Conformal Mapping for Registration and Visualization: Laplace-Beltrami Approach

Surface Deformations and Flattening

- Conformal and Area-Preserving Maps
  - Optical Flow
- Gives Parametrization of Surface
  - Registration
- Shows Details Hidden in Surface Folds
- Path Planning
  - Fly-Throughs
- Medical Research
  - Brain, Colon, Bronchial Pathologies
  - Functional MR and Neural Activity
- Computer Graphics and Visualization
  - Texture Mapping

Mathematical Theory of Surface Mapping

❑ Conformal Mapping:

- One-one
- Angle Preserving
- Fundamental Form $(E, F, G) \rightarrow \rho(E, F, G)$

❑ Examples of Conformal Mappings:

- One-one Holomorphic Functions
- Spherical Projection

❑ Uniformization Theorem:

- Existence of Conformal Mappings
- Uniqueness of Mapping

Deriving the Mapping Equation

Let p be a point on the surface Σ. Let

$$z : \Sigma \to S^2$$

be a conformal equivalence sending p to the North Pole.

Introduce **Conformal Coordinates** (u, v) near p,
with $u = v = 0$ at p.

In these coordinates, $ds^2 = \lambda(u,v)^2 \left(du^2 + dv^2 \right)$

We can ensure that $\lambda(p) = 1$.

In these coordinates, the Laplace Beltrami operator takes the form

$$\Delta = \frac{1}{\lambda(u,v)^2} \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right).$$

Deriving the Equation-Continued

Set $w = u + iv$. The mapping $z = z(w)$ has a simple pole at $w = 0$, i.e. at p.

Near p, we have a Laurent series $z(w) = \frac{A}{w} + B + C + Dw^2 + \ldots$

Apply Δ to get $\Delta z = A\Delta\left(\frac{1}{w}\right)$.

Taking $A = \frac{1}{2\pi}$,

$$\begin{aligned}
\Delta z &= \frac{1}{2\pi}\Delta\left(\frac{1}{w}\right) \\
&= \frac{1}{2\pi}\Delta\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)\log|w| \\
&= \frac{1}{2\pi}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)\Delta \log|w| \\
&= \frac{1}{2\pi}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)(2\,\pi\,\delta_p)
\end{aligned}$$

The Mapping Equation

$$\Delta z = \left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right) \delta_p \; .$$

Simply a second order linear PDE. Solvable by standard methods.

Finite Elements-I

Σ is a triangulated surface. Start with

$$\Delta z = \left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right) \delta_p$$

Multiply by an arbitrary smooth f and integrate by parts. For all f we want:

$$\int\int_\Sigma \nabla z \cdot \nabla f \; dS = \int\int_\Sigma \left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right) \delta_p \; f \; dS$$

$$= \frac{\partial f}{\partial u}(p) - i \frac{\partial f}{\partial v}(p)$$

Let $z, f \in PL(\Sigma)$, the space of piecewise linear functions.

Finite Elements-II

For each vertex $P \in \Sigma$, let ϕ_p be the continuous function such that:

$$\begin{cases} \phi_P(P) = 1 \\ \phi_P(Q) = 0, Q \neq P, Q \text{ a vertex,} \\ \phi_{\mathrm{P}} \text{ is linear on each triangle.} \end{cases}$$

These functions form a basis for the finite dimensional space $PL(\Sigma)$.

Then $z = \Sigma_P \; z_P \phi_P$.
And we want, for all Q,

$$\Sigma_P z_P \int\int \nabla \phi_P \cdot \nabla Q dS = \frac{\partial \phi_Q}{\partial u}(p) - \frac{\partial \phi_Q}{\partial v}(p)$$

This is simply a matrix equation.

Finite Elements-III

Set $D = \left(D_{PQ} \right)$, $D_{PQ} = \int\int \nabla \phi_P \cdot \nabla \phi_Q dS$.

Define vectors

$$a = \left(a_Q \right) = \left(\frac{\partial \phi_Q}{\partial u}(p) \right),$$

$$b = \left(b_Q \right) = \left(\frac{\partial \phi_Q}{\partial v}(p) \right).$$

Our equation becomes simply $Dz = a - ib$.

$$D_{PQ} = -\frac{1}{2}\{\cot \angle R + \cot \angle S\},$$

$$D_{PP} = -\Sigma_{P \neq Q} D_{PQ}.$$

Need formulas for

a, b.

Finite Elements-IV

Suppose the point p lies on a triangle with vertices ABC.

Since $a = \left(a_Q \right) = \left(\frac{\partial \phi_Q}{\partial u}(p) \right)$,

and $b = \left(b_Q \right) = \left(\frac{\partial \phi_Q}{\partial v}(p) \right)$,

we have $a_Q - ib_Q = 0$ if $Q \notin \{A, B, C\}$.

Finite Elements-V

If $Q \in \{A, B, C\}$, then considering that ϕ_Q is linear on ABC:

$$a_Q - ib_Q := \begin{cases} \dfrac{-1}{\|B-A\|} + i\dfrac{1-\theta}{\|C-E\|} & Q = A, \\ \dfrac{1}{\|B-A\|} + i\dfrac{\theta}{\|C-E\|} & Q = B, \\ i\dfrac{-1}{\|C-E\|} & Q = C, \end{cases}$$

$$\theta = \frac{\langle C-A, B-A \rangle}{\|B-A\|^2}$$

Finite Elements-VI

If we set $z = x + iy$, then our system $Dz = a - ib$ becomes $Dx = a$ and $Dy - b$.

D is sparse, real, symmetric and positive semi-definite. Its kernel is the space of constant vectors, and it is positive definite on the space orthogonal to its kernel.

These properties of D allow us to use the conjugate gradient method to solve the system.

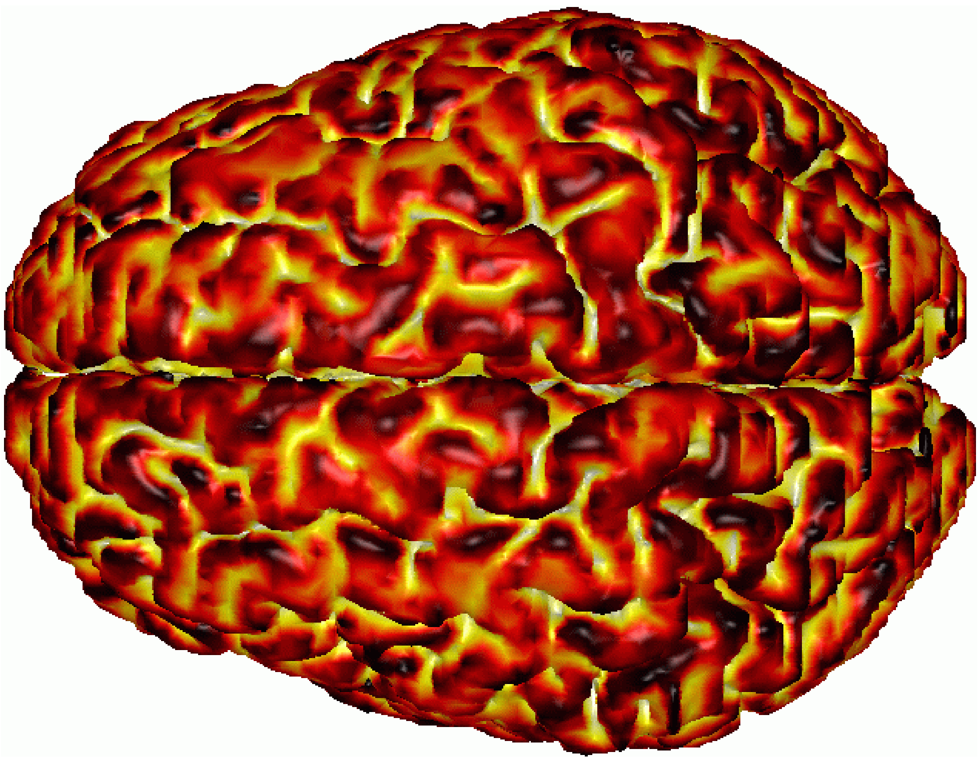
Summary of Flattening

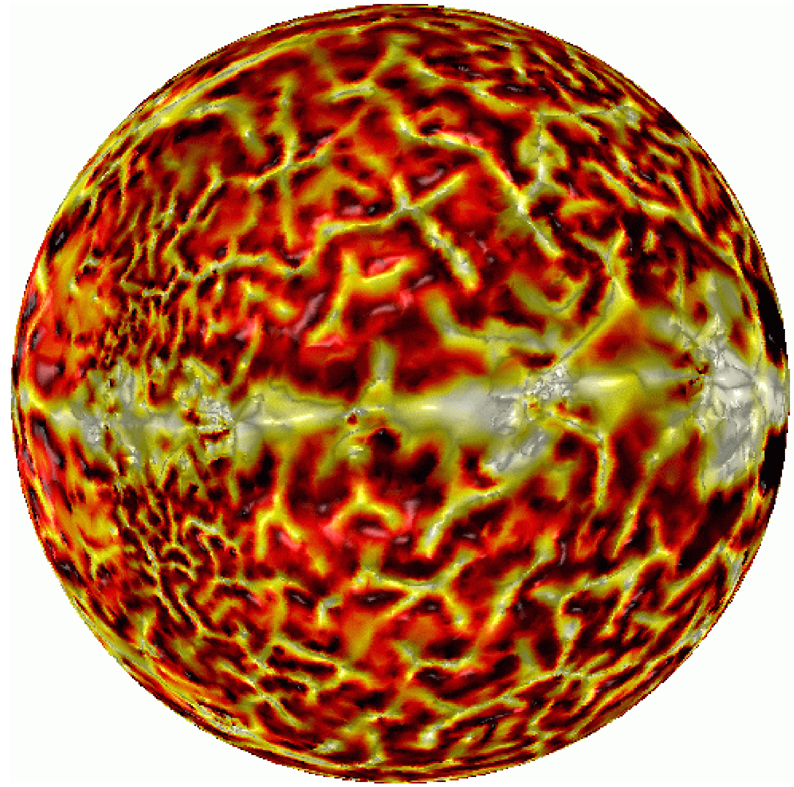
Flattening:
Calculate the elements of the matrices $D, a,$ and b.

Use the conjugate gradient method to solve $Dx = a$ and $Dy = -b$.
The resulting $z = x + iy$ is the conformal mapping to the complex plane.

Compose z with inverse stereo projection to get a conformal map to the unit sphere.

Cortical Surface Flattening-Normal Brain





White Matter Segmentation and Flattening

Conformal Mapping of Neonate Cortex

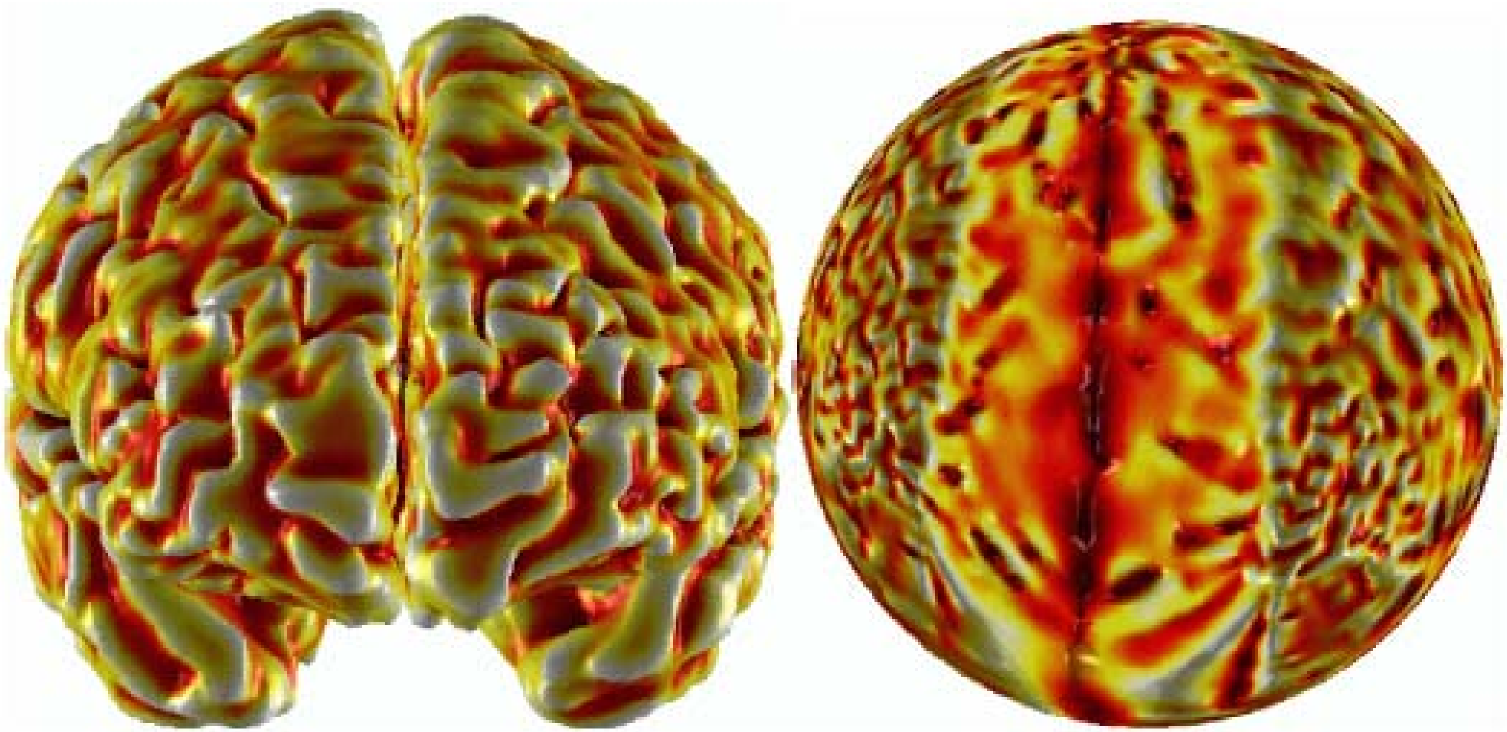
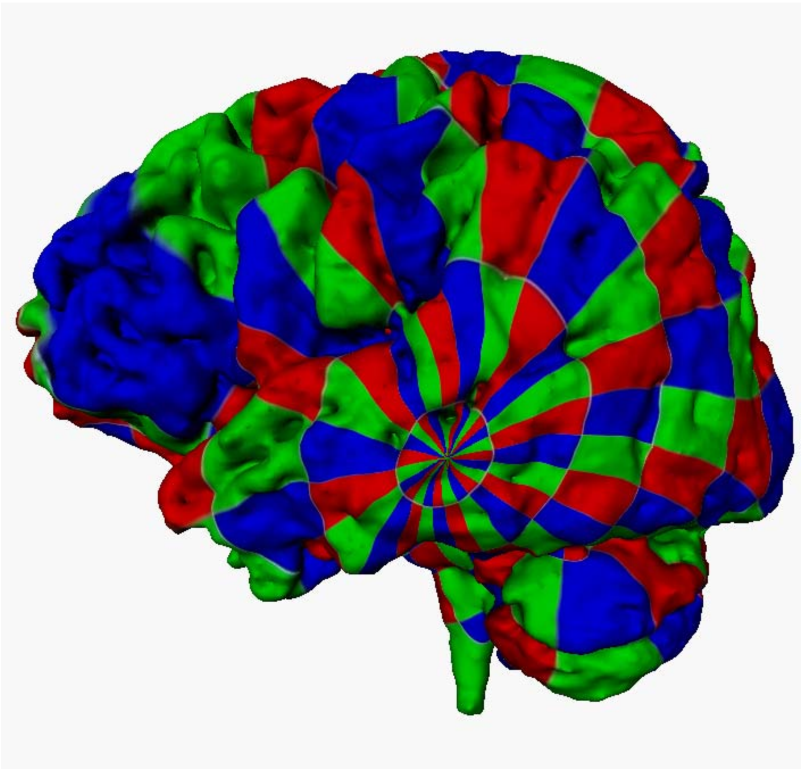
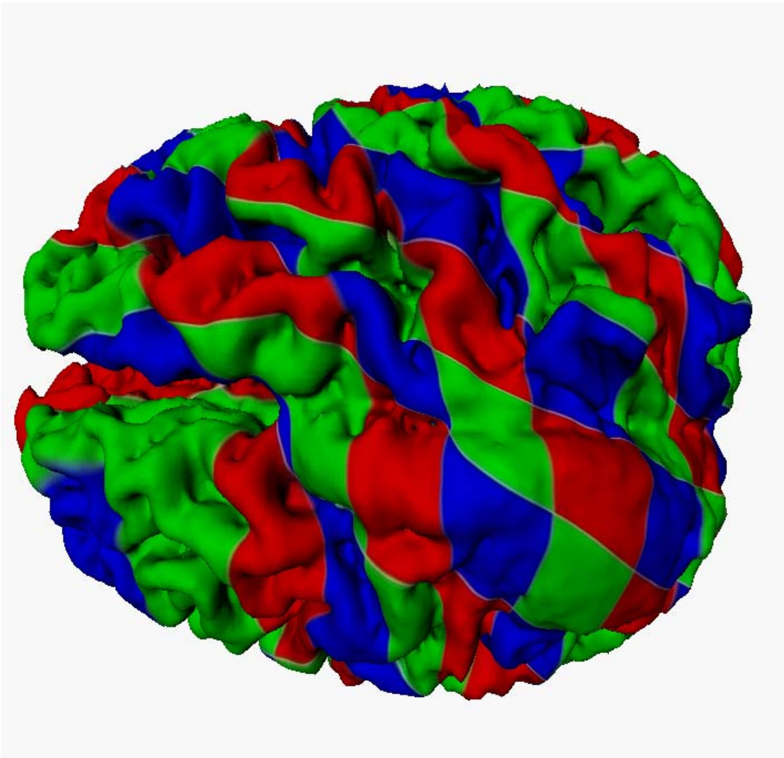


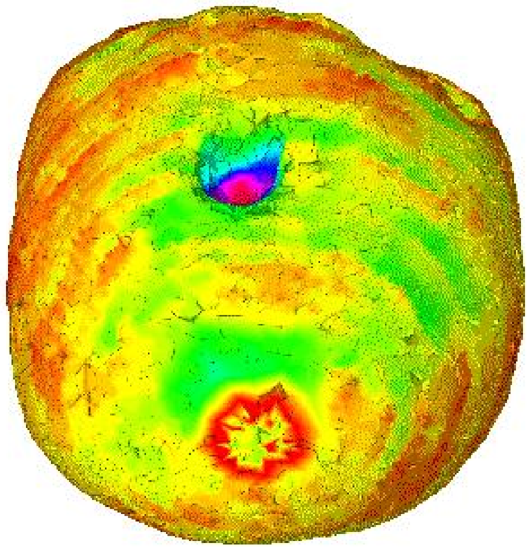
Figure 8.4.5-12
Conformal mapping of the neonate cortical surface to the sphere. The shading scheme represents mean curvature.

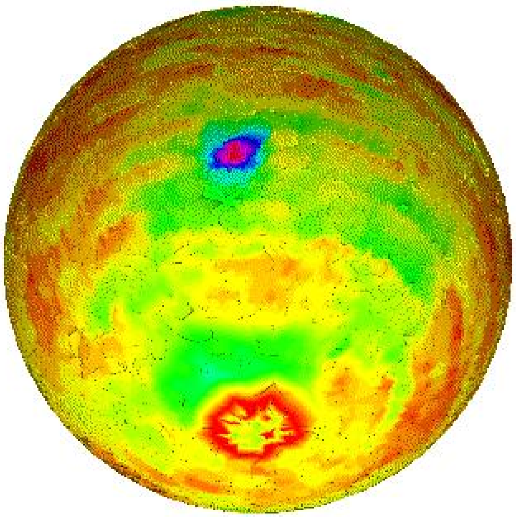
Coordinate System on Cortical Surface



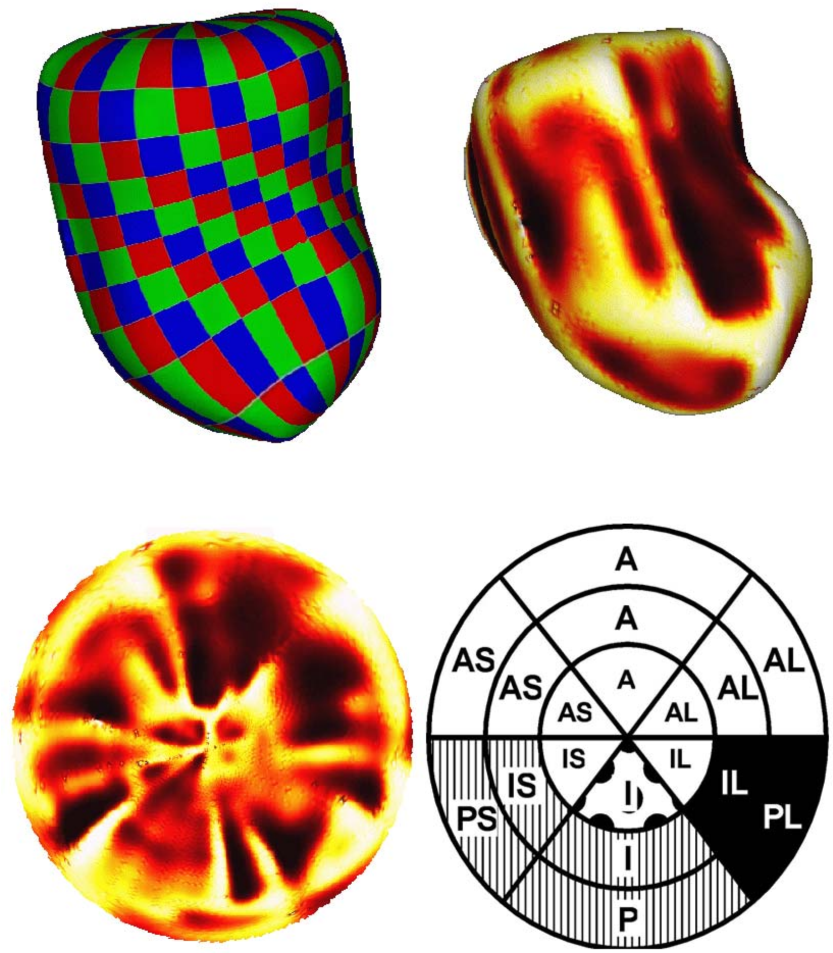


Bladder Flattening





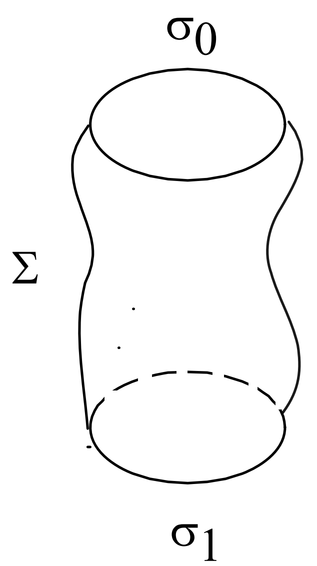
3D Ultrasound Cardiac Heart Map



Flattening a Tube

(1) Solve

$$\Delta u = 0 \quad on \quad \Sigma \setminus (\sigma_0 \cup \sigma_1)$$
$$u = 0 \quad on \quad \sigma_0$$
$$u = 1 \quad on \quad \sigma_1$$



(2) Make a cut from σ_0 to σ_1.

Make sure u is increasing along the cut.

Flattening a Tube-Continued

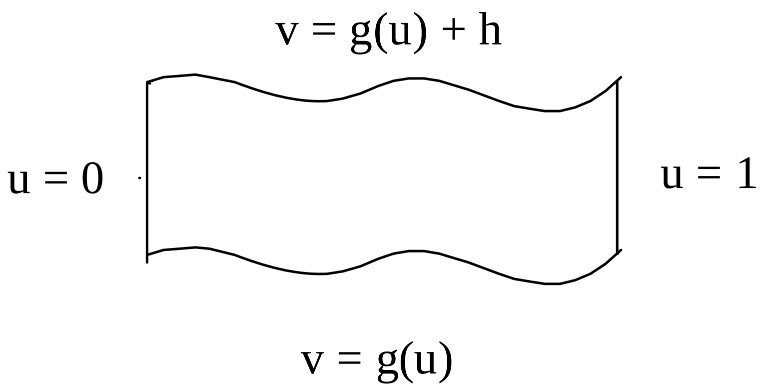
(3) Calculate v **on the boundary loop**

$$\sigma_0 \rightarrow cut \rightarrow \sigma_1 \rightarrow cut \rightarrow \sigma_0$$

by integration

$$v(\xi) = \int^{\xi} \frac{\partial v}{\partial s} ds = \int^{\xi} \frac{\partial u}{\partial n} ds$$

(4) Solve Dirichlet problem using boundary values of v **.**



If you want, scale so $h = 2\pi$, **take** e^{u+iv} **to get an annulus.**

Flattening Without Distortion-I

In practice, once the tubular surface has been flattened into a rectangular shape, it will need to be visually inspected for pathologies. We present a simple technique by which the entire colon surface can be presented to the viewer as a sequence of images or cine. In addition, this method allows the viewer to examine each surface point without distortion at some time in the cine. Here, we will say a mapping is without distortion at a point if it preserves the intrinsic distance there.

It is well known that a surface cannot in general be flattened onto the plane without some distortion somewhere. However, it may be possible to achieve a surface flattening which is free of distortion along some curve. A simple example of this is the familiar Mercator projection of the earth, in which the equator appears without distortion. In our case, the distortion free curve will be a level set of the harmonic function (essentially a loop around the tubular colon surface), and will correspond to the vertical line through the center of a frame in the cine. This line is orthogonal to the "path of flight" so that every point of the colon surface is exhibited at some time without distortion.

Flattening Without Distortion-II

Suppose we have conformally flattened the colon surface onto a rectangle

$R = [0, u_{\max}] \times [-\pi, \pi]$.

Let F be the inverse of this mapping, and let $\phi^2 = \phi^2(u,v)$ be the amount by Which F scales a small area near (u,v), i.e. let $\phi > 0$ be the "conformal factor" for F.

Fix $w > 0$, and for each $u_0 \in [0, u_{\max}]$ define a subset $R_0 = ([u_0 - w, u_0 + w] \times [-\pi, \pi]) \cap R$ which will correspond to the contents of a cine frame. We define a mapping

$$(\hat{u}, \hat{v}) = G(u,v) = \left(\int_{u_0}^{u} \phi(\mu, v)\, d\mu, \int_{0}^{v} \phi(u_0, v)\, dv \right).$$

Flattening Without Distortion-III

We have

$$dG(u,v) = \begin{pmatrix} \hat{u}_u & \hat{u}_v \\ \hat{v}_u & \hat{v}_v \end{pmatrix} = \begin{pmatrix} \phi(u,v) & \int_{u_0}^{u} \phi_v(\mu,v)\,d\mu \\ 0 & \phi(u_0,v) \end{pmatrix},$$

$$dG(u_0,v) = \phi(u_0,v) \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This implies that composition of the flattening map with G sends level set loop $\{u = u_0\}$ on the surface to the vertical line $\{\hat{u} = 0\}$ in the $\hat{u} - \hat{v}$ plane without distortion. In addition, it follows from the formula for dG that lengths measured in the $\hat{u}$ direction accurately reflect the lengths of corresponding curves on the surface.

Introduction: Colon Cancer

- US: 3rd most common diagnosed cancer
- US: 3rd most frequent cause of death
- US: 56.000 deaths every year

- Most of the colorectal cancers arise from preexistent adenomatous polyps

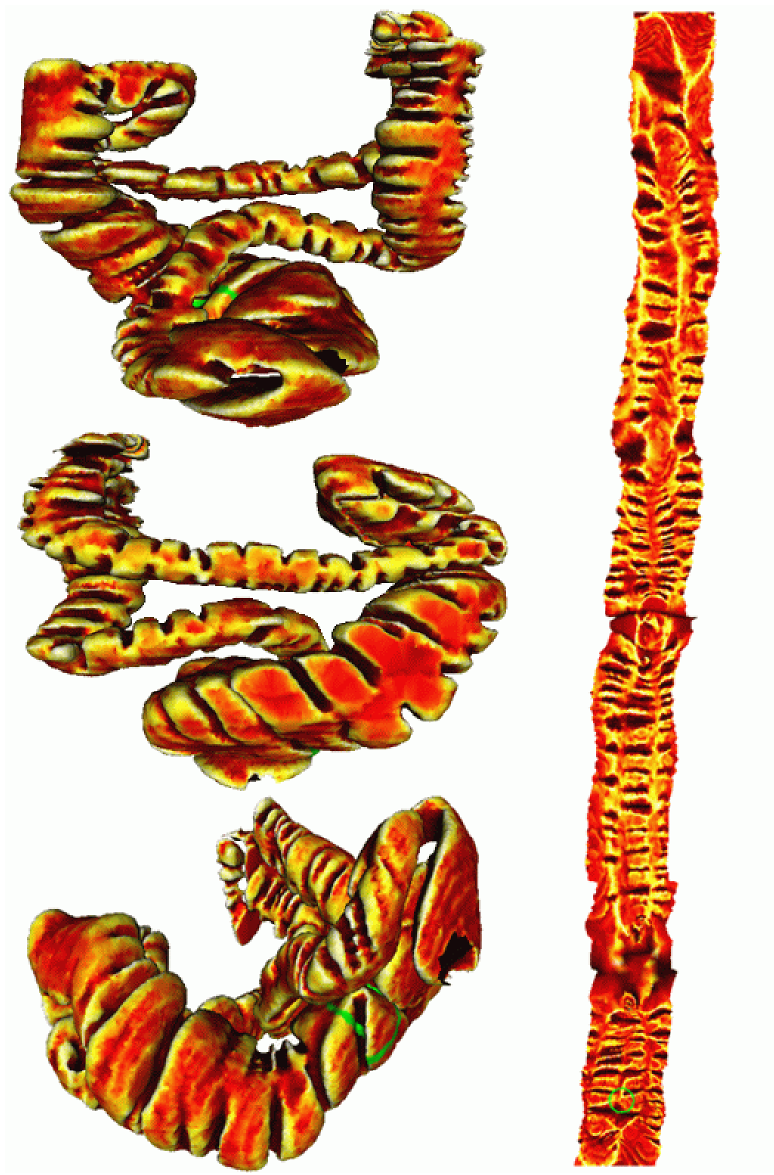
- Landis S, Murray T, Bolden S, Wingo Ph.Cancer Statitics 1999. Ca Cancer J Clin. 1999; 49:8-31.

Problems of CT Colonography

- Proper preparation of bowel
- How to ensure complete inspection

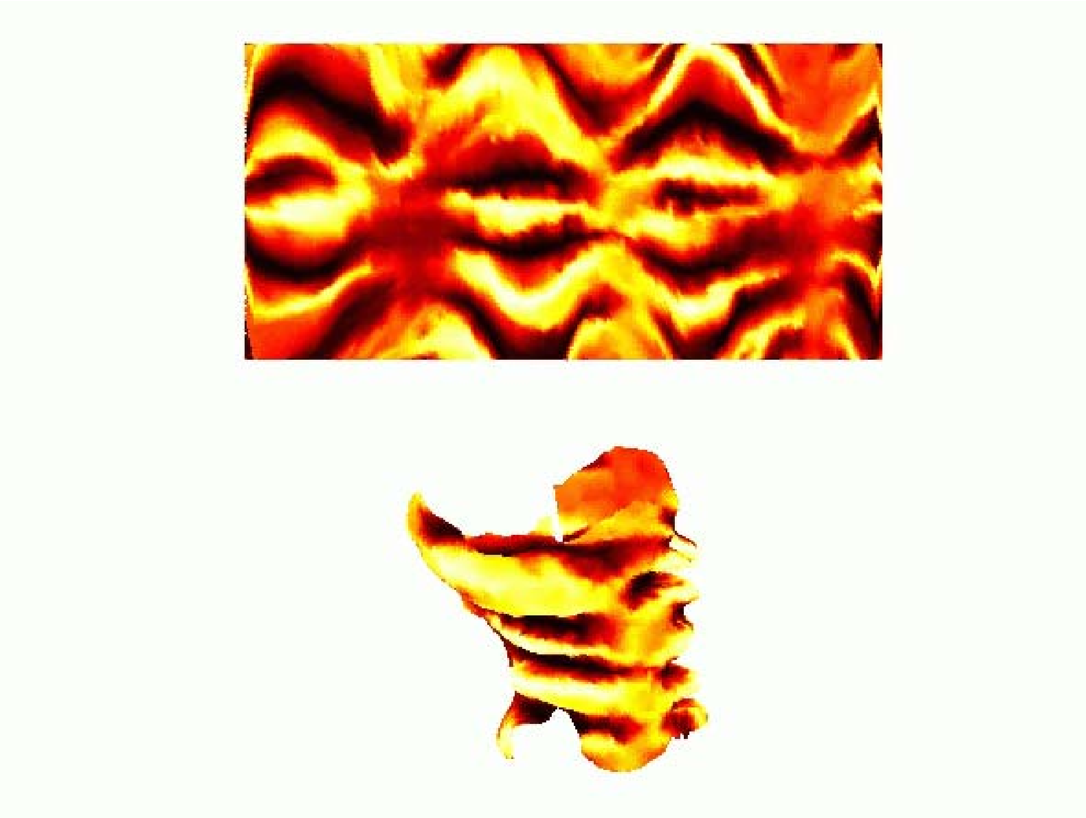
→ Nondistorting colon flattening program

Colon Segmentation and Flattening



Nondistorting colon flattening

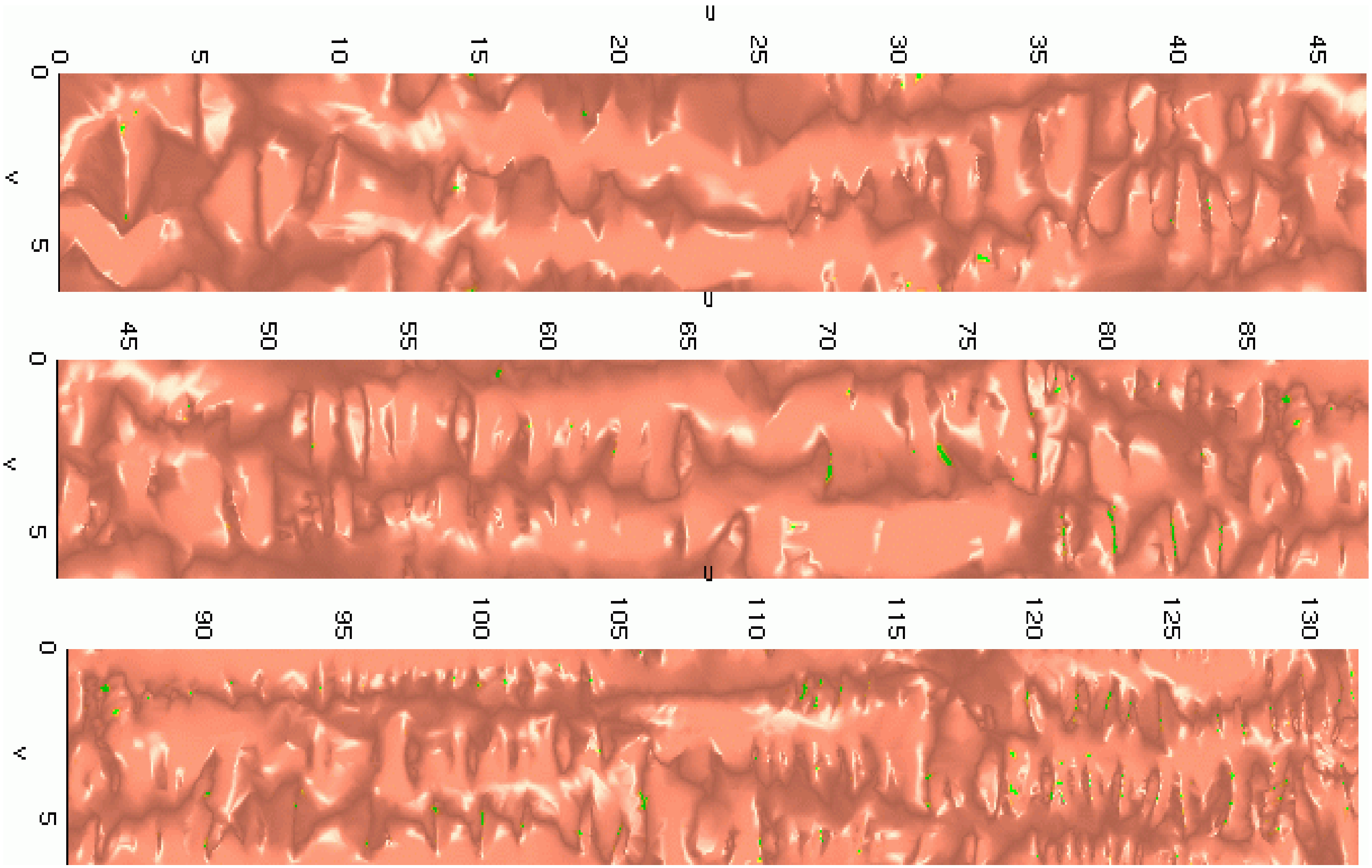
- Simulating pathologist' approach
- No Navigation is needed
- Entire surface is visualized



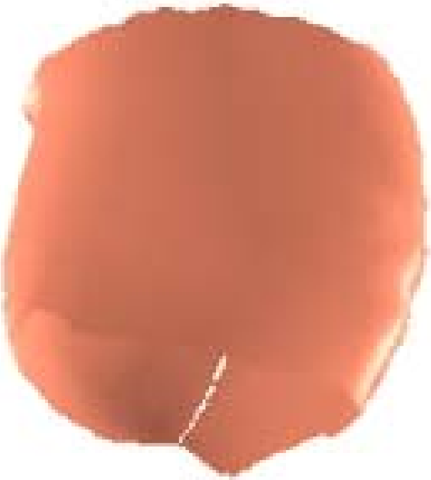
Nondistorting Colon Flattening

- Using CT colonography data
- Standard protocol for CT colonography
- 43 patients (28 m, 15 f)
- Mean age 70.2 years (from 50 to 82)

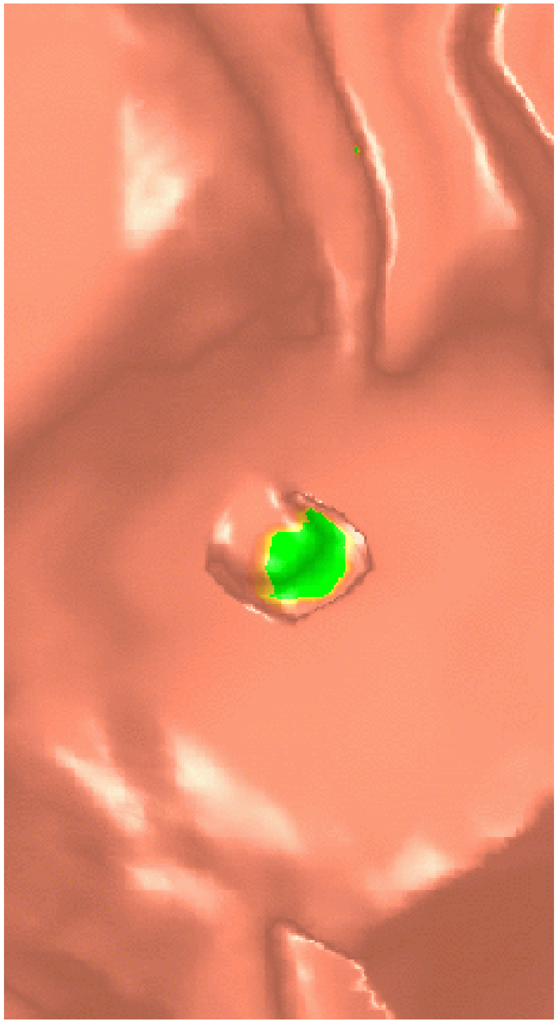
Flattened Colon

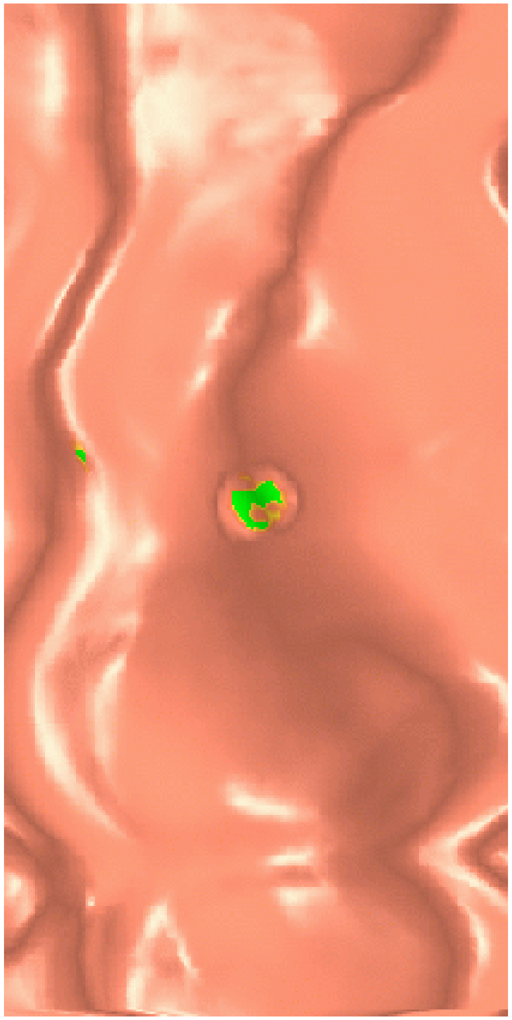


Colon Fly-Through Without Distortion

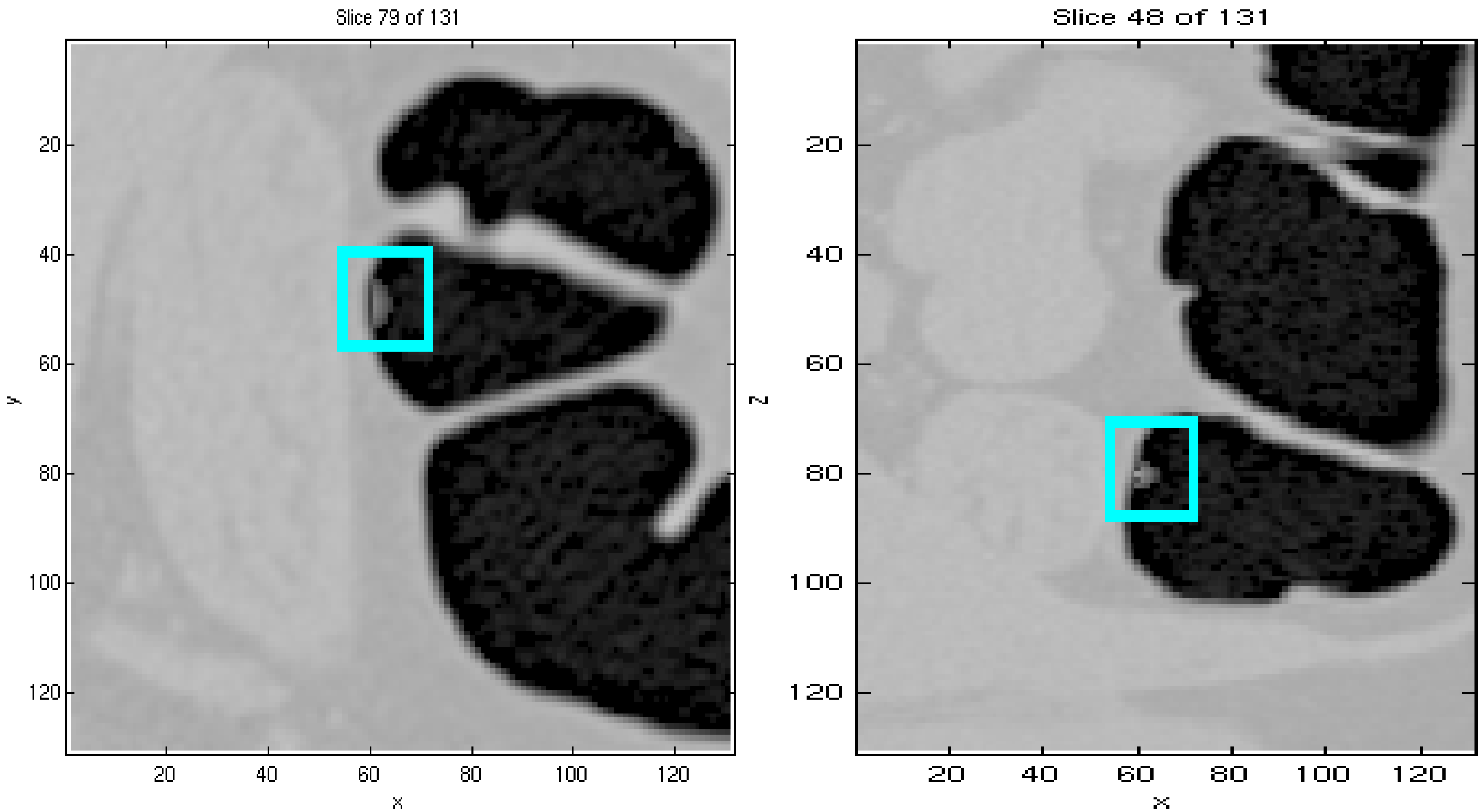


Polyps Rendering

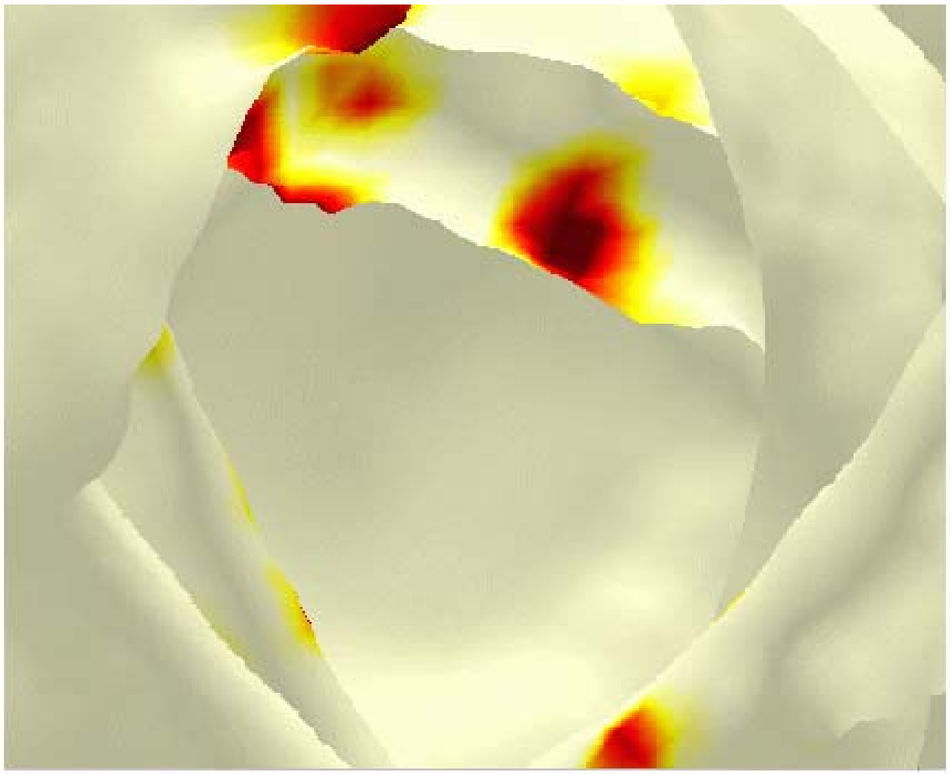


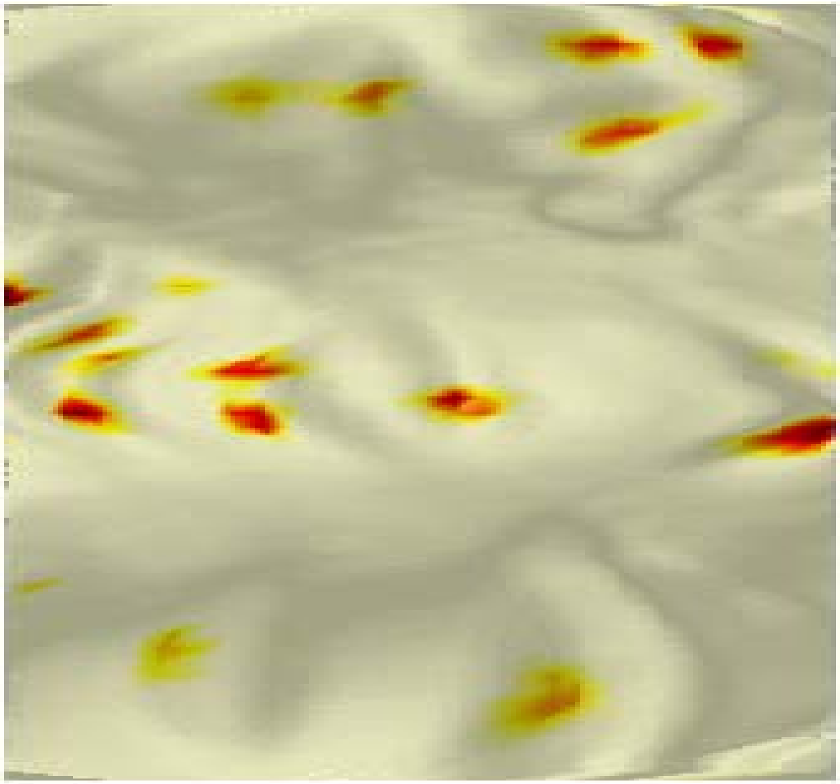


Finding Polyps on Original Images

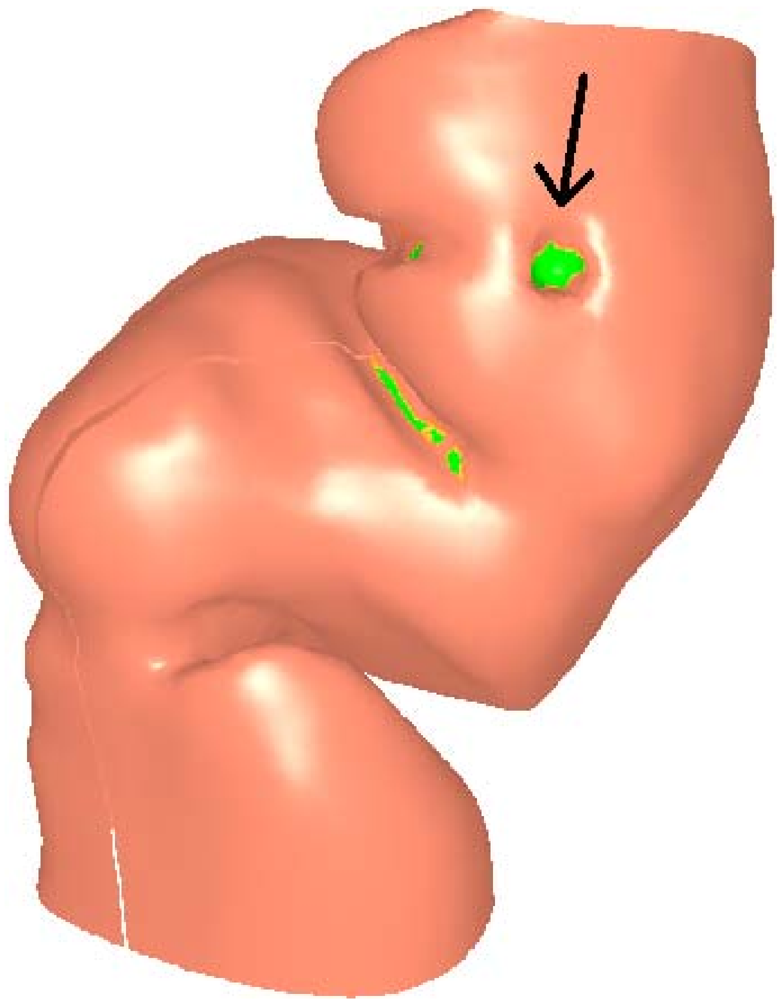


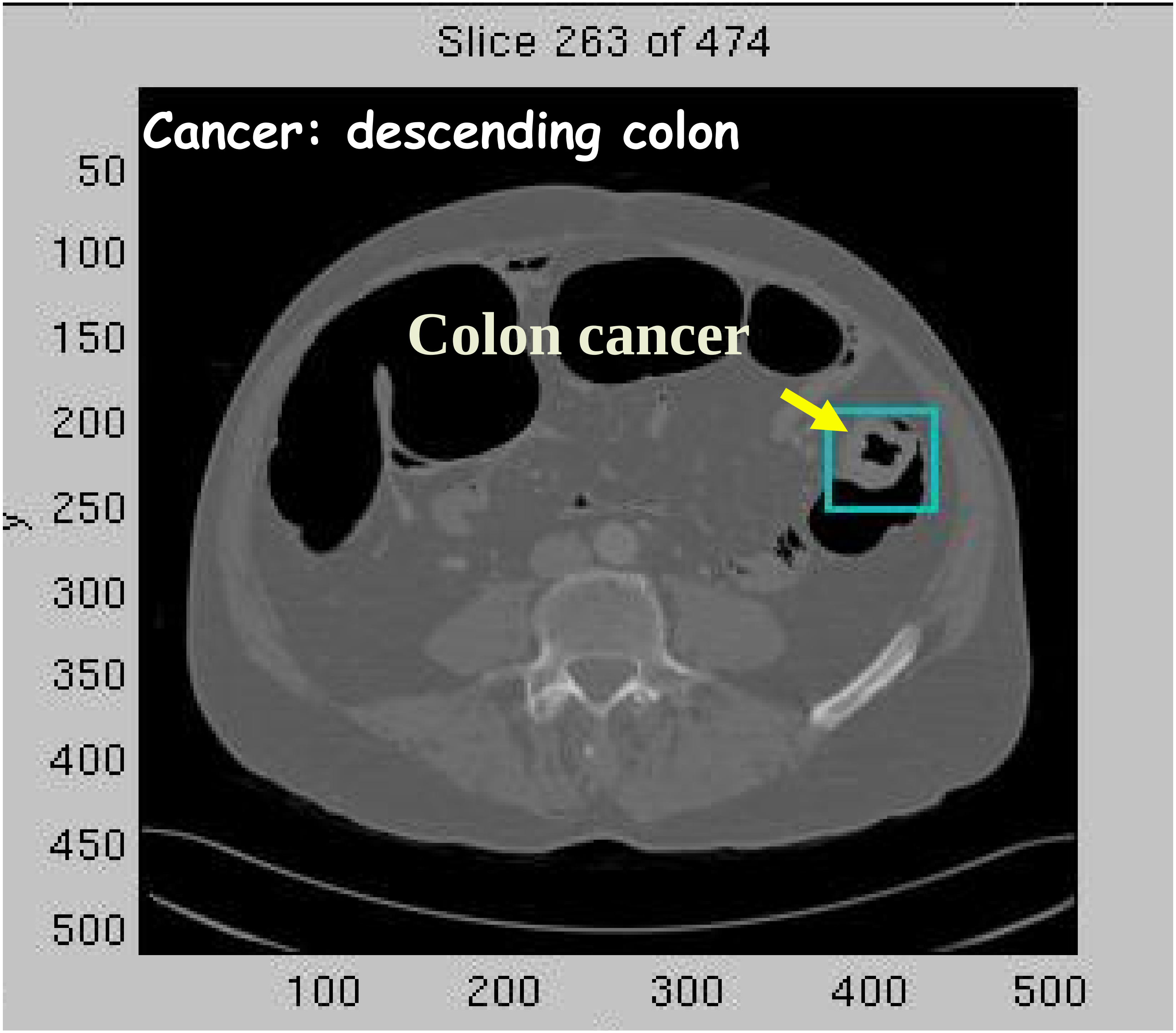
Polyp Detection





Polyp Highlighted




Slice 263 of 474
Cancer: descending colon
Colon cancer
50
100
150
200
250
300
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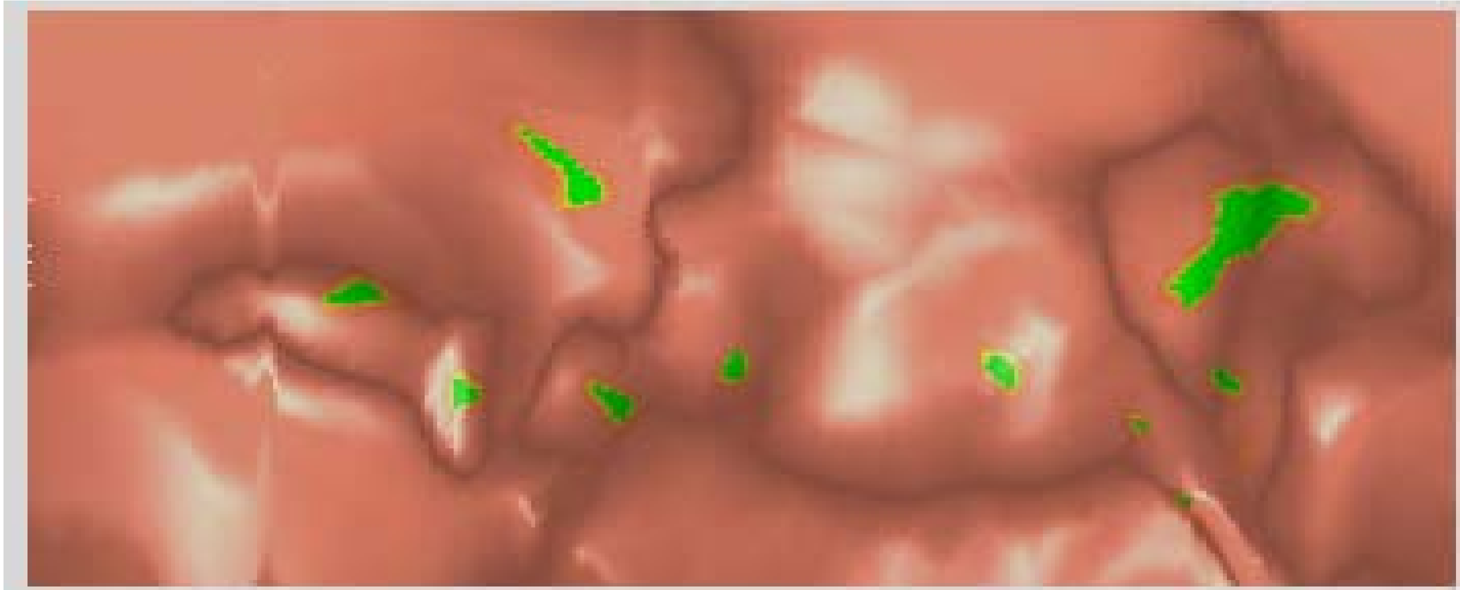
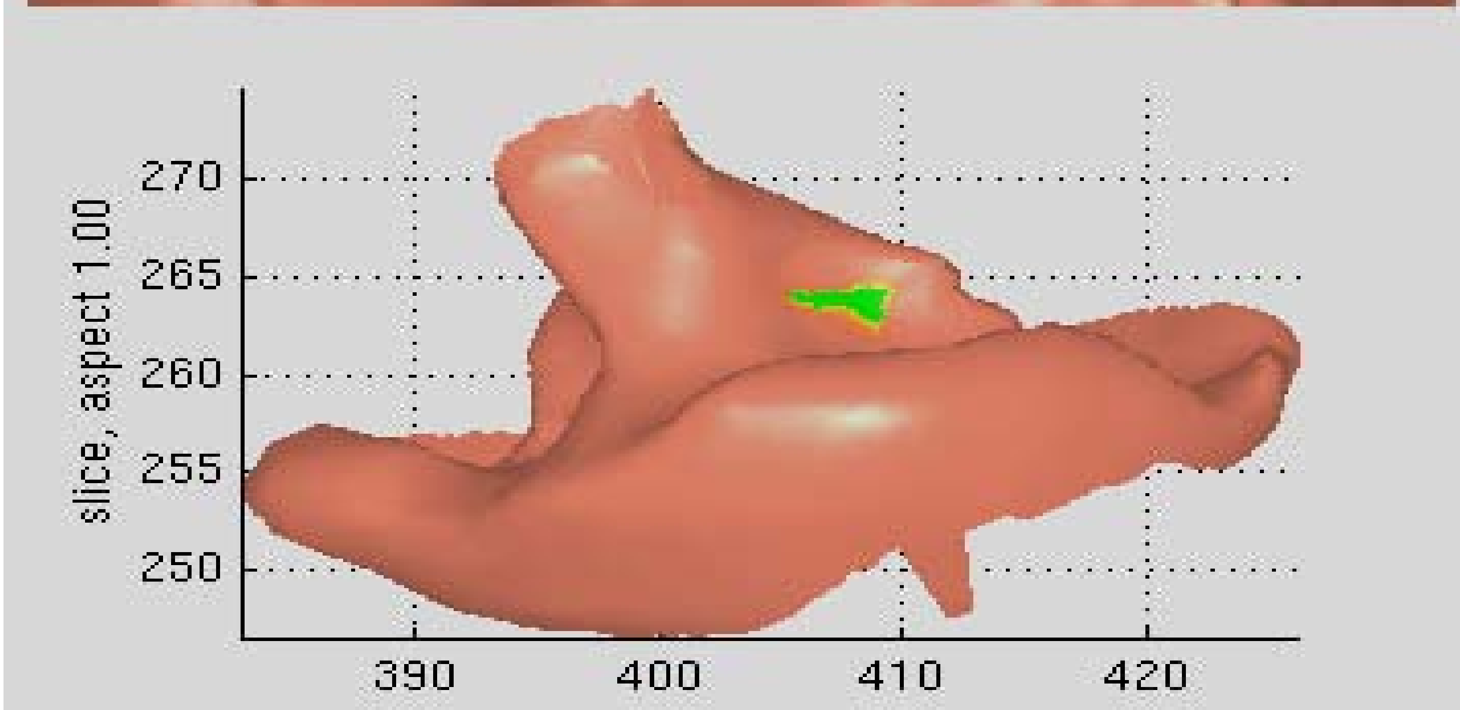
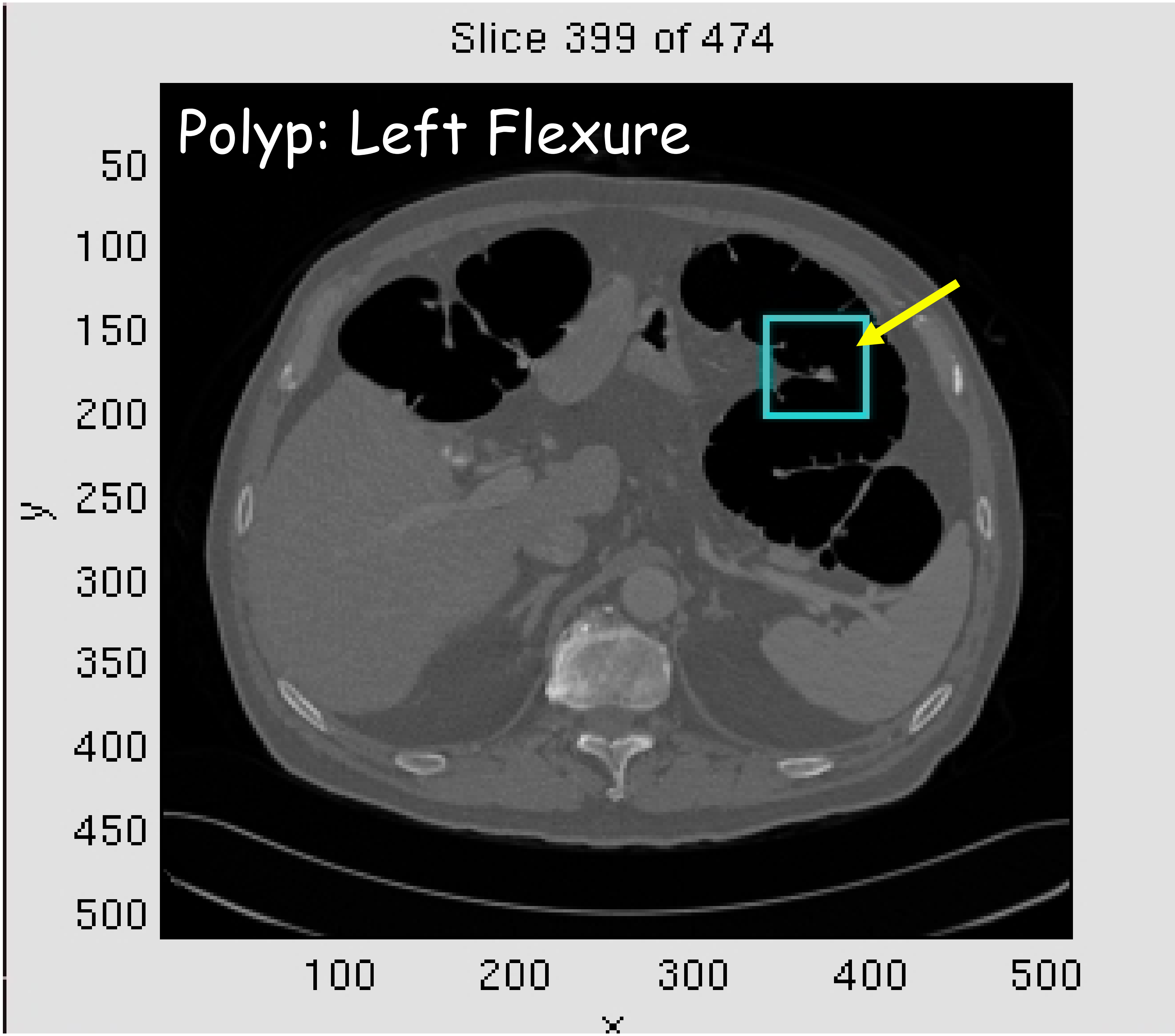
500
100
200
300
400
500





Cancer: Descending Colon


Slice 399 of 474
Polyp: Left Flexure
50
100
150
200
250
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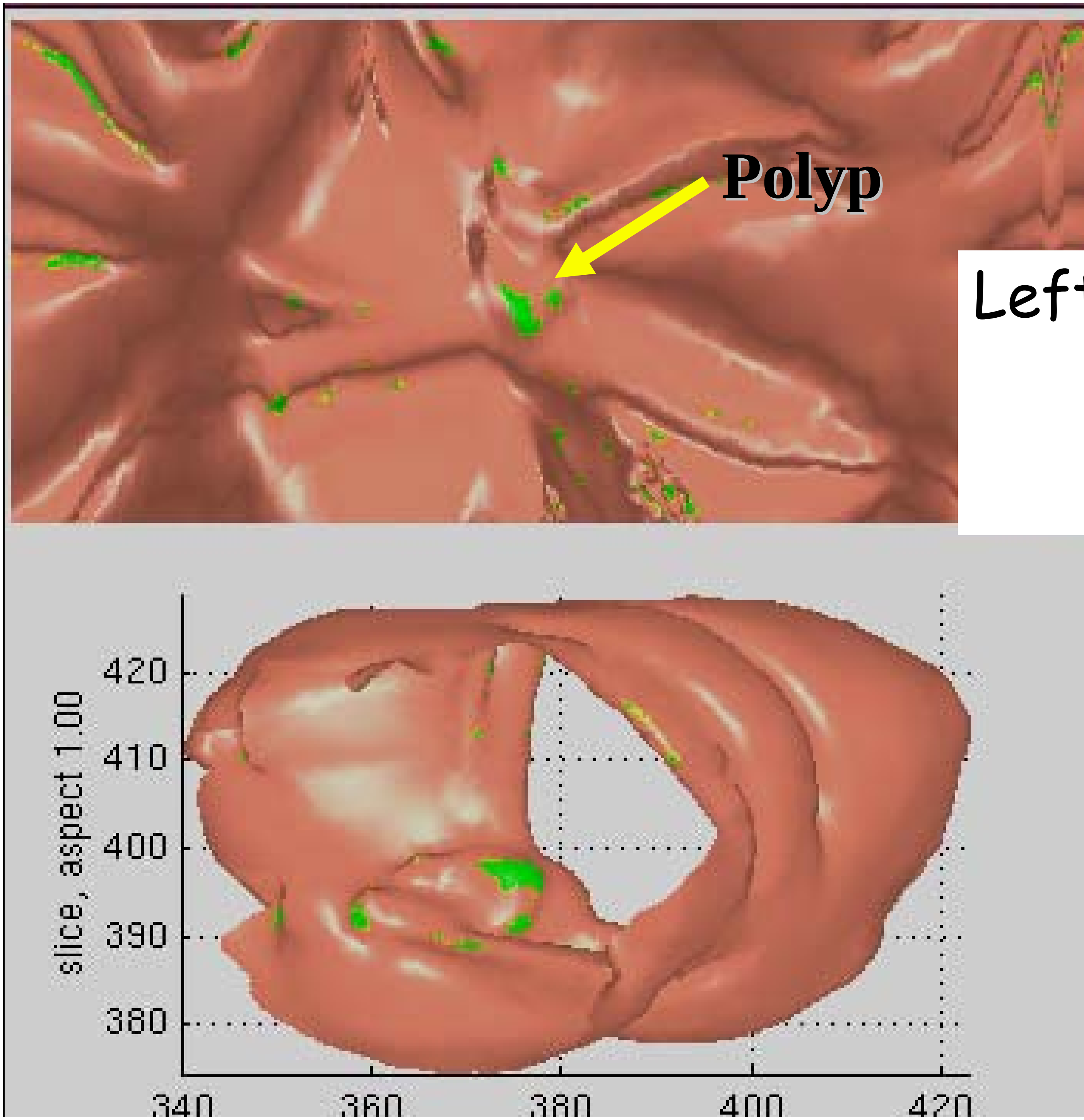
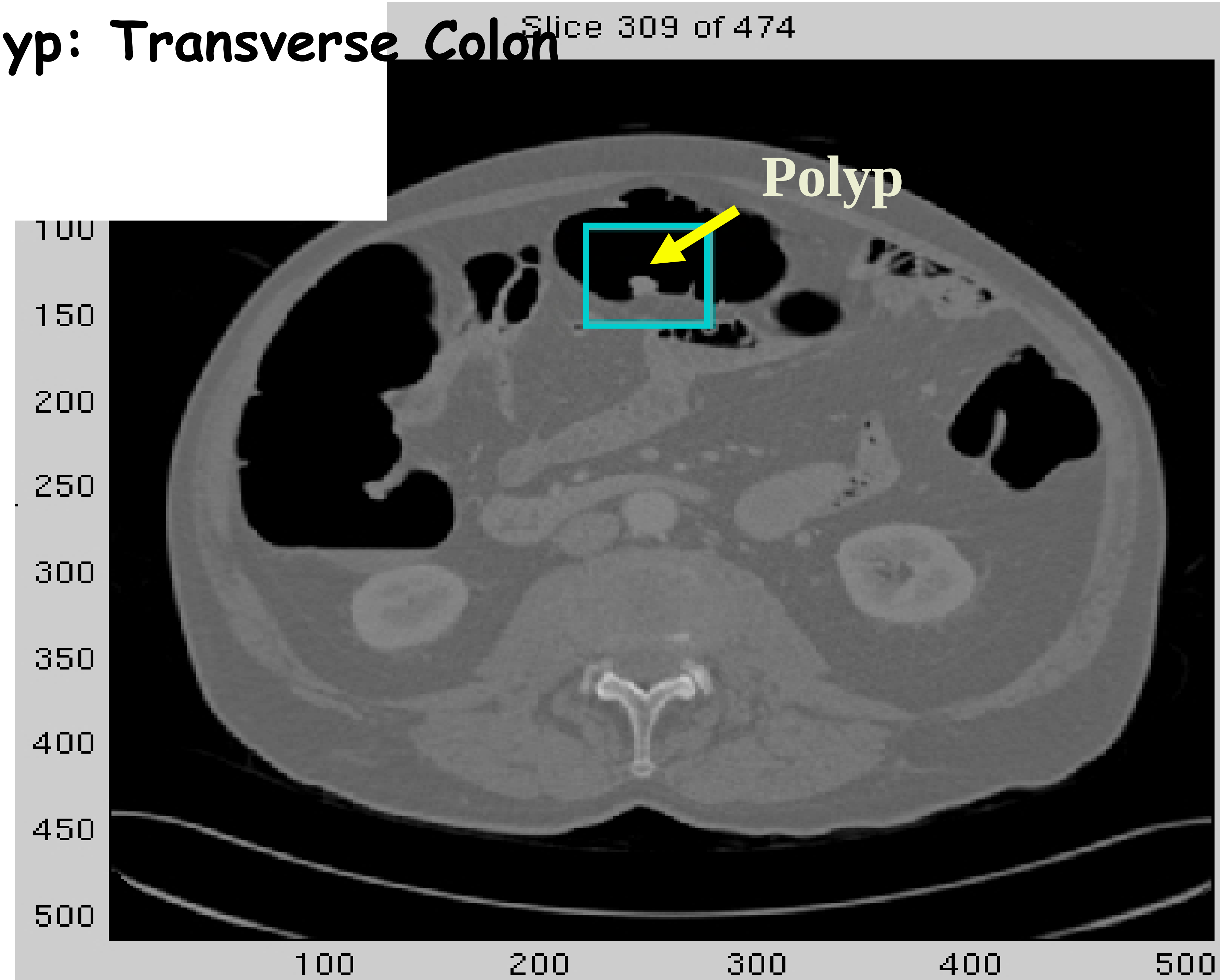
350
400
450
500
y
100
200
300
400
500
x

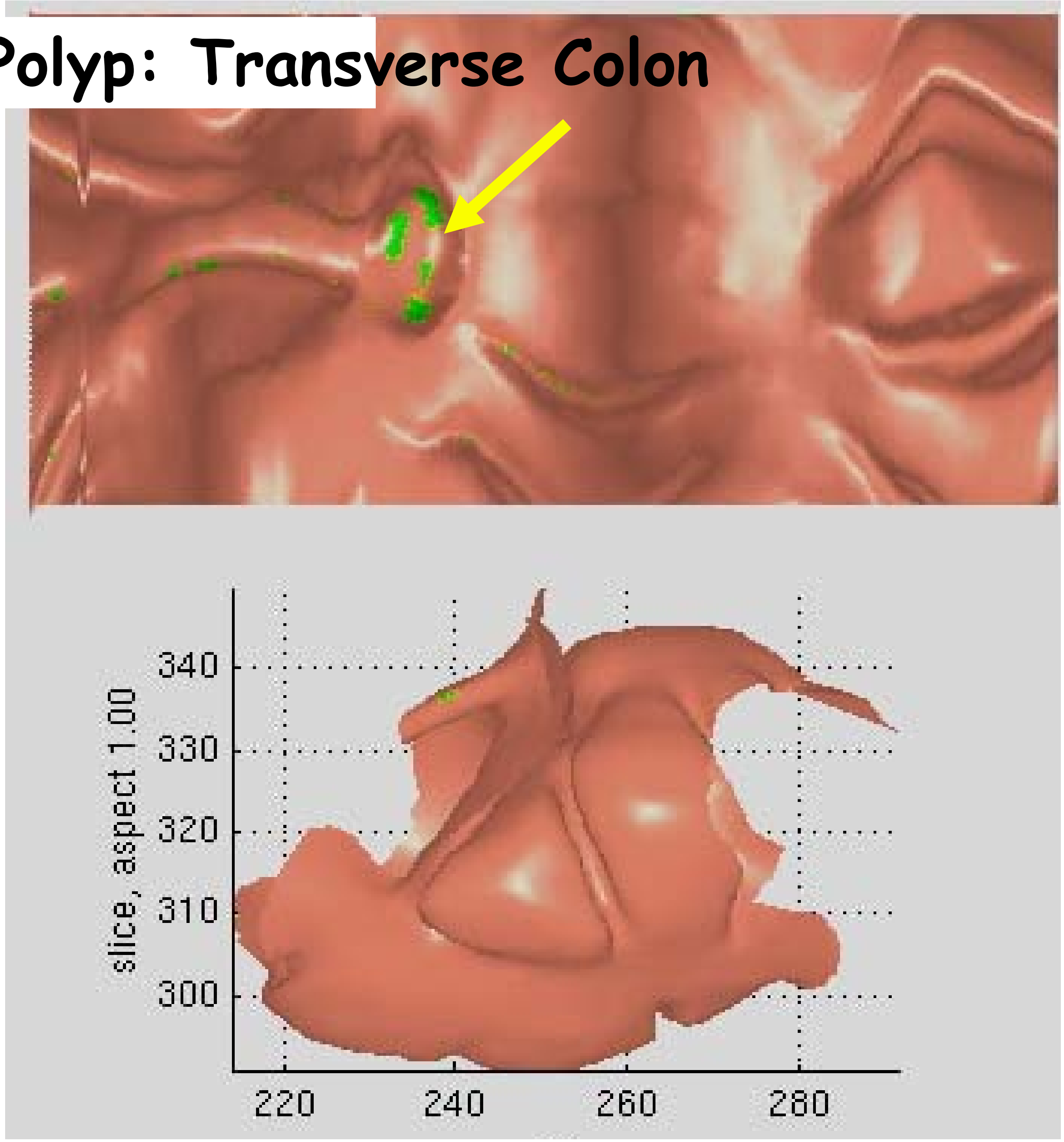
Left Flexur

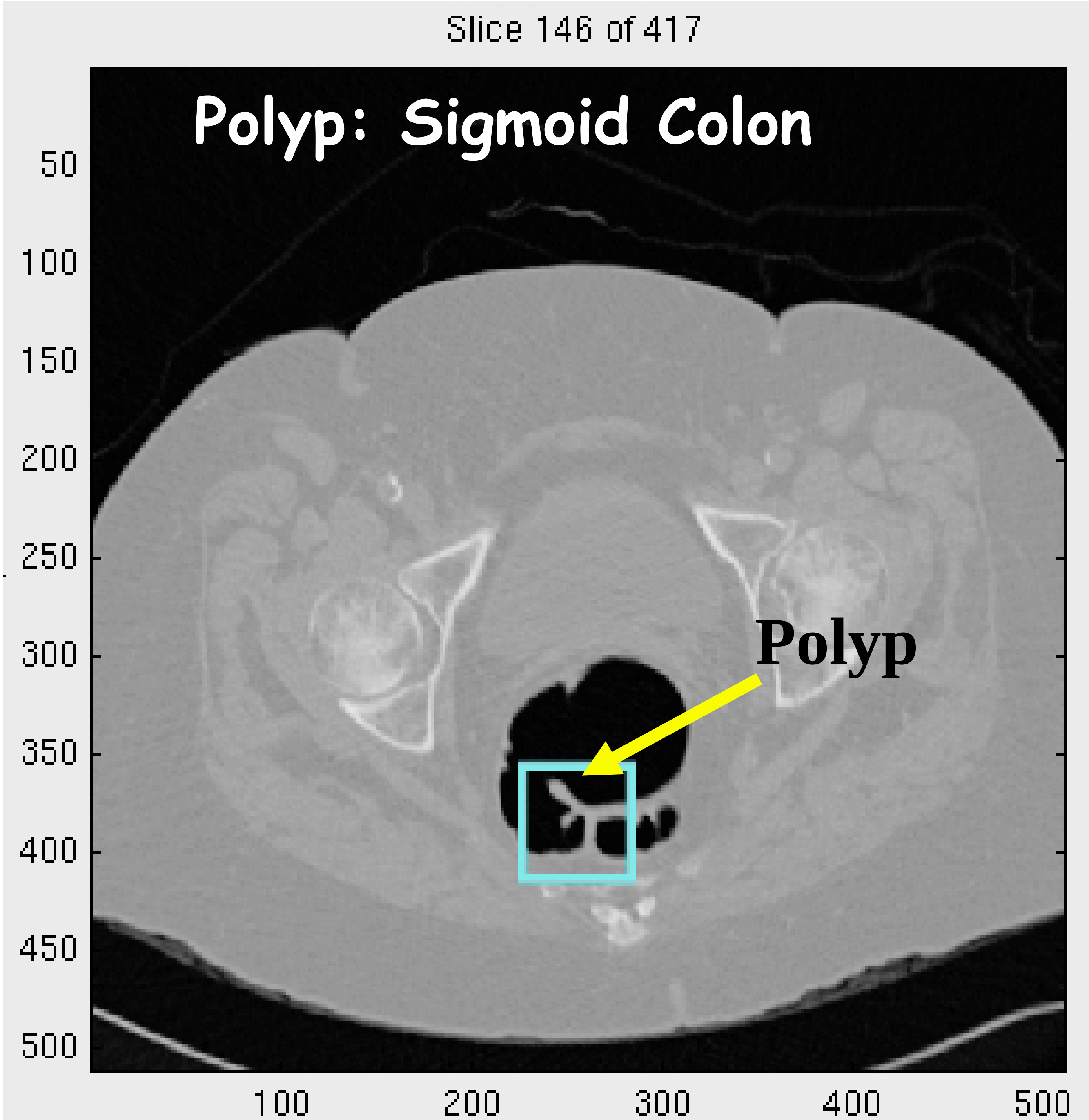


Polyp: Transverse Colon



Polyp: Transverse Colon




Slice 146 of 417
Polyp: Sigmoid Colon
Polyp
50
100
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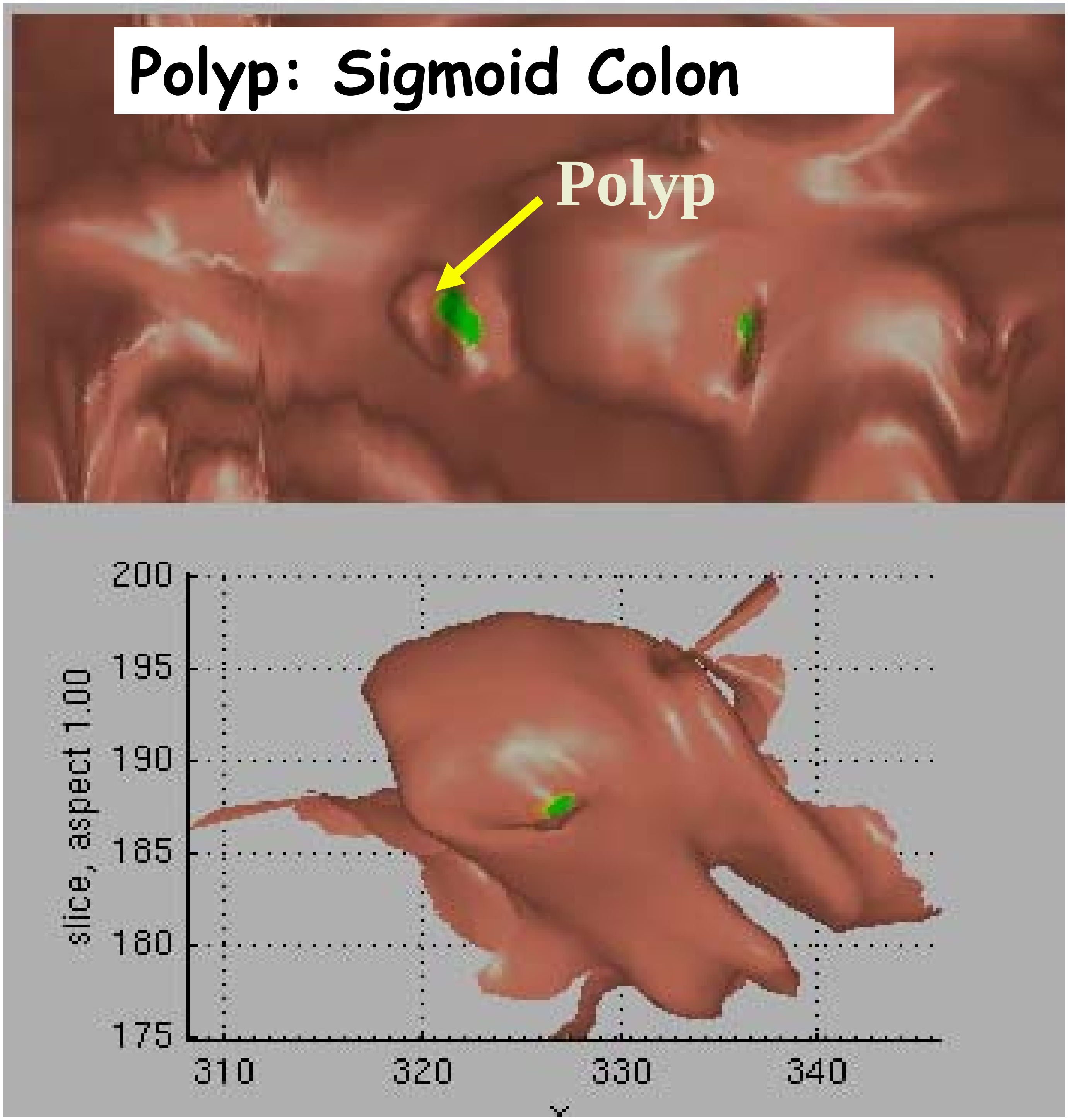
Polyp: Sigmoid Colon
Polyp
200
195
190
185
180
175
slice, aspect 1.00
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320
330
340

Path-Planning Deluxe