CHAPTER 2
SUMS

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More SUMS

Problem from Book, page 39
Let’s **EVALUATE** the following sum

\[
S_n = \sum_{1 \leq j < k \leq n} \frac{1}{k - j}
\]

We denote \( P(j, k) : 1 \leq j < k \leq n \) and re-write the sum as

\[
S_n = \sum_{P(j,k)} a_{k,j}
\]

for

\[
a_{k,j} = \frac{1}{k - j}
\]
Special SUM

Consider case \( n=1 \)

Remember that \( a_{k,j} = \frac{1}{k-j} \)

We get that \( S_1 = \sum_{1 \leq j < k < 1} a_{k,j} \) is undefined.

Book defines \( S_1 = 0 \)

Consider \( S_2 = \sum_{1 \leq j < k \leq 2} a_{k,j} = \sum_{1 \leq j < k \leq 2} \frac{1}{k-j} \)

Evaluate \( S_2 = a_{2,1} = \frac{1}{2-1} = 1, \quad S_2 = 1 \)
Special SUM

Evaluate $S_3$

$$S_3 = \sum_{1 \leq j < k \leq 3} a_{k,j} = a_{3,2} + a_{3,1} + a_{2,1} = \frac{1}{3 - 2} + \frac{1}{3 - 1} + \frac{1}{2 - 1}$$

$$= \frac{1}{1} + \frac{1}{2} + 1 = \frac{5}{2}$$

$S_3 = \frac{5}{2}$

$$S_3 = \sum_{1 \leq j < k \leq 3} \frac{1}{k - j} = \frac{5}{2}$$
Special SUM

Now we want to express \( P(j, k) = 1 \leq j < k \leq n \) as

\[
P(j, k) \equiv P_1(k) \cap P_2(j)
\]

in order to use definition of the multiple sum below for our sum

\[
\sum_{P(j,k)} a_{k,j} \overset{\text{def}}{=} \sum_{P_1(k)} \sum_{P_2(j)} a_{k,j} = \sum_{P_2(j)} \sum_{P_1(k)} a_{k,j}
\]
Special SUM

Step 1  APPROACH 1

We consider $P(j, k) = 1 \leq j < k \leq n$

\[(\star) \quad 1 \leq j < k \leq n \equiv 1 < k \leq n \cap 1 \leq j < k\]

$P(j, k) \equiv P_1(k) \cap P_2(j)$

We get from (\star) that

$$S_n = \sum_{1 < k \leq n} \sum_{1 \leq j < k} \frac{1}{k - j}$$
Special SUM

We substitute \( j := k - j \) and evaluate \( S_n \) and new boundaries for \( S_n \)

**Boundaries:** we substitute \( j := k - j \) in \( 1 \leq j < k \)

\[
1 \leq k - j < k \quad \text{iff} \quad 1 - k \leq -j < 0 \quad \text{iff} \quad k - 1 \geq j > 0
\]

Remark that

\[
0 < j \leq k - 1 \quad \text{iff} \quad 1 \leq j \leq k - 1
\]

so the new boundaries for \( S_n \) are

\[
1 < k \leq n \quad \text{and} \quad 1 \leq j \leq k - 1
\]
Special SUM

We substitute $j := k - j$ and evaluate $S_n$ with new boundaries $1 < k \leq n$ and $1 \leq j \leq k - 1$

\[
S_n = \sum_{1 < k \leq n} \sum_{1 \leq j < k} \frac{1}{k - j} = \sum_{1 < k \leq n} \sum_{1 \leq j \leq k - 1} \frac{1}{j}
\]

\[
= \sum_{1 < k \leq n} \sum_{j=1}^{k-1} \frac{1}{j} = \sum_{1 < k \leq n} H_{k-1}
\]

Now we evaluate new boundaries for the last sum

We put $k := k + 1$ in $1 < k \leq n$ and get

$1 < k + 1 \leq n$ iff $0 < k \leq n - 1$ iff $1 \leq k \leq n - 1$ and

\[
\sum_{1 < k \leq n} H_{k-1} = \sum_{k=1}^{n-1} H_k
\]
Special SUM Formula

We developed a new formula for $S_n$

$$
\sum_{1 \leq j < k \leq n} \frac{1}{k-j} = \sum_{k=1}^{n-1} H_k
$$

We now check our result for few $n$

$S_1 = \sum_{k=1}^{0} H_1$ undefined, $S_1 = \sum_{1 \leq j < k \leq 1} \frac{1}{k-j}$ is also undefined

Book puts (page 39) $S_1 = 0$

Remark that the BOOK formula for $S_n$

$S_n = \sum_{k=0}^{n} H_k$ is not correct unless we define $H_0 = 0$
Observe that we got just another formula for our sum, not a "sum closed" formula; we have expressed one double sum by another that uses $H_n$

**Step 2  APPROACH 2**

Let’s now re-evaluate the $S_n$ by expressing its boundaries differently.

We have as before $P(j, k) \equiv 1 \leq j < k \leq n$ and want to write is now as

$$P(j, k) \equiv R_1(k) \cap R_2(j)$$

for some $R_1(k)$, $R_2(j)$ and evaluate the sum

$$S_n = \sum_{1 \leq j < k \leq n} \frac{1}{k-j} = \sum_{R_2(j)} \sum_{R_1(k)} \frac{1}{k-j}$$
Special SUM Approach 2

We write now

\[ 1 \leq j < k \leq n \equiv (1 \leq j < n) \cap (j < k \leq n) \equiv R_1(k) \cap R_2(j) \]

and evaluate

\[ S_n = \sum_{1 \leq j < k \leq n} \frac{1}{k-j} = \sum_{1 \leq j < n} \sum_{j < k \leq n} \frac{1}{k-j} \]

We substitute now \( k := k+j \) and re-work boundaries

\[ j < k \leq n \quad \text{iff} \quad j < k+j \leq n \quad \text{iff} \quad 0 < k \leq n-j \]

\[ \text{iff} \quad 1 \leq k \leq n-j \quad \text{and the} \quad S_n \quad \text{becomes now} \]

\[ S_n = \sum_{1 \leq j < n} \sum_{1 \leq k \leq n-j} \frac{1}{k} = \sum_{1 \leq j < n} H_{n-j} \]
Special SUM Approach 2

We have now

\[ S_n = \sum_{1 \leq j < n} H_{n-j} \]

We substitute now \( j := n - j \) and re-work boundaries:

\[ 1 \leq j < n \quad \text{iff} \quad 1 \leq n - j < n \quad \text{iff} \quad 1 - n \leq -j < 0 \]

\[ \text{iff} \quad n - 1 \geq j > 0 \quad \text{iff} \quad 0 < j \leq n - 1 \quad \text{iff} \quad 1 \leq j \leq n - 1 \]

and the \( S_n \) becomes now

\[ S_n = \sum_{j=1}^{n-1} H_j \]

All the work - and nothing new!!
Step 3  APPROACH 3

We want to find a closed formula $CF$ for

$$S_n = \sum_{1 \leq j < k \leq n} \frac{1}{k - j}$$

We substitute $k := k + j$ and now

$$S_n = \sum_{1 \leq j < k + j \leq n} \frac{1}{k}$$
Special SUM Approach 3

PLAN of ACTION

(1) We prove: \( P(k, j) \equiv Q_1(k) \cap Q_2(j) \) expressed as follows

\[
1 \leq j < k + j \leq n \equiv (1 \leq k \leq n - 1) \cap 1 \leq j \leq n - k
\]

(2) We evaluate:

\[
S_n = \sum_{1 \leq j < k + j \leq n} \frac{1}{k} = \sum_{(1 \leq k \leq n - 1) \cap (1 \leq j \leq n - k)} \frac{1}{k}
\]
Special SUM Approach 3

**Proof** of (1)

We evaluate:

\[(1 \leq j < k + j \leq n) \equiv \]
\[\equiv (1 \leq j) \cap (1 \leq n) \cap (j \leq n - k) \cap (j < k + j \leq n) \]
\[\equiv (1 \leq j \leq n - k) \cap (0 < k \leq n - j)\]

Now look at \( (0 < k \leq n - j) \equiv (1 \leq k \leq n - j) \) for \( j = 1, 2, \ldots, n - k \)

and get that \( 1 \leq k \leq n - 1 \)

Hence

\[(1 \leq j < k + j \leq n) = (1 \leq j \leq n - k) \cap (1 \leq k \leq n - 1) \]

end of the proof
Special SUM Approach 3

We evaluate now (2)

\[ S_n = \sum_{1 \leq k \leq n-1 \wedge 1 \leq j \leq n-k} \frac{1}{k} = \sum_{k=1}^{n-1} \left( \sum_{j=1}^{n-k} \frac{1}{k} \right) \]

\[ = \sum_{k=1}^{n-1} \frac{1}{k} \sum_{j=1}^{n-k} 1 = \sum_{k=1}^{n-1} \frac{1}{k} (n-k) \]

\[ = \sum_{k=1}^{n-1} \frac{n}{k} - \sum_{k=1}^{n-1} 1 = n \left( \sum_{k=1}^{n-1} \frac{1}{k} \right) - (n-1) \]
We have now
\[ S_n = n \sum_{k=1}^{n-1} \frac{1}{k} - (n - 1) \]

We note:
\[ \sum_{k=1}^{n-1} \frac{1}{k} = H_{n-1} \] and \[ H_{n-1} = H_n - \frac{1}{n} \]

\[ S_n = nH_{n-1} - n + 1 = n(H_n - \frac{1}{n}) - n + 1 = nH_n - 1 - n + 1 \]

Our \( H_n \) CF formula for \( S_n \) is
\[ S_n = \sum_{1 \leq j < k \leq n} \frac{1}{k - j} = nH_n - n \]
Evaluation in Book

\[ S_n = \sum_{k=1}^{n} \sum_{1 \leq j \leq n-k} \frac{1}{k} = \sum_{k=1}^{n} \sum_{j=1}^{n-k} \frac{1}{k} = \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{n-k} 1 \]

\[ = \sum_{k=1}^{n} \frac{1}{k} (n-k) = \sum_{k=1}^{n} \left( \frac{n}{k} - 1 \right) = n \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} 1 = nH_n - n \]

\[ S_n = nH_n - n \]

Justify all the steps
Extra Bonuses

We proved in Steps 1, 2 that

\[ S_n = nH_n - n \quad \text{and} \quad S_n = \sum_{k=1}^{n} H_k \]

We get an an Extra Bonus

\[ \sum_{k=1}^{n} H_k = nH_n - n \]

And also because Book sum = Our sum we get

\[ \sum_{1 \leq k \leq n, 1 \leq j \leq n-k} \frac{1}{k} = \sum_{1 \leq k \leq n-1, 1 \leq j \leq n-k} \frac{1}{k} \]

and we have also proved as a bonus Book Remark on page 41