cse547, math547
DISCRETE MATHEMATICS

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LECTURE 9
CHAPTER 2
SUMS

Part 4: Finite and Infinite Calculus
Part 5: Infinite Sums
Negative Exponent Falling Powers

**Definition:** (of negative exponent falling powers)

\[
x^{-1} = \frac{1}{x + 1}
\]

\[
x^{-2} = \frac{1}{(x + 1)(x + 2)}
\]

\[
x^{-3} = \frac{1}{(x + 1)(x + 2)(x + 3)}
\]

**General:**

\[
x^{-m} = \frac{1}{(x + 1)(x + 2) \cdots (x + m)} \quad m > 0
\]
Problems

Homework: $x^{n+m} = x^n \cdot x^m$

Prove:

\[ x^{m+n} = x^m(x - m)^n \]

Homework: (we proved for $m \geq 0$)

Prove: (for $m < 0$)

\[ \Delta x^m = mx^{m-1} \]
Example:

\[
\Delta x^{-2} = \frac{1}{(x+2)(x+3)} - \frac{1}{(x+1)(x+2)}
\]

\[
= \frac{(x+1) - (x+3)}{(x+1)(x+2)(x+3)}
\]

\[
= -2x^{-3}
\]

Fact:

\[
\sum_{a}^{b} x^m \delta x = \frac{x^{m+1}}{m+1} \bigg|_{a}^{b} \quad \text{all } m \neq -1
\]

What about case \( m = -1 \)?
Example

Case \( m = -1 \):

Infinite Integral
\[
\int_{a}^{b} x^{-1} \, dx = \int_{a}^{b} \frac{1}{x} \, dx = \ln |x| \bigg|_{a}^{b}
\]

We want to have a finite analog:
\[
x^{-1} = \frac{1}{x+1} \quad \Delta f = f(x+1) - f(x)
\]

Take:

\[
f(x) = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{x} = \sum_{k=1}^{x} \frac{1}{k} = H_{x}
\]

\[
\Delta f(x) = \left( \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{x} + \frac{1}{x+1} \right) - \left( \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{x} \right)
\]
\[
= \frac{1}{x+1}
\]
Example

**Case** \( m = -1 \)

\[
\sum_a^b x^{-1} \delta x = \sum_a^b \frac{1}{x+1} \delta x = H_x \bigg|_a^b
\]

We will prove (Ch. 9) that for large \( x \):

\[
H_x - \ln x \approx 0.577 + \frac{1}{2x}
\]

\( H_x \sim \ln x \) as do \( \int_a^b \) and \( \sum_a^b \).
Theorem

**Theorem: Sums of falling powers**

\[
\sum_a^b x^m \delta x = \begin{cases} 
\frac{x^{m+1}}{m+1} \bigg|_a^b & m \neq -1 \\
H_x \bigg|_a^b & m = -1 
\end{cases}
\]

all \( m \in \mathbb{Z} \)

and \( \int_a^b \frac{1}{x} \, dx = \ln|x| \bigg|_a^b \) is similar to \( \sum_a^b x^{-1} = H_x \bigg|_a^b \)
More Similarities

More Similarities
We know \((e^x)' = e^x\).

\[
\begin{align*}
D e^x &= e^x, \\
D f &= f \text{ for } f(x) = e^x
\end{align*}
\]

Q. Which function \( f \) has a similar property for \( \Delta \)?

i.e. \( \Delta f(x) = f(x) \)

\[
\Delta f(x) = f(x + 1) - f(x) = f(x)
\]

\( f \) is such that: \( f(x + 1) = 2f(x) \) Recurrence!
Example

Example of solution:

\[ f(x) = 2^x \]

\[ f(x + 1) - f(x) = 2^{x+1} - 2^x \]
\[ = 2 \cdot 2^x - 2^x \]
\[ = 2^x = f(x) \]

We proved \( \Delta(2^x) = 2^x \)

\( (e^x)' = e^x \)

Find a formula for \( \Delta f \), where \( f(x) = c^x \)

\[ \Delta(c^x) = c^{x+1} - c^x = c \cdot c^x - c^x \]
\[ = c^x(c - 1) \]
Difference

Difference:

\[ \Delta(c^x) = (c - 1)c^x \quad c \in \mathbb{N}^+ \quad \text{“derivative”} \]

\[ \sum_a^b c^x \delta x = \frac{c^x}{c - 1} \bigg|_a^b \quad c \neq 1 \quad \text{“antiderivative”} \]
Geometric Progression

We prove:

Theorem: Geometric Progression

\[
\sum_{a \leq k < b} c^k = \sum_{a}^{b} c^x \delta x
\]

\[
= \left. \frac{c^x}{c-1} \right|_{a}^{b} = \frac{c^b - c^a}{c-1}, \quad c \neq 1
\]

General Formula for Geometric Progression

\[
\sum_{a \leq k < b} c^k = \frac{c^b - c^a}{c-1}, \quad c \neq 1
\]

\[
\sum_{k=a}^{b-1} c^k = \frac{c^b - c^a}{c-1}
\]
Chain Rule

**Infinite:** “chain rule”

\[ Df(g(x)) = Df \cdot Dg(x) \]

**Finite:** no such rule

Can’t relate \( \Delta f(g(x)) \) to \( \Delta g(x) \)
Integration by Parts

Infinite

\[ D(uv) = uDv + vDu \]

Integration by parts

\[ \int u \, dv = uv - \int v \, du \]

Does it have an analog for \( \Delta \)?
Integration by Parts

Can we have

\[ \Delta( uv ) = u \Delta v + v \Delta u \]

and \( \sum u \delta v = uv - \sum v \delta u \)?

Not exactly, but close!

Evaluate:

\[ \Delta(u(x)v(x)) = u(x+1)v(x+1) - u(x)v(x) \]

\[ = u(x+1)v(x+1) - u(x)v(x+1) + u(x)v(x+1) - u(x)v(x) \]

\[ = u(x)v(x+1) - u(x)v(x) + u(x+1)v(x+1) - u(x)v(x+1) \]

\[ = u(x)(v(x+1) - v(x)) + (x+1)(u(x+1) - u(x)) \]

\[ = u(x)\Delta v(x) + v(x+1)\Delta u(x) \]
Summation by Parts

Shift Operator: \( Ev = v(x + 1) \)

We proved \( \Delta(uv) = u\Delta v + Ev\Delta u \)

**Summation by parts**

\[ \sum u \delta v = uv - \sum Ev \delta u \]

\[ \sum_{a}^{b} u \delta v = uv\bigg|_{a}^{b} - \sum_{a}^{b} Ev \delta u \]
Summation by Parts

Integration

\[ \int xe^x \, dx = xe^x - \int 1 \cdot e^x \, dx = e^x(x - 1) + C \]

Summation

\[ \sum x2^x \delta x = x2^x - \sum 2^{x+1} \delta x = x2^x - 2^{x+1} + C(x) \]

for \( C(x) = C(x + 1) \)

Evaluate

\( u(x) = x, \quad v(x) = 2^x, \quad Ev(x) = 2^{x+1} \)

\( \Delta u(x) = 1, \quad \Delta v(x) = 2^x \)

Fact: \( \Delta(2^{x+1}) = 2^{x+1} \)
Summation by Parts

In particular, evaluate:

$$\sum_{k=0}^{n} k2^k = 1 \cdot 2^1 + 2 \cdot 2^2 + \cdots + n \cdot 2^n$$

$$\sum_{k=0}^{n} k2^k = \sum_{x=0}^{n+1} x2^x \delta x = (x2^x - 2^{x+1})|_{x=0}^{x=n+1}$$

$$= ((n+1)2^{n+1} - 2^{n+2}) - (0 \cdot 2^0 - 2)$$

$$= (n+1)2^{n+1} - 2 \cdot 2^{n+1} + 2$$

$$= (n+1-2)2^{n+1} + 2 = (n-1)2^{n+1} + 2$$

$$\sum_{k=0}^{n} k2^k = (n-1)2^{n+1} + 2$$
Use finite calculus to evaluate:

\[ \sum_{k=0}^{n-1} kH_k \]

"sum" by parts

Analog:

\[ \int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx \]

\[ = \frac{x^2}{2} \ln x - \left( \frac{x^2}{2} \right) \cdot \frac{1}{2} \cdot \frac{x^2}{2} \]

\[ = \frac{x^2}{2} \left( \ln x - \frac{1}{2} \right) \]
Summation by Parts

Use
\[ \sum u \delta v = uv - \sum Ev \delta u \]
\[ Ev(x) = v(x + 1) \]
\[ \sum_{0 \leq x, n} xH_x \delta x = uv - \sum Ev \delta u \]

\[ \Delta v(x) = x = x^1 \]
\[ v(x) = \frac{x^2}{2}, \quad v(x + 1) = \frac{(x + 1)^2}{2} \]
\[ \Delta u(x) = \Delta H_x = x^{-1} \]

\[ u(x) = H_x \]
\[ v(x) = \frac{x^2}{2} \]
\[ Ev(x) = \frac{(x + 1)^2}{2} \]
\[ \Delta u(x) = x^{-1} \]
Summation by Parts

\[
\sum_{k=0}^{n-1} kH_k = \sum_{0 \leq x < n} xH_x \delta x = \sum_{0}^{n} xH_x \delta x
\]

\[
= \left( \frac{x^2}{2} H_x - \sum \frac{(x+1)^2}{2} \cdot x^{-1} \delta x \right) \bigg|_{0}^{n}
\]
Summation by Parts

Evaluate:

\[
\frac{(x + 1)^2}{2} \cdot x^{-1} = \frac{1}{2} x(x + 1) \cdot \frac{1}{x + 1} = \frac{1}{2} x
\]

\[
= \frac{1}{2} x^1
\]
Summation by Parts

\[ \sum_{0}^{n-1} kH_k = \sum_{0 \leq x < n} xH_x \delta x \]

\[ = \left( \frac{x^2}{2} H_x - \frac{1}{2} \sum x^1 \delta x \right) \bigg|_0^n \]

\[ = \left( \frac{x^2}{2} H_x - \frac{1}{2} \frac{x^2}{2} \right) \bigg|_0^n \]

\[ = \frac{x^2}{2} \left( H_x - \frac{1}{2} \right) \bigg|_0^n = \frac{n^2}{2} \left( H_n - \frac{1}{2} \right) \]
FC Formula

\[ \sum_{k=0}^{n-1} kH_k = \frac{n^2}{2} \left( H_n - \frac{1}{2} \right) \]