Introduction to Predicate Logic

Cse537

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Predicate Logic Introduction

Part 1

Predicate Logic Language

Translations to Logic Formulas

Translations to AI Logic Formulas

Predicate Logic Language

Symbols:

1. P, Q, R… **predicates symbols**, denote relations in "real life", countably infinite set
2. x,y,z…. **variables**, countably infinite set
3. c1, c2, … **constants**, countably infinite set
4. f, g, h … **functional symbols**, may be empty, denote functions in "real life"
5. **Propositional connectives:**

   $\vee, \wedge, \Rightarrow, \neg, \Leftrightarrow$
6. **Symbols for quantifiers**

   $\forall$ **x** – universal quantifier reads: For all x…

   $\exists$ **x** – existential quantifier reads: There is x…

Formulas of Predicate Logic

We use symbols **1** - **6** to build **formulas** of predicate logic as follows

1. P(x), Q(x,f(y)), R(x)… R(c1), Q(x, c3), Q(g(x,y), c), …
   are called **atomic formulas** for any variables x, y,… ,
   functions f, g …. and constants c, c1, c2, …
2. **All atomic formulas are formulas ;**
3. If **A,B** are **formulas** then (like in propositional logic):
   $(A \vee B)$, $(A \wedge B)$, $(A \Rightarrow B)$, $(A \Leftrightarrow B)$, $\neg A$
   are **formulas**
4. **$\forall x\, A$, $\exists y\, A$** are **formulas,** for any variables x, y
5. The set **F** of **all formulas** is the **smallest** set that fulfills the conditions 1 -4.

Free and Bound Variables

Quantifiers **bound** variables within formulas

For example: A is a formula:

$\exists \mathbf{x} (P(\mathbf{x}) \Rightarrow \neg Q(\mathbf{x}, y))$

all the **x**'s in **A** are **bounded** by **∃ x**

y is a **free variable** in **A** and we write **A=A(y)**

A(y) can be bounded by a quantifier, for example

$\forall \mathbf{y}\ \exists x (P(x) \Rightarrow \neg Q(x, \mathbf{y}))$

y got **bounded** and there are **no free** variables in **A** now

A **formula without free variables** is called a **sentence**

Examples

For example: let

$P(y)$, $Q(x,c)$, $R(z)$, $P_1(g(x, y), z)$ be **atomic** formulas, i.e.

$\mathbf{P(x), \; Q(x,c), \; R(z), \; P_1(g(x, y), z) \in F}$

Then we form **some** other formulas out of them as follows:

$\mathbf{(P(y) \lor \neg Q(x, c)) \in F}$

It is a **formula** **A** with two **free** **variables** x, y

We denote it as a formula **A(x,y)**

$\mathbf{\exists x \, (P(y) \lor \neg Q(x, c)) \in F}$ – y is a free variable

We denote it as a formula **B(y)**

$\mathbf{\forall y \, (P(y) \lor \neg Q(x, c)) \in F}$ – x is a free variable

We denote it as a formula **C(x)**

$\mathbf{\forall y \, \exists x (P(y) \lor \neg Q(x, c)) \in F}$ – no free variables

Logic and Mathematical Formulas

We often use **logic symbols** while writing **mathematical statements** in a more **symbolic** way

Example of a Mathematical Statement **written** with logical symbols

$\forall x \in N \, (x > 0 \wedge \exists y \in N \, (y = 1))$

1. Quantifier $\forall x \in N$ is a quantifier with **restricted domain**

Logic uses only $\forall x$, $\exists y$

2. $x > 0$ and $y = 1$ are mathematical statements about **real relations** $>$ and $=$

Logic uses **symbols** P, Q, R… for relations

For example – we write

$R(y, c_1)$ for $y = 1$ and $P(x, c_2)$ for $x > 0$ where c_1 and c_2 are **constants** representing **numbers 1 and 0**, respectively

Translation of Mathematical Statements to **Logic Formulas**

Consider a Mathematical Statement written with logical symbols

$$\forall x \in N \, (x > 0 \land \exists y \in N \, (y = 1))$$

$x \in N$ – we translate it as **one** argument predicate $Q(x)$
$x > 0$ – we translate as $P(x, c_1)$, and $y = 1$ as $R(y, c_2)$ and get

$$\forall Q(x) \, (P(x, c_1) \land \exists Q(y) \, R(y, c_2) \,)$$

↑ Logic formula with **restricted domain** quantifiers

But this is **not yet a proper formula** since **we cannot** have quantifiers $\forall Q(x)$, $\exists Q(y)$ **in LOGIC,** but only quantifiers $\forall x$, $\exists x$

$\forall Q(x)$, $\exists Q(y)$ are called **quantifiers with restricted domain**

Logic Formula Corresponding to a **Mathematical Statement**

We need to "**get rid**" of **quantifiers with restricted domain** i.e. to **translate them** into **logic quantifiers: $\forall x$, $\exists y$**

$\exists x \in N$, $\exists y \in N$ **are restricted quantifiers**

↑ certain **predicate** P(x)

General: restricted domain quantifiers are :

$$\forall_{A(x)}, \quad \exists_{B(x)}$$

for A(x), B(x) any formulas, in particular atomic formulas (predicates) P(x), Q(x)

Restricted Domain **Existential** Quantifiers

Translation for **existential** quantifier

$$\exists_{A(x)} B(x) \equiv \exists x(A(x) \land B(x))$$

↑ **restricted** ↑**logic, not restricted**

Example (mathematical formulas):

$\exists\ x\neq1\ (x>0 \Rightarrow x + y > 5)$ - restricted

$\exists\ x\ ((x\neq1) \land (x > 0 \Rightarrow x + y > 5))$ - not restricted

↑ B(x, y)

English statement:

Some students are good

Logic Translation (restricted domain):

$$\exists_{S(x)} G(x)$$

Predicates are :

S(x) – x is a student

G(x) – x is good

TRANSLATION:

$$\exists x(S(X) \land G(x))$$

Restricted Domain **Universal** Quantifiers

Translation for universal quantifier

Restricted | Logic (non-restricted)

$\forall_{A(x)} B(x) \equiv \forall x\, (A(x) \Rightarrow B(x))$

Example (mathematical statement)

$\forall x \in N\, (x = 1 \lor x < 0)$ restricted domain

$\equiv \forall x\, (x \in N \Rightarrow (x = 1 \lor x < 0))$ – non-restricted

Translation of Mathematic Statements to Logic Formulas

Mathematical statement:

$$\forall x\, (x \in N \Rightarrow (x = 1 \vee x < 0))$$

$x \in N$ – translates to $N(x)$

$x < 0$ – translates to $P(x, c_1)$

$x < y$ – $<$ is a 2 argument relation translates to two argument predicate $P(x, y)$, x, y are variables

0 – is a constant – denote by c_1

$x=1$: relation $=$ translates to is a two argument predicate **$Q(x,y)$**

$x = 1$: 1 translates to a is **constant** denoted by c_2

$x=1$ translates to $Q(x, c_2)$

Corresponding logic formula:

$$\forall x\, (N(x) \Rightarrow (Q(x, c_2) \vee P(x, c_1)))$$

Remark

Mathematical statement: **x + y = 5**

We re-write it as

= (+ (x, y), 5)

Given **x = 2, x = 1**, we get **+(2,1) = 3** and the statement:

= (3,5) which is **FALSE (F)**

Predicates always returns logical values F or T

We really need also **function** symbols (like +, etc..) to **translate** mathematical statements to **logic**, even if we could use only relations as functions are special relations

This is why in **formal** definition of the predicate language we often we have **2 sets of symbols**

1. **Predicates** symbols which can be true or false in proper domains under certain **interpretation**
2. **Functions** symbols

Translations to Logic

Rules:

1. **Identify** the domain: always a set $X \neq \phi$
2. **Identify** predicates (simple: atomic)
3. **Identify** functions (if needed)
4. **Identify** the connectives $\vee, \wedge, \Rightarrow, \neg, \Leftrightarrow$
5. **Identify** the quantifiers $\forall x, \exists x$

   Write a logic formula using only symbols for 2 , 3, 4

6. **Use restricted domain quantifier translation rules,** where needed

Translations Examples

Translate:

For every bird there are some birds that are white

Predicates:

B(x) – x is a bird

W(x) – x is white

Restricted:

$\forall_{B(x)} \exists_{B(x)} W(x)$

Logic

$\forall x(B(x) \Rightarrow \exists x\,(B(x) \wedge W(x)))$

Re-name variables

$\forall x(B(x) \Rightarrow \exists y(B(y) \wedge W(y)))$

By **Laws of Quantifiers** - we will study the laws later, we can re-write it as

$\forall x \exists y\,(B(x) \Rightarrow (B(y) \wedge W(y)))$

AI: Intended Interpretation

Translate:

For every bird there are some birds that are white

Predicates:

In **AI** we usually deal only with INTENDED INTERPRETATION so we use proper names for predicates and functions, i.e. we write

Bird(x) for x is a bird

White(x) for x is white

Restricted:

$$\forall_{Bird(x)} \exists_{Bird(x)} White(x)$$

AI Logic

$$\forall x(Bird(x) \Rightarrow \exists x\, (Bird(x) \wedge White(x)))$$

Re-name variables

$$\forall x(Bird(x) \Rightarrow \exists y(Bird(y) \wedge White(y)))$$

By **Laws of Quantifiers** - we will **study the laws later**, we can re-write it as

$$\forall x \exists y\, (Bird(x) \Rightarrow (Bird(y) \wedge White(y)))$$

Example

Translate into LOGIC:

For every student there is a student that is an elephant

B(x) for x is a student

W(x) for x is an elephant

$\forall_{B(x)} \exists_{B(x)} W(x)$ - restricted

$\forall_{B(x)} \exists x(B(x) \land W(x))$

$\forall x(B(x) \Rightarrow \exists x(B(x) \land W(x)))$ (logic formula)

Example

Translate into **AI LOGIC**

For every student there is a student that is an elephant

Student(x) for x is a student

Elephant(x) for x is an elephant

$\forall_{Student(x)} \exists_{Student(x)} Elephant(x)$ - restricted

$\forall_{B(x)} \exists x(B(x) \wedge W(x))$

$\forall x(Student(x) \Rightarrow \exists x(Student(x) \wedge Elephant(x)))$

(AI logic formula)

Translations Example

Translate into **Logic**

Some patients like all doctors

Predicates:

P(x) – x is a patient

D(x) – x is a doctor

L(x,y) – x likes y

$$\exists_{P(x)} \forall_{D(y)} L(x,y)$$

There is a **patient(x)**, such that for all **doctors(y)**, **x** likes **y**

$$\exists x(P(x) \land \forall y(D(y) \Rightarrow L(x,y)))$$

By **laws of quantifiers** to be studied later we can "pull out $\forall y$")

$$\exists x \forall y(P(x) \land (D(y) \Rightarrow L(x,y)))$$

Translations Exercise

Here is a mathematical statement S:

For all natural numbers n the following hold:

IF $n < 0$, then there is a natural number m, such that $m + n < 0$

1. Re-write S as a "formula" SF that only uses mathematical and logical symbols
2. Translate your SF to a correct logic formula LF
3. Argue whether the statement S it true of false
4. Give an interpretation of the logic formula LF (in a non-empty set X) under which LF is false

Predicate Logic Introduction

Part 2

Predicate Logic Tautologies

Intuitive Semantics for Predicate Logic

Equational Laws for Quantifiers

- **Renaming the Variables**
- Let $A(x)$ be a formula with a free variable x
  Let y be a variable that **does not occur** in $A(x)$
- Let $A(x/y)$ be a result of **replacement** of each occurrence of x by y, then the following holds

$$\forall x\, A(x) \equiv \forall y\, A(x/y)$$

$$\exists x\, A(x) \equiv \exists y\, A(x/y)$$

Equational Laws for Quantifiers

- **Renaming the Variables**
- Let **B** be any formula in which there is a subformula **∀x A(x)** or **∃x A(x)**
- Let **B*** be a result of **replacement** of each occurrence of **∀x A(x)** or **∃x A(x))** by
- **∀y A(x/y)** or **∃y A(x/y)**, respectively
- Then the following equivalence holds
- **B ≡ B***

Equational Laws for Quantifiers

- **Definition**
- We say that a formula **B** has its variables named apart if **no two quantifiers** in **B** bind the **same variab**le and no bound variable is also free
- **Theorem**
- Every formula is logically equivalent to one in which the variables are named apart

Example

Consider a formula B

$$B = \forall x(B(x) \Rightarrow \exists x\, (B(x) \wedge W(x)))$$

We rename variables

Substituting **y** for **x** in $A(x) = (B(x) \wedge W(x))$ we get

$A(x/y) = (B(y) \wedge W(y))$ and

$$\exists x\, (B(x) \wedge W(x)) \equiv \exists y\, (B(y) \wedge W(y))$$

Substituting in **B** we get a formula **B***

$$B^* = \forall x(B(x) \Rightarrow \exists y(B(y) \wedge W(y)))$$

logically equivalent to **B** in which the variables are **named apart**

Equational Laws for Quantifiers

De Morgan Laws

$$\neg \forall x\, A(x) \equiv \exists x\, \neg A(x)$$

$$\neg \exists x\, A(x) \equiv \forall x\, \neg A(x)$$

where $A(x)$ is any formula with free variable x,

$\equiv$ means “logically equivalent”

Definability of Quantifiers

$$\forall x\, A(x) \equiv \neg \exists x\, \neg A(x)$$

$$\exists x\, A(x) \equiv \neg \forall x\, \neg A(x)$$

Application Example

De Morgan and other Laws Application in Mathematical Statements

$$\neg \forall x((x>0 \Rightarrow x+y > 0) \wedge \exists y (y<0))$$

$\equiv$ (by De Morgan's Law)

$$\exists x \neg ((x>0 \Rightarrow x+y > 0) \wedge \exists y (y < 0))$$

$\equiv$ (by De Morgan's Law and 1., 2., 3., 4.)

$$\exists x((x>0 \wedge x+y \leq 0) \vee \forall y(y \geq 0))$$

We used

1. $\neg (A \Rightarrow B) \equiv (A \wedge \neg B)$,   2. $\neg(A \wedge B) \equiv (\neg A \vee \neg B)$
3. $\neg(x + y) > 0) \equiv x+y \leq 0$
4. $\neg \exists y (y <0) \equiv \forall y \neg (y < 0)$

   $\equiv \exists y (y \geq 0)$

Math Statement -to -Logic Formula

Mathematical statement

$\neg \forall x((x<0 \Rightarrow x+y > 0) \wedge \exists y (y<0))$

Corresponding Logic Formula is

$\neg \forall x((P(x,c) \Rightarrow R(f(x,y),c)) \wedge \exists y P(y,c))$

More general; $A(x)$, $B(x)$ any formulas

$\neg \forall x((A(x) \Rightarrow B(x,y)) \wedge \exists y A(y))$

$\equiv \exists x \neg((A(x) \Rightarrow B(x,y)) \wedge \exists y A(y))$

$\equiv \exists x((A(x) \wedge \neg B(x,y)) \vee \neg \exists y A(y))$

$\equiv \exists x ((A(x) \wedge \neg B(x,y)) \vee \forall y \neg A (y))$

Distributivity Laws

1. $\exists x(A(x) \lor B(x)) \equiv (\exists x\, A(x) \lor \exists x\, B(x))$

   Existential quantifier is distributive over $\lor$

What we write as $(\exists x, \lor)$

2. $\forall x\, (A(x) \land B(x)) \equiv (\forall x\, A(x) \land \forall x\, B(x))$

Universal quantifier is distributive over $\land$, what we write as $(\forall x\, , \land)$

Existential quantifier is distributive over $\land$ **only in one direction:**

3. $\exists x(A(x) \land B(x)\,) \Rightarrow (\exists x\, A(x) \land \exists x\, B(x))$

Distributivity Laws

We show the inverse implication

$(\exists x\, A(x) \wedge \exists x\, B(x)) \Rightarrow \exists x(A(x) \wedge B(x))$

is NOT a predicate tautology;

It means that **it is not true**, that the implication

$(\exists x\, A(x) \wedge \exists x\, B(x)) \Rightarrow \exists x(A(x) \wedge B(x))$

holds for **any** $X \neq \phi$ and for **any** $A(x), B(x)$ defined in the set X

To prove it **we have to show** that

there are $X \neq \phi$, $A(x)$, $B(x)$ defined in $X \neq \phi$ for which this implication is **FALSE**

Not a Tautology

The formula

$(\exists x\, A(x) \wedge \exists x\, B(x)) \Rightarrow \exists x(A(x) \wedge B(x))$

Is not a predicate tautology

Here is a **counter- example**

Take: X = R (real numbers),

A(x): $x > 0$ and B(x): $x<0$ we get that

$\exists x\, (x>0) \wedge \exists x(x<0)$ is a **true** statement in R

and

$\exists x(x>0 \wedge x< 0)$ is a **false** statement in R

Distributivity Laws

Universal quantifier is distributive over $\vee$ in **only one direction**:

4. $((\forall x\, A(x) \vee \forall x\, B(x)) \Rightarrow \forall x(A(x) \vee B(x)))$

Here is the other direction implication counter- example

Take: **X=R** and **A(x): x < 0, B(x): x ≥ 0**

$\forall x\, (x<0 \vee x \geq 0)$ is a **true** statement in **R** (real numbers) and

$\forall x(x<0) \vee \forall x(x \geq 0)$ is a **false** statement in **R**

Distributivity Laws

Universal quantifier is distributive over $\Rightarrow$ in **one direction only**:

5. $(\forall x(A(x) \Rightarrow B(x)) \Rightarrow (\forall x\, A(x) \Rightarrow \forall x\, B(x)))$

Other direction implication **counter example**:

Take: **X = R**, **A(x): x < 0** and **B(x): x+1 >0**

$(\forall x(x < 0) \Rightarrow \forall x(x+1 > 0))$ is a **True** statement in set **R** of real numbers and

$\forall x(x < 0 \Rightarrow x+1 > 0)$ is a **False** statement:

take $x = -2$, we get $(-2 < 0 \Rightarrow -2+1 > 0)$ False

Introduction and Elimination Laws

B - **Formula without free variable x**

6. $\forall x(A(x) \Rightarrow B) \equiv (\exists x\, A(x) \Rightarrow B)$
7. $\exists x(A(x) \Rightarrow B) \equiv (\forall x\, A(x) \Rightarrow B)$
8. $\forall x(B \Rightarrow A(x)) \equiv (B \Rightarrow \forall x\, A(x))$
9. $\exists x(B \Rightarrow A(x)) \equiv (B \Rightarrow \exists x\, A(x))$

Introduction and Elimination Laws

B - Formula without free variable x

10. $\forall x(A(x) \lor B) \equiv (\forall x\, A(x) \lor B)$

11. $\forall x(A(x) \land B) \equiv (\forall x\, A(x) \land B)$

12. $\exists x(A(x) \lor B) \equiv (\exists x\, A(x) \lor B)$

13. $\exists x(A(x) \land B) \equiv (\exists x\, A(x) \land B)$

Remark: we prove **6 -9** from **10 – 13** + de Morgan + definability of implication

TRUTH SETS

We use truth sets for predicates to define an intuitive semantics for predicate logic

Given a set $X \neq \phi$ and a predicate $P(x)$, the set

$$\{x \in X: P(x)\}$$

is called a truth set for the predicate $P(x)$ in the domain $X \neq \phi$

Truth Sets, Interpretations

Example

Take $P(x)$ as $x+1 = 3$

– **it is called an interpretation of** $P(x)$ in a set **$X \neq \phi$**

Let $X=\{1, 2, 3\}$ then the **truth set** for $P(x)$ is

$$\{x \in X: P(x)\} = \{x \in X: x+1 = 3\} = \{2\}$$

and we say that $P(x)$ is **TRUE** in the set X **under the interpretation** $P(x): x+1 = 3$

TRUTH SETS semantics for Connectives

We use truth sets for predicates **always for** $X \neq \phi$

Conjunction:

$$\{x \in X: (P(x) \wedge Q(x))\} = \{x: P(x)\} \wedge \{x: Q(x)\}$$

Truth set for conjunction $(P(x) \wedge Q(x))$ is the set **intersection** of truth sets for its components.

Disjunction:

$$\{x \in X: (P(x) \vee Q(x))\} = \{x: P(x)\} \vee \{x: Q(x)\}$$

Truth set for disjunction $(P(x) \vee Q(x))$ is the set **union** of **truth sets** for its components.

Negation:

$$\{x \in X: \neg P(x)\} = X - \{x \in X: P(x)\}$$

$\neg$ is the negation

and $-$ is the **set complement** relative to X

Truth sets semantics for Connectives

Implication:

$\{x \in X: (P(x) \Rightarrow Q(x))\}$ = $X - \{x:P(x)\} \vee \{x : Q(x)\}$

$= \{x: \neg P(x)\} \vee \{x : Q(x)\}$

Example:

$\{x \in N: n>0 \Rightarrow n^2 <0\} = \{x \in N\ x \leq 0\} \vee \{x \in N : n^2<0\}$

$= \phi \vee \phi = \phi$

Truth Sets Semantics for Universal Quantifier

Definition:

$$\forall x\, A(x) = T \quad \text{iff} \quad \{x \in X : A(x)\} = X$$

where

$X \neq \phi$ and $A(x)$ is any formula with a free variable x

Definition:

$$\forall x\, A(x) = F \quad \text{iff} \quad \{x \in X : A(x)\} \neq X$$

where

$X \neq \phi$ and $A(x)$ is any formula with a free variable x

Truth Sets semantics for Existential Quantifier

Definition:

$\exists x\, A(x) = T$ (in $x \neq \phi$) iff $\{x \in X : A(x)\} \neq \phi$

Definition:

$\exists x\, A(x) = F$ (in $x \neq \phi$) iff $\{x \in X : A(x)\} = \phi$

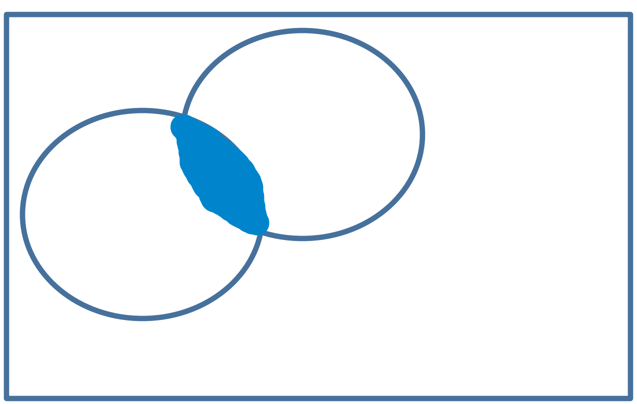
Where $X \neq \phi$ and **$A(x)$ is a formula with a free variable x**

Venn Diagrams For **Existential Quantifier** and **Conjunction**

$\exists x(A(x) \wedge B(x))=T$ iff **$\{x:A(X)\} \wedge \{x:B(x)\} \neq \Phi$**

Picture

$X \neq \Phi$ observe that **$\{x:A(X)\}$** $\neq \Phi$ and **$\{x:B(x)\} \neq \Phi$**

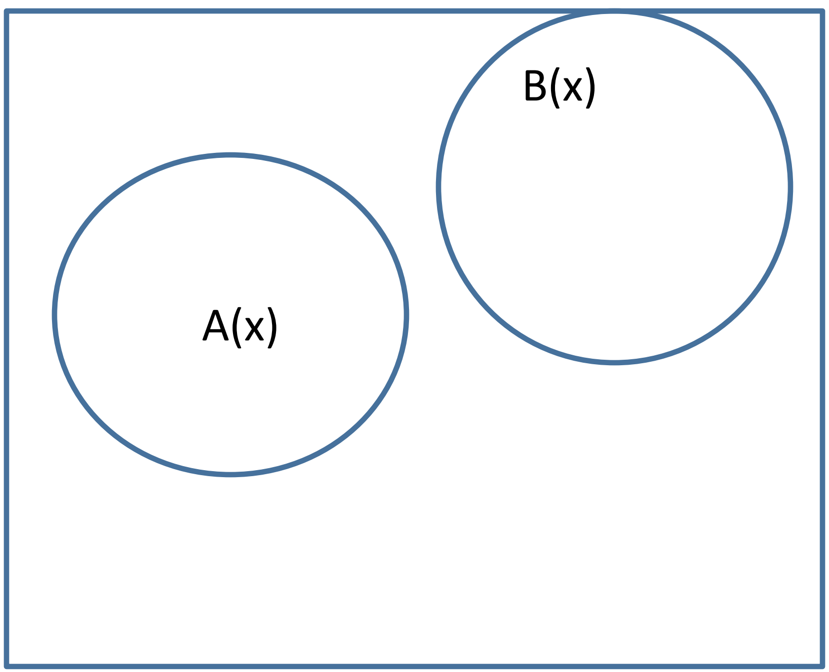


Venn Diagrams For Existential Quantifier and Conjunction

$\exists x(A(x) \wedge B(x)) = F$ iff $\{x:A(x)\} \wedge \{x:B(x)\} = \Phi$

Picture

$X \neq \Phi$



Remember {x:A(x)}, {x:B(x)} now can be Φ!

$X \neq \Phi$

Venn Diagrams For Universal Quantifier and Implication

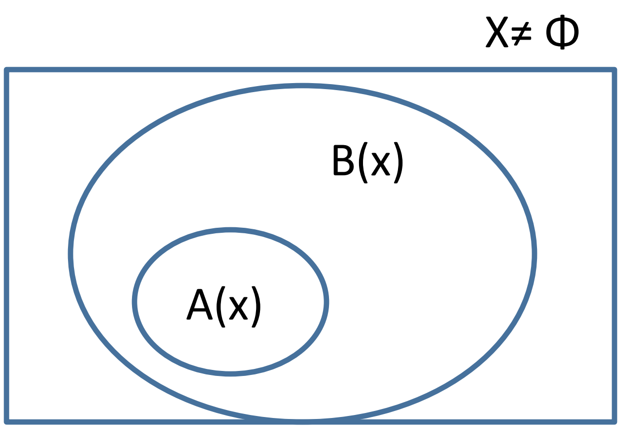
Observe that

$\forall x\,(A(x) \Rightarrow B(x)) = T$ iff $\{x \in X : A(x) \Rightarrow B(x)\} = X$

Iff

$\{x : A(x)\} \subseteq \{x : B(x)\}$

Picture



Remember that {x:A(x)}, {x:B(x)} now can be Φ!

Exercise

Draw a picture for a situation where (in $X \neq \Phi$)

1. $\exists x\, P(x) = T$
2. $\exists x\, Q(x) = T$
3. $\exists x (P(x) \wedge Q(x)) = F$
4. $\forall x\, (P(x) \vee Q(x) = F$

Exercise Solution

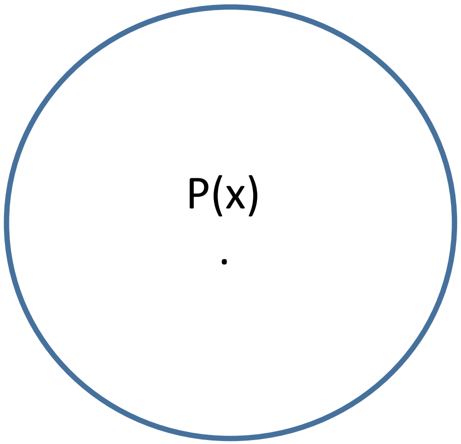
1. $\exists x\, P(x) = T$ iff $\{ x:P(x)\} \neq \Phi$

2. $\exists x\, Q(x) = T$ iff $\{x:Q (x)\} \neq \Phi$

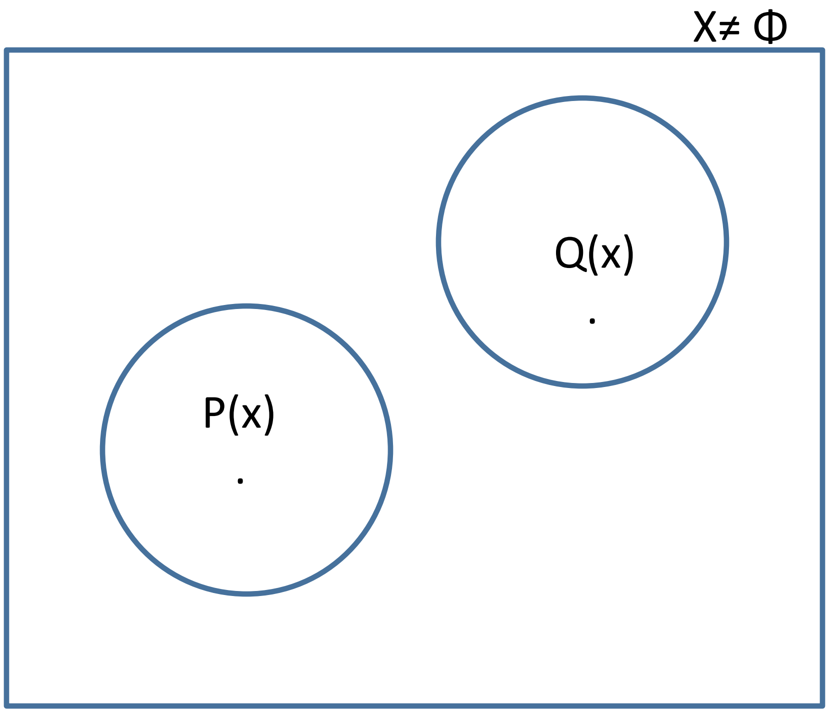
3. $\exists x(P(x) \wedge Q(x)) = F$ iff $\{x: P(x)\} \wedge \{x: Q(x)\} = \Phi$

4. $\forall x\, (P(x) \vee Q(x) = F$ iff $\{x:P(x)\} \vee \{ x:Q(x)\} \neq X$

Picture:



Denotes {x: P(x)} ≠ Φ



Proving Predicate Tautologies with **TRUTH Sets**

Prove that

$$\models (\forall x\, A(x) \Rightarrow \exists x\, A(x))$$

Proof:

Assume that not true

(Proof by contradiction) i.e. that there are $X \neq \Phi$, $A(x)$ such that.

$(\forall x\, A(x) \Rightarrow \exists x\, A(x)) = \mathbf{F}$

iff $\forall x\, A(x) = T$ and $\exists x\, A(x) = F$ $\quad (A \Rightarrow B) = F$

iff $X \neq \phi$ and

$\{x \in X : A(x)\} = X$ and $\{x \in X : A(x)\} = \phi$

iff $X = \phi$

Contradiction with $X \neq \phi$, hence proved.

Proving Predicate Tautologies with TRUTH Sets

Prove:

$$\neg \forall x\, A(x) \equiv \exists x\, \neg A(x)$$

Case1: $\exists x\, \neg A(x) = T$ in $X \neq \phi$ **iff** $\{x: \neg A(x)\} \neq \phi$ **iff** $X - \{x: A(x)\} \neq \phi$ **iff** $\{x: A(x)\} \neq X$ **iff** $\forall x\, A(x) = F$ **iff** $\neg \forall x\, A(x) = T$

Case2: $\exists x\, \neg A(x) = F$ in $X \neq \phi$ **iff** $\{x: \neg A(x)\} = \phi$ **iff** $X - \{x: A(x)\} = \phi$ **iff** $\{x: A(x)\} = X$ **iff** $\forall x\, A(x) = T$ **iff** $\neg \forall x\, A(x) = F$

Prove

$\exists x(A(x) \vee B(x)) \equiv \exists x\, A(x) \vee \exists x\, B(x)$

Case 1: $\exists x(A(x) \vee B(x)\} = T$ iff

$\{x: (A(x) \vee B(x)\} \neq \phi$ (definition)

$= \{x: (A(x)\} \vee \{x: (B(x)\} \neq \phi$ iff

$\{x: A(x)\} \neq \phi$ or $\{x: B(x)\} \neq \phi$ iff

$= \exists x\, A(x)=T$ or $\exists x\, B(x)=T$

We used: for any sets, $A \vee B \neq \phi$ iff

$A \neq \phi$ or $B \neq \phi$

Case2 – similar

Russell's Paradox

We assumed in our approach that for any statement $A(x)$

the TRUTH set

$$\{x \in X: A(x)\} \quad \text{exists}$$

Russell Antinomy showed that that technique of TRUTH sets is **not sufficient**

This is why we need a proper semantics!