Calculus of Variations

Example: Shortest path between two points

Choose points (0,*a*) and (1,*b*).

$$F(u,u') = \sqrt{1+(u')^2}$$

Euler-Lagrange equation:

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = 0$$

$$||$$

$$\frac{1}{2}\left(1+u'^2\right)^{-1/2}.2u'$$

$$||$$

$$\frac{d}{dx}\left(\frac{u'}{\sqrt{1+u'^2}}\right)$$

$$||$$

$$\underbrace{\frac{u_{xx}}{\left(1+u_x^2\right)^{3/2}}} = 0$$

Therefore:

$$\begin{aligned} u_{xx} &= 0 \\ u(x) &= \alpha x + \beta \\ u(0) &= \beta = a \\ u(1) &= \alpha + \beta \\ &= \alpha + a \quad \alpha = b - a \\ &= b \end{aligned}$$

$$\boxed{u(x) = (b-a)x + a}$$

Two dimension Problems: Works the same

$$P(u) = \iint \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 dxdy$$

$P(u+v) > P(u)$ with proper boundary conditions. **Use integration by parts**.
Euler-Lagrange derivative is

$$\Delta u = 0.$$