cse547, math547
DISCRETE MATHEMATICS

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LECTURE 8

CHAPTER 2
SUMS

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Part 3: Multiple Sums (2) - Lecture 8

More SUMS

Problem from Book, page 39

Let's **EVALUATE** the following sum

$$S_n = \sum_{1 \le j < k \le n} \frac{1}{k - j}$$

We denote $P(j,k) : 1 \le j < k \le n$ and re-write the sum as

$$S_n = \sum_{P(j,k)} a_{k,j}$$

for $a_{k,j} = \frac{1}{k-j}$

Special SUM

Consider case n=1

Remember that $a_{k,j} = \frac{1}{k-j}$

We get that $S_1 = \sum_{1 \le j < k < 1} a_{k,j}$ is undefined.

Book defines $\boxed{S_1 = 0}$

Consider $S_2 = \sum_{1 \le j < k \le 2} a_{k,j} = \sum_{1 \le j < k \le 2} \frac{1}{k-j}$

Evaluate $S_2 = a_{2,1} = \frac{1}{2-1} = 1, \quad S_2 = 1$

Special SUM

Evaluate S_3

$$S_3 = \sum_{1 \leq j < k \leq 3} a_{k,j} = a_{3,2} + a_{3,1} + a_{2,1} = \frac{1}{3-2} + \frac{1}{3-1} + \frac{1}{2-1}$$

$$= \frac{1}{1} + \frac{1}{2} + 1 = \frac{5}{2}$$

$$\boxed{S_3 = \tfrac{5}{2}}$$

$$\boxed{S_3 = \sum_{1 \leq j < k \leq 3} \frac{1}{k-j} = \frac{5}{2}}$$

Special SUM

Now we want to express $P(j,\ k) = 1 \leq j < k \leq n$ as

$$\boxed{P(j,\ k) \equiv P_1(k) \cap P_2(j)}$$

in order to use definition of the multiple sum below for out sum

$$\boxed{\sum_{P(j,k)} a_{k,j} =^{def} \sum_{P_1(k)} \sum_{P_2(j)} a_{k,j} = \sum_{P_2(j)} \sum_{P_1(k)} a_{k,j}}$$

Special SUM

Step 1 APPROACH 1

We consider $P(j,k) = 1 \leq j < k \leq n$

$$(\star) \qquad \boxed{1 \leq j < k \leq n \equiv 1 < k \leq n \cap 1 \leq j < k}$$

$$P(j,k) \equiv P_1(k) \cap P_2(j)$$

We get from $(\star)$ that

$$\boxed{S_n = \sum_{1<k\leq n} \sum_{1\leq j<k} \frac{1}{k-j}}$$

Special SUM

We substitute $j := k - j$ and evaluate S_n and new boundaries for S_n

Boundaries: we substitute $j := k - j$ in $1 \leq j < k$

$1 \leq k - j < k$ iff $1 - k \leq -j < 0$ iff $k - 1 \geq j > 0$

Remark that

$\boxed{0 < j \leq k - 1}$ *iff* $\boxed{1 \leq j \leq k - 1}$

so the new **boundaries** for S_n are

$1 < k \leq n$ and $1 \leq j \leq k - 1$

Special SUM

We substitute $j := k - j$ and evaluate S_n with **new boundaries** $1 < k \leq n$ and $1 \leq j \leq k-1$

$$S_n = \sum_{1<k\leq n} \sum_{1\leq j<k} \frac{1}{k-j} = \sum_{1<k\leq n} \sum_{1\leq j\leq k-1} \frac{1}{j}$$

$$= \sum_{1<k\leq n} \sum_{j=1}^{k-1} \frac{1}{j} = \sum_{1<k\leq n} H_{k-1}$$

Now we evaluate **new boundaries** for the last sum

We put $k := k+1$ in $1 < k \leq n$ and get

$1 < k+1 \leq n$ iff $0 < k \leq n-1$ iff $1 \leq k \leq n-1$ and

$$\sum_{1<k\leq n} H_{k-1} = \sum_{k=1}^{n-1} H_k$$

Special SUM Formula

We developed a new formula for S_n

$$\sum_{1\leq j<k\leq n} \frac{1}{k-j} = \sum_{k=1}^{n-1} H_k$$

We now check our result for few n

$S_1 = \sum_{k=1}^{0} H_1$ undefined, $S_1 = \sum_{1\leq j<k\leq 1} \frac{1}{k-j}$ is also undefined

Book puts (page 39) $S_1 = 0$

Remark that the BOOK formula for S_n

$S_n = \sum_{k=0}^{n} H_k$ is **not correct** unless we define $H_0 = 0$

Special SUM Approach 2

Observe that we got just another formula for our sum, not a "sum closed" formula; we have expressed one double sum by another that uses H_n

Step 2 APPROACH 2

Let's now **re-evaluate** the S_n by expressing its **boundaries differently**

We have as before $P(j,k) \equiv 1 \leq j < k \leq n$ and want to write is now as

$$P(j,k) \equiv R_1(k) \cap R_2(j)$$

for some $R_1(k)$, $R_2(j)$ and **evaluate** the sum

$$S_n = \sum_{1 \leq j < k \leq n} \frac{1}{k-j} = \sum_{R_2(j)} \sum_{R_1(k)} \frac{1}{k-j}$$

Special SUM Approach 2

We write now

$$1 \le j < k \le n \equiv (1 \le j < n) \cap (j < k \le n) \equiv R_1(k) \cap R_2(j)$$

and evaluate

$$S_n = \sum_{1 \le j < k \le n} \frac{1}{k-j} = \sum_{1 \le j < n} \sum_{j < k \le n} \frac{1}{k-j}$$

We substitute now $k := k + j$ and re-work boundaries

$j < k \le n$ iff $j < k + j \le n$ iff $0 < k \le n - j$

iff $1 \le k \le n - j$ and the S_n becomes now

$$S_n = \sum_{1 \le j < n} \sum_{1 \le k \le n-j} \frac{1}{k} = \sum_{1 \le j < n} H_{n-j}$$

Special SUM Approach 2

We have now

$$S_n = \sum_{1 \leq j < n} H_{n-j}$$

We substitute now $j := n - j$ and re-work boundaries

$1 \leq j < n$ iff $1 \leq n - j < n$ iff $1 - n \leq -j < 0$

iff $n - 1 \geq j > 0$ iff $0 < j \leq n - 1$ iff $1 \leq j \leq n - 1$

and the S_n becomes now

$$S_n = \sum_{j=1}^{n-1} H_j$$

All the work - and nothing new!!

Special SUM Approach 3

Step 3 APPROACH 3

We want to find a closed formula CF for

$$S_n = \sum_{1 \leq j < k \leq n} \frac{1}{k - j}$$

We substitute $\boxed{k := k + j}$ and now

$$S_n = \sum_{1 \leq j < k+j \leq n} \frac{1}{k}$$

Special SUM Approach 3

PLAN of ACTION

(1) **We prove:** $P(k,j) \equiv Q_1(k) \cap Q_2(j)$ expressed as folows

$$1 \leq j < k+j \leq n \equiv (1 \leq k \leq n-1) \cap 1 \leq j \leq n-k$$

(2) **We evaluate:**

$$S_n = \sum_{1 \leq j < k+j \leq n} \frac{1}{k} = \sum_{(1 \leq k \leq n-1) \cap (1 \leq j \leq n-k)} \frac{1}{k}$$

Special SUM Approach 3

Proof of (1)

We evaluate:

$(1 \le j < k + j \le n) \equiv$

$\equiv (1 \le j) \cap (1 \le n) \cap (j \le n - k) \cap (j < k + j \le n)$

$\equiv (1 \le j \le n - k) \cap (0 < k \le n - j)$

Now look at $(0 < k \le n - j) \equiv (1 \le k \le n - j)$ for $j = 1, 2, ..., n - k$

and get that $\boxed{1 \le k \le n - 1}$

Hence

$$\boxed{(1 \le j < k + j \le n) = (1 \le j \le n - k) \cap (1 \le k \le n - 1)}$$

end of the proof

Special SUM Approach 3

We evaluate now (2)

$$S_n = \sum_{1\le k\le n-1 \cap 1\le j\le n-k} \frac{1}{k}$$

$$= \sum_{k=1}^{n-1} \sum_{\boxed{j}=1}^{n-k} \frac{1}{k}$$

k is a constant on j

$$= \sum_{k=1}^{n-1} \frac{1}{k} \sum_{j=1}^{n-k} 1 = \sum_{k=1}^{n-1} \frac{1}{k}(n-k)$$

$$= \sum_{k=1}^{n-1} \frac{n}{k} - \sum_{k=1}^{n-1} 1 = n \boxed{\sum_{k=1}^{n-1} \frac{1}{k}} - (n-1)$$

Sum CF formula

We have now

$S_n = n \boxed{\sum_{k=1}^{n-1} \frac{1}{k}} - (n-1)$

We note: $\boxed{\sum_{k=1}^{n-1} \frac{1}{k} = H_{n-1}}$ and $\boxed{H_{n-1} = H_n - \frac{1}{n}}$

$S_n = nH_{n-1} - n + 1 = n(H_n - \frac{1}{n}) - n + 1 = nH_n - 1 - n + 1$

Our H_n CF formula for S_n is

$$S_n = \sum_{1 \leq j < k \leq n} \frac{1}{k-j} = nH_n - n$$

Book Computation

Evaluation in Book

$$\boxed{S_n} \triangleq \sum_{k=1}^{n} \sum_{1 \le j \le n-k} \frac{1}{k} = \sum_{k=1}^{n} \sum_{j=1}^{n-k} \frac{1}{k} = \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{n-k} 1$$

$$= \sum_{k=1}^{n} \frac{1}{k}(n-k) = \sum_{k=1}^{n} (\frac{n}{k} - 1) = n \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} 1 = nH_n - n$$

$$\boxed{S_n = nH_n - n}$$ Book Sum CF Formula

Justify all the steps

Extra Bonuses

We proved in Steps 1,2 that

$$S_n = nH_n - n \quad \text{and} \quad S_n = \sum_{k=1}^{n} H_k$$

We get an an **Extra Bonus**

$$\sum_{k=1}^{n} H_k = nH_n - n$$

And also because Book sum = Our sum we get

$$\sum_{1\leq k\leq n,\ 1\leq j\leq n-k} \frac{1}{k} = \sum_{1\leq k\leq n-1,\ 1\leq j\leq n-k} \frac{1}{k}$$

and we have also proved as a bonus Book Remark on page 41