bind, must be satisfied, respectively.

As we said before, the equivalences (5)-(12) are not independent, some of them are the consequences of the others. Assuming that we have already proved (5) and (6), the proofs of (7)-(12) are the following.

Proof of (7) $\exists x(A(x) \cup B)$ is logically equivalent, by the Definability Law 4 to $\neg\forall x\neg(A(x) \cup B)$. By the reasoning presented in the example 4, we have that $\neg\forall x\neg(A(x) \cup B) \equiv \neg\forall x(\neg A(x) \cap \neg B)$. By the Introduction Law 6, $\neg\forall x(\neg A(x) \cap \neg B) \equiv \neg(\forall x\neg A(x) \cap \neg B)$. Substituting $\forall x\neg A(x)$ for a and $\neg B)$ for b in propositional equivalence $\neg(a \cap \neg b) \equiv (\neg a \cup \neg\neg b)$, we get, by the **Fact1** that $\neg(\forall x\neg A(x) \cap \neg B) \equiv \neg\forall x\neg A(x) \cup \neg\neg B$. In a similar way we prove that $\neg\neg B \equiv B$, by the Definability Law (4) $\neg\forall x\neg A(x) \equiv \exists x A(x)$, hence $\neg\forall x\neg A(x) \cup \neg\neg B \equiv (\exists x A(x) \cup B)$ and finally, $\exists x(A(x) \cup B) \equiv (\exists x A(x) \cup B)$, what end the proof.

We can write this proof in a shorter, symbolic way as follows:

$$\begin{aligned}
\exists x(A(x) \cup B) &\overset{\text{law 4}}{\equiv} \neg\forall x\neg(A(x) \cup B) \\
&\overset{\text{thm 1, 2}}{\equiv} \neg\forall x(\neg A(x) \cap \neg B) \\
&\overset{\text{law 6}}{\equiv} \neg(\forall x\neg A(x) \cap \neg B) \\
&\overset{\text{fact 1}}{\equiv} \neg\forall x\neg A(x) \cup \neg\neg B \\
&\overset{\text{law 1}}{\equiv} (\exists x A(x) \cup B)
\end{aligned}$$

Distributivity Laws
Let $A(x), B(x)$ be any formulas with a free variable x.
Law of distributivity of **universal quantifier** over **conjunction**

$$\forall x \ (A(x) \cap B(x)) \ \equiv \ (\forall x A(x) \cap \forall x B(x)) \tag{13}$$

Law of distributivity of **existential quantifier** over **disjunction**.

$$\exists x \ (A(x) \cup B(x)) \ \equiv \ (\exists x A(x) \cup \exists x B(x)) \tag{14}$$

Alternations of Quantifiers Let $A(x, y)$ be any formula with a free variables x,y.

$$\forall x \forall y \ (A(x, y) \ \equiv \ \forall y \forall x \ (A(x, y) \tag{15}$$

$$\exists x \exists y \ (A(x, y) \ \equiv \ \exists y \exists x \ (A(x, y) \tag{16}$$

Renaming the Variables
Let $A(x)$ be any formula with a free variable x and let y be a variable that **does not occur** in A(x).
Let $A(x/y)$ be a result of **replacement** of each occurrence of x by y, then the following holds.

$$\forall x A(x) \equiv \forall y A(y), \tag{17}$$

$$\exists x A(x) \equiv \exists y A(y). \tag{18}$$