Calculus of Variations

Example: Shortest path between two points

Choose points (0, a) and (1, b).

\[ F(u, u') = \sqrt{1 + (u')^2} \]

**Euler-Lagrange equation:**

\[
\frac{\partial F}{\partial u} - \frac{d}{dx} \left( \frac{\partial F}{\partial u'} \right) = 0
\]

\[
\frac{1}{2} \left( 1 + u'^2 \right)^{-1/2} \cdot 2u' - \frac{d}{dx} \left( \frac{u'}{\sqrt{1 + u'^2}} \right) = 0
\]

\[
\frac{u_{xx}}{\left( 1 + u'^2 \right)^{3/2}} = 0
\]

Therefore:

\[ u_{xx} = 0 \]
\[ u(x) = \alpha x + \beta \]
\[ u(0) = \beta = a \]
\[ u(1) = \alpha + \beta = \alpha + a \quad \alpha = b - a \]
\[ = b \]

Two dimension Problems: Works the same

\[ P(u) = \iint \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \, dx \, dy \]

\[ P(u + v) > P(u) \] with proper boundary conditions. **Use integration by parts.**

Euler-Lagrange derivative is

\[ \Delta u = 0. \]