# cse547, math547
# DISCRETE MATHEMATICS

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# LECTURE 9a

# CHAPTER 2
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CHAPTER 2
SUMS

Part 4: Finite and Infinite Calculus (1) - Lecture 9a

## Finite and Infinite Calculus

**Infinite Calculus** review

We define a **derivative OPERATOR** D as

$$Df(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivative operator D is defined for **some** functions on real numbers R, called **differentiable** functions.

We denote Df(x) = f'(x) and call the result a derivative f' of a **differentiable** function f

# Derivative Operator

D is called an operator because it is a function that **transforms** some functions into different functions

D is a **PARTIAL function** on the set $R^R$ of all functions over R, i.e.

$$D: \quad R^R \longrightarrow \quad R^R$$

where $R^R = \{f: \quad R \longrightarrow \quad R\}$

D is a partial function because the domain of D consists of the **differentiable functions** only, i.e. functions f for which $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ **exists**

# FINITE CALCULUS

**Difference operator** $\Delta$

Let f be any function on real numbers R (may be partial)
$f: \ R \longrightarrow R$

We **define**

$$\Delta f(x) = f(x+1) - f(x)$$

$\Delta$ transforms **ANY function** f into another function
$g(x) = f(x+1) - f(x)$, so we have that

$$\Delta : \ R^R \longrightarrow R^R$$

# Difference Operator Example

Let $f: R \longrightarrow R$ be given by a formula $f(x) = x^m$

**We evaluate:**

$$Df(x) = mx^{m-1}$$

**Reminder:** $D(x^m) = mx^{m-1}$

**What about** $\Delta$???

Evaluate

$\Delta(x^3) = (x+1)^3 - x^3 = 3x^2 + 3x + 1, \qquad Dx^3 = 3x^2$

$$\Delta \neq D$$

## Difference Operator Question

Q: Is there a function $f$ for which $\Delta f = Df$

Yes there is a "new power" of x, which transforms as nicely under $\Delta$, as $x^m$ does under $D$

# Falling Factorial Power

**Definition** of **Falling** Factorial Power

Let $f: R \longrightarrow R$ be given by a formula

$$f(x) = x^{\underline{m}}$$

for $x^{\underline{m}} = x(x-1)(x-2)\cdots(x-m+1)$ and $m > 0$

Wa also define in a similar way a notion of a rising factorial power

# Rising Factorial Power

**Definition** of **Rising** Factorial Power

Let $f: R \longrightarrow R$ be given by a formula

$$f(x) = x^{\overline{m}}$$

for $x^{\overline{m}} = x(x+1)\cdots(x+m-1)$ and $m > 0$

Let now see what happens when the domain of f is **restricted** to the set of natural numbers N

# Factorial Powers

Let now $f: \; N \longrightarrow N, \quad f(x) = x^{\underline{m}}$

We evaluate

$$\boxed{n^{\underline{n}} = n(n-1)(n-2)\cdots(n-n+1) = n!}$$

We evaluate

$$\boxed{1^{\overline{n}} = 1 \cdot 2 \cdots (1+n-1) = n!}$$

We got

$$n^{\underline{n}} = n! \quad \text{and} \quad 1^{\overline{n}} = n!$$

## Factorial Powers

We **define** case $m = 0$

$\boxed{x^{\underline{0}} = x^{\overline{0}} = 1}$ PRODUCT OF NO FACTORS $\boxed{0! = 1}$ $\boxed{x \in R}$

$$1^{\overline{0}} = 0! = 1 \qquad 0^{\underline{0}} = 0! = 1$$

We have already proved:

$$\boxed{n! = n^{\underline{n}} = 1^{\overline{n}}} \qquad \text{for any } n \geq 0$$

# Factorial Powers

Let's now **evaluate**

$$\boxed{\Delta(x^{\underline{m}}) = (x+1)^{\underline{m}} - x^{\underline{m}}}$$

in order to **PROVE** the formula:

$$\boxed{\Delta(x^{\underline{m}}) = mx^{\underline{m-1}}}$$

It means that $\Delta$ on $x^{\underline{m}}$ "behaves" like D on $x^m$:

$$D(x^m) = mx^{m-1}$$

## Factorial Powers

Evaluate

$$\boxed{(x+1)^{\underline{m}}} = (x+1)x(x-1)\cdots(x+1-m+1)$$
$$= \boxed{(x+1)x(x-1)\cdots(x-m+2)}$$

Evaluate

$$\boxed{x^{\underline{m}}} = \boxed{x(x-1)\cdots(x-m+2)(x-m+1)}$$

# Factorial Powers

Evaluate

$$\boxed{\Delta(x^{\underline{m}}) = (x+1)^{\underline{m}} - x^{\underline{m}}}$$

$$\begin{aligned}
&= (x+1)x(x-1)\cdots(x-m+2) - x(x-1)\cdots(x-m+2)(x-m+1) \\
&= x(x-1)\cdots(x-m+2)((x+1)-(x-m+1)) \\
&= x(x-1)\cdots(x-m+2)\cdot m \\
&= \boxed{mx^{\underline{m-1}}}
\end{aligned}$$

We proved:

$$\Delta(x^{\underline{m}}) = mx^{\underline{m-1}}$$

Hwk Problem 7 is about $x^{\overline{m}}$

## Infinite Calculus: Integration

**Reminder**: differentiation operator D is

$$D : R^R \rightarrow R^R \qquad Df(x) = g(x) = f'(x)$$

$D$ is **partial function**

Domain $D =$ **all differentiable functions**

$D$ is not 1-1; $D(c) = 0$ all $c \in R$

So **inverse** function to D **does not exist**

BUT we define a **reverse** process to **DIFFERENTIATION** that is called **INTEGRATION**

**(1)** We define a notion of a **primitive function**

**(2)** We use it to give a general definition of **indefinite integral**

## Infinite Calculus: Integration

**Definition**

A function $F(x) = F$ such that $DF = DF(x) = F'(x) = f(x)$ is called a **primitive function** of $f(x)$, or simply a **primitive of** f
Shortly,

$$F \text{ is a } \textbf{primitive of } f \quad \text{iff} \quad DF = f$$

$F$ is a **primitive of** $f$ iff $f$ is obtained from $F$ by **differentiation**

The process of finding primitive of $f$ is called **integration**

# Fundamental Theorem

**Problem:** given function f, find **all primitive** function of f (if exist)

**Fundamental Theorem** of differential and integral calculus

The difference of two primitives $F_1(x)$, $F_2(x)$ of the same function $f(x)$ is a constant C, i.e.

$$F_1(x) - F_2(x) = C$$

for any $F_1, F_2$ such that

$$DF_1(x) = f(x), \qquad DF_2(x) = f(x)$$

## Fundamental Theorem

The **Fundamental Theorem** says that

**(1)** From any primitive function $F(x)$ we obtain all the other in the form $\boxed{F(x) + C}$ (suitable $C$)

**(2)** For every value of the constant $C$, the function $\boxed{F_1(x) = F(x) + C}$ represents a **PRIMITIVE** of $f$

# Indefinite Integral

**Definition** of Indefinite Integral as a general form of a primitive function of $f$

$$\int f(x)dx = F(x) + C \qquad C \in R$$

where $F(x)$ is **any primitive** of $f$

i.e. $DF(x) = f(x)$ $\qquad F' = f$ ( for short)

# Proof of Fundamental Theorem

**Proof** of the **FUNDAMENTAL THEOREM** of differential and Integral calculus has two parts

**(2) We prove:** if $F(x)$ is primitive to $f(x)$, so is $F(x)+C$; i.e. $D(F(x)+C)=f(x)$, where $DF(x)=f(x)$

**(1) We prove:** $F_1(x)-F_2(x)=C$ i.e. from any primitive $F(x)$ we obtain all others in the form $F(x)+C$
We first prove (2)

## Proof of Fundamental Theorem

**Proof** of Fundamental Theorem part **(2)**

**(2)** Let $G(x) = F(x) + C$

$$\boxed{D(F(x)+C)} = \lim_{h\to 0} \frac{(F(x+h)+C) - (F(x)+C)}{h}$$
$$= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = F'(x) = \boxed{f(x)}$$

as $F(x)$ is a primitive of $f(x)$

## Proof of Fundamental Theorem

(1) **Consider** $\boxed{F_1(x) \; - \; F_2(x) = G(x)}$

such that $F_1' = f, \quad F_2' = f$

We want to **show** that $G(x) = C$ for all $x \in R$

We use the definition of the derivative to evaluate $G'(x) = DG(x)$

## Proof of Fundamental Theorem

$$\begin{aligned}
D(G(x)) &= \lim_{h\to 0} \frac{(F_1(x+h) - F_2(x+h)) - (F_1(x) - F_2(x))}{h} \\
&= \lim_{h\to 0} \Bigg( \underbrace{\frac{F_1(x+h) - F_1(x)}{h} - \frac{F_2(x+h) - F_2(x)}{h}}_{\text{Both limits exist, as } F_1, F_2, \text{primitive of } f.} \Bigg) \\
&= \lim_{h\to 0} \frac{F_1(x+h) - F_1(x)}{h} - \lim_{h\to 0} \frac{F_2(x+h) - F_2(x)}{h} \\
&= f(x) - f(x) = 0 \qquad \text{for all } x \in R
\end{aligned}$$

## Proof of Fundamental Theorem

We proved that

$F_1(x) - F_2(x) = G(x)$ and

$\boxed{G'(x) = 0}$ for all $x \in R$

But the function whose derivative is everywhere zero must have a graph whose tangent is everywhere parallel to x-asis;

i.e. must be constant;

and therefore we have $\boxed{G(x) = C}$ for all $x \in R$

This is an intuitive, nor a formal proof.

The formal proof uses the Mean Value Theoremm

## Proof of Fundamental Theorem

**Formal proof**

Apply the MEAN VALUE THEOREM to $G(x)$, i.e.

$$G(x_2) - G(x_1) = (x_2 - x_1)G'(\xi) \qquad x_1 < \xi < x_2$$

but $G'(x) = 0$ for all $x$, hence $G'(\xi) = 0$

and $G(x_2) - G(x_1) = 0$ , for any $x_1, x_2$

i.e. $G(x_2) = G(x_1)$ for all $x_1, x_2$ i.e. $\boxed{G(x) = C}$

This ((1)+(2)) justifies the following

**Definition:** INDIFINITE INTEGRAL

$$\boxed{\int f(x) = F(x) + C, \quad \text{where} \quad DF(x) = F'(x) = f(x)}$$

FINITE CALCULUS

# Finite Calculus

**Reminder**: Difference Operator $\Delta$

$$\Delta : R^R \to R^R$$

For any $f \in R^R$ we **define**:

$$\Delta f(x) = f(x+1) - f(x)$$

$\Delta$ is a **total function** on $R^R$

**Remark** : INVERSE to $\Delta$ does not exist! because

$\Delta$ is not 1-1 function

# Finite Calculus

**Example** ; $\Delta$ is not 1-1 function

Take $f_1(x) = c_1, \; f_2(x) = c_2$ for $c_1 \neq c_2$

We have that $f_1(x) \neq f_2(x)$ for all x, i.e. $f_1 \neq f_2$
We evaluate

$$\Delta f_1(x) = f_1(x+1) - f_1(x) = c_1 - c_1 = 0$$

$$\Delta f_2(x) = f_2(x+1) - f_2(x) = c_2 - c_2 = 0$$

$$\boxed{\Delta f_1 = \Delta f_2 \quad \text{for } f_1 \neq f_2}$$

We **proved** that $\Delta$ is not 1-1 function

# Finite Integration

**Question**:
Do we have a REVERSE **operation to** $\Delta$ similar to the one we had for D?

**Answer** YES!

We proceed as the case of **Infinite Calculus**

**Definition**

A function $F = F(x)$ is a **finite primitive** of $f = f(x)$ iff $\Delta F(x) = f(x)$ for all $x \in R$

We write $\Delta F = f$

The process of finding a **finite primitive (FP)** of a function $f = f(x)$ is called a **finite integration**

# Fundamental Theorem

**Problem**:
Given a function f = f(x), **find all finite primitives** of f= f(x)

**Fundamental Theorem** of Finite Calculus

The difference of two **finite primitives** $F_1(x)$, $F_2(x)$ of the **same function** $f(x)$ is a function $C(x)$ , such that $C(x+1) = C(x)$ for all $x \in R$, i.e.

$$F_1(x) - F_2(x) = C(x)$$

and

$$C(x+1) = C(x) \quad \text{for all} \quad x \in R$$

## Fundamental Theorem

The **Fundamental Theorem** says that

(1) Given a **finite primitive** function $F(x)$ of $f(x)$ we obtain all others in the form $F(x) + C(x)$, where the function $C : R \longrightarrow R$ **fulfills a condition**

$$C(x+1) = C(x) \quad \text{for all} \quad x \in R$$

(2) For every function $C(x)$, such that

$$C(x+1) = C(x) \quad \text{for all} \quad x \in R$$

the function $F_1(x) = F(x) + C(x)$ is a **finite primitive** of $f(x)$

## Proof of Fundamental Theorem

**Proof**

(1) Consider $F_1(x) - F_2(x) = C(x)$ such that $\Delta F_1 = f, \Delta F_2 = f$

We want to show that $C(x+1) = C(x)$, i.e.

$$\Delta C(x) = C(x+1) - C(x) = 0$$

Evaluate

$$\begin{aligned}
\Delta C(x) &= \Delta(F_1(x) - F_2(x)) \\
&= (F_1(x+1) - F_2(x+1)) - (F_1(x) - F_2(x)) \\
&= \underbrace{(F_1(x+1) - F_1(x))}_{\Delta F_1} - \underbrace{(F_2(x+1) - F_2(x))}_{\Delta F_2} \\
&= f(x) - f(x) = \boxed{0}
\end{aligned}$$

## Proof of Fundamental Theorem

(2) Let $F_1(x) = F(x) + C(x)$ and

$$\Delta F(x) = f(x), \quad C(x+1) = C(x)$$

**We prove** that $F_1(x)$ is a **finite primitive** of $f$

$$\boxed{\Delta F_1(x)} = (F(x+1) + C(x+1)) - (F(x) + C(x)) \quad (\Delta F_1 = f)$$

$$= F(x+1) - F(x) + 0 = \Delta F(x) = \boxed{f(x)} \quad \text{yes!}$$

## Indefinite Sum Definition

**Definition** of INDEFINITE SUM
as a general form of a **finite primitive** of $f = f(x)$

$$\sum g(x)\delta(x) = f(x) + C(x)$$

if and only if

$$g(x) = \Delta f(x) \quad \text{and} \quad C(x+1) = C(x)$$

for $g: R \longrightarrow R$; $f: R \longrightarrow R$, $C: R \longrightarrow R$

Remark : in paticular case: we can put

$$C(x) = C \quad \text{for all} \quad x \in R$$

as in the case of Indefinite Integral because
$C(x+1) = C = C(x)$

## Example

EXAMPLE of a "CONSTANT" function $C = P(x)$ under $\Delta$

$$P(x) = \sin 2\pi x \qquad \text{(PERIODIC function)}$$

Evaluate

$$P(x+1) = \sin(2\pi(x+1)) = \sin(2\pi x + 2\pi) = P(x)$$

We proved

$$\boxed{P(x) = P(x+1)} \quad \text{for all} \quad x \in R$$

## Definite Integral and Definite Sum

**Infinite Calculus**: DEFINITE INTEGRAL

$$\int_a^b g(x)dx = f(x)\Big|_a^b = f(b) - f(a)$$ where $$f'(x) = g(x)$$

DEFINITION **Finite Calculus**: DEFINITE SUM

$$\sum_a^b g(x)\delta_x = f(x)\Big|_a^b = f(b) - f(a)$$

where $\Delta f(x) = g(x)$

## Definite Sum

We defined

$$\sum_a^b g(x)\delta_x = f(x)\Big|_a^b = f(b) - f(a)$$

for $f(x)$ such that

$$g(x) = \Delta f(x) \qquad g = \Delta f$$

What is the MEANING of our new "INTEGRAL"

$$\sum_a^b g(x)\delta_x ?$$

# Definite Sum

Reminder: $\sum_a^b g(x)\delta_x = f(x)\Big|_a^b = f(b) - f(a)$ for

$$g(x) = \Delta f(x) = f(x+1) - f(x)$$

Let's consider a case: $b = a$

$$\sum_a^a g(x)\delta_x = f(a) - f(a) = 0$$

TAKE now $b = a + 1$

$$\sum_a^{a+1} g(x)\delta_x = f(a+1) - f(a) = \Delta f(a) = g(a)$$

## Definite Sum

We proved that

$$\Sigma_a^a g(x)\delta_x = 0$$

$$\Sigma_a^{a+1} g(x)\delta_x = g(a)$$

Evaluate

$$\Sigma_a^{a+2} g(x)\delta_x \stackrel{\text{def}}{=} f(a+2) - f(a)$$

where

$$g(x) = f(x+1) - f(x) = \Delta f(x)$$

# Definite Sum

Consider

$$\begin{aligned}
& \sum_a^{a+2} g(x)\delta_x - \sum_a^{a+1} g(x)\delta_x \\
= \;& f(a+2) - f(a) - (f(a+1) - f(a)) \qquad \text{(by definition)} \\
= \;& f(a+2) - f(a) - f(a+1) + f(a) \\
= \;& f(a+2) - f(a+1) = g(a+1) \\
\sum_a^{a+2} g(x)\delta_x = \;& \sum_a^{a+1} g(x)\delta_x + g(a+1) \\
= \;& g(a) + g(a+1)
\end{aligned}$$

# Definite Sum

**We proved**

$$\begin{aligned} \sum_a^{a+1} g(x)\delta_x &= g(a) \\ \sum_a^{a+2} g(x)\delta_x &= g(a)+g(a+1) \end{aligned}$$

**Evaluate**

$$\sum_a^{a+3} g(x)\delta_x \stackrel{\text{def}}{=} f(a+3)-f(a)$$

## Definite Sum

**Compute**

$$\begin{aligned}
& \textstyle\sum_a^{a+3} g(x)\delta_x - \sum_a^{a+2} g(x)\delta_x \\
&= f(a+3) - f(a) - (f(a+2) - f(a)) \quad \text{(by definition)} \\
&= f(a+3) - f(a+2) = g(a+2) \\
\textstyle\sum_a^{a+3} g(x)\delta_x &= \textstyle\sum_a^{a+2} g(x)\delta_x + g(a+2) \\
&= g(a) + g(a+1) + g(a+2)
\end{aligned}$$

# Definite Sum

**GUESS** (proof by math. induction over $k$)

$b \geq a$

$$\sum_{a}^{a+k} g(x)\delta_x = g(a) + g(a+1) + \cdots + g(a+k-1)$$

where a+k=b, and a+k-1=b-1

## Definite Sum

For $b \geq a$

$$\underbrace{\sum_a^b g(x)\delta_x}_{\text{DEFINITE SUM}} = \underbrace{\sum_{a \leq k < b} g(k)}_{\text{NORMAL SUM}}$$

$$= \underbrace{\sum_{k=a}^{b-1} g(k)}_{\text{NORMAL SUM}}$$

$\sum_a^b g(x)\delta_x = f(b) - f(a)$, where $\Delta f(x) = g(x)$

# Definite Sum and Normal Sums

**Relationship** between DEFINITE and NORMAL sums

**We defined** $g(x) = \Delta f(x)$

$$\Sigma_a^b g(x)\delta_x \stackrel{\text{def}}{=} f(x)\Big|_a^b = f(b) - f(a)$$

**WE PROVED:**

$$\Sigma_a^b g(x)\delta_x = \Sigma_{k=a}^{b-1} g(k)$$

For $b \geq a$

## Definite and Normal SumsTheorem

**THEOREM**

For $b \geq a$

$$\begin{aligned} \sum_{k=a}^{b-1} g(k) &= \sum_a^b g(x)\delta_x \\ &= f(x)\Big|_a^b = f(b) - f(a) \end{aligned}$$

We write it as

$$\underbrace{\sum_{k=a}^{b-1} g(k)}_{\text{SUM}} \overset{\text{Thm}}{=} \sum_a^b g(x)\delta_x$$

Reminder: $g(x) = \Delta f(x) = f(x+1) - f(x)$

## Definite and Normal SumsTheorem

When asked of evaluating a SUM, we can evaluate a ”SUM INTEGRAL”

$$\underbrace{\sum_{a}^{b} g(x)\delta_x}_{\text{INTEGRAL}} = f(x)\Big|_a^b = f(b) - f(a)$$

where $\Delta f(x) = g(x)$

Very easy if you know how to integrate $\sum_{a}^{b} g(x)\delta_x$

## Example

$$\int x^m dx = \frac{x^{m+1}}{m+1} \qquad \text{INFINITE}$$

$$\sum x^{\underline{m}} d_x = \frac{x^{\underline{m+1}}}{m+1} \qquad \text{FINITE}$$

because

$$\begin{aligned} \Delta\left(\frac{x^{\underline{m+1}}}{m+1}\right) &= \frac{1}{m+1}\Delta(x^{\underline{m+1}}) \\ &= \frac{1}{m+1}(m+1)x^{\underline{m}} = x^{\underline{m}} \end{aligned}$$

where

$$x^{\underline{m}} = x(x-1)(x-2)\cdots(x-m+1)$$

## Example

**Evaluate**

$$\sum_{0\leq k<n} k^{\underline{m}} = 0^{\underline{m}} + 1^{\underline{m}} + 2^{\underline{m}} + \cdots + (n-1)^{\underline{m}} \qquad \text{BAD}$$

$$\underbrace{\sum_{k=0}^{n-1} k^{\underline{m}}}_{\text{SUM}} \overset{\text{Thm}}{=} \underbrace{\sum_{0}^{n} k^{\underline{m}}\delta_k}_{\text{INTEGRAL}}$$

We know that $\sum_{k=0}^{n-1} k^{\underline{m}}\delta_k = \frac{k^{\underline{m+1}}}{m+1}\Big|_0^n$

## Example

**We evaluate**

$$\sum_{k=0}^{n} k^{\underline{m}} \overset{\text{Thm}}{=} \underbrace{\sum_{0}^{(n+1)} k^{\underline{m}}\delta_k}_{\text{Integral}}$$

$$= \sum_0^{n+1} k^{\underline{m}}\delta_k = \frac{k^{\underline{m+1}}}{m+1}\Big|_0^{n+1}$$

$$= (n+1)^{\underline{m+1}} - \frac{0^{\underline{m+1}}}{1}$$

$$= (n+1)^{\underline{m+1}}$$

**Answer**

$$\sum_{k=0}^{n} k^{\underline{m}} = (n+1)^{\underline{m+1}}$$