cse547, math547
DISCRETE MATHEMATICS

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CHAPTER 2
SUMS

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Part 1: Introduction
Sequences and Sums of Sequences
Sequences

Definition

A sequence of elements of a set $A$ is any function $f$ from the set of natural numbers $\mathbb{N}$ into $A$

$$f : \mathbb{N} \to A$$

Any $f(n) = a_n$ is called $n$-th term of the sequence $f$.

Notations:

$$f = \{a_n\}_{n \in \mathbb{N}}, \quad \{a_n\}_{n \in \mathbb{N}}, \quad \{a_n\}$$
Example
We define a sequence \( f \) of real numbers \( R \) as follows

\[
f : N \rightarrow R
\]

Given by a formula

\[
f(n) = n + \sqrt{n}
\]

We also use a shorthand notation for the sequence \( f \) and write

\[
a_n = n + \sqrt{n}
\]
Sequences Example

We often write \( f = \{ a_n \} \) in an even shorter and more informal form as

\[
a_0 = 0, \quad a_1 = 1 + 1 = 2, \quad a_2 = 2 + \sqrt{2}
\]

\[
0, \quad 2, \quad 2 + \sqrt{2}, \quad 3 + \sqrt{3}, \quad \ldots \ldots n + \sqrt{n} \ldots
\]
Observations

Observation 1: A Sequence is always INFINITE (countably infinite) as by definition, the domain of the sequence (function f) is a set of N of natural numbers.

Observation 2: card N = card N-K, for K is any finite subset of N, so we can enumerate elements of a sequence by any infinite subset of N.

Definition: A set T is called countably infinite iff card T = card N, i.e. there is a one to one (1-1) function f that maps N onto T, i.e.

\[ f : N \rightarrow T \]
Observations

**Observation 3:** We can choose as a SET of INDEXES of a sequence any COUNTABLY infinite set $T$, not only the set $N$ of natural numbers.

**In our Book:** $T = N - \{0\} = N^+$, i.e. we consider sequences that "start" with $n = 1$.

We usually write sequences as

$$a_1, \ a_2, \ a_3, \ \ldots \ a_n, \ \ldots$$

$$\{a_n\}_{n \in N^+}$$
Finite Sequences

Definition

A finite sequence of elements of a set $A$ is any function $f$ from a finite set $K$ into $A$

In case when $K$ is a non-empty finite subset of natural numbers $N$ we write, for simplicity $K = \{1, 2, \ldots, n\}$ and call $n$ the length of the sequence

We write sequence function $f$ as

$$f : \{1, 2, \ldots, n\} \longrightarrow A \quad f(n) = a_n, \quad f = \{a_k\}_{k=1}^{n}$$

Case $n=0$: the function $f$ is empty we call it an empty sequence and denote by $e$
Example

Example 1
Let
\[ a_n = \frac{n}{(n - 2)(n - 5)} \]

Domain of the sequence \( f(n) = a_n \) is \( N - \{2, 5\} \) and
\[ f : N - \{2, 5\} \to \mathbb{R} \]

Example 2  Let \( T = \{-1, -2, 3, 4\} \)
\( f(n) = a_n \) for \( n \in T \) is now a finite sequence with the domain \( T \)
FINITE SUMS

In Chapter 2, we consider only finite sums of consecutive elements of sequences \( \{a_n\} \) of rational numbers.

**Definition**

Given a sequence \( f \) of rational numbers

\[
f : N^+ \longrightarrow R \quad f(n) = a_n
\]

We write a finite sum as

\[
\sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n
\]
Sums of elements of sequences

We also use notations:

\[ \sum_{k=1}^{n} a_k = \sum_{1 \leq k \leq n} a_k = \sum_{k \in \{1, \ldots, n\}} a_k \]

\[ \sum_{k=1}^{n} a_k = \sum_{K} a_k \]

for \( K = \{1, \ldots, n\} \)
Sums of elements of sequences

Given a sequence of numbers:

\[ f : \mathbb{N}^+ \rightarrow \mathbb{R}, \quad f(n) = a_n \] ← FULL DEFINITION

\[ a_1 a_2 \ldots a_n, \quad a_k \in \mathbb{R} \] ← SHORTHAND

We sometimes evaluate a sum of some sub-sequence of \( \{a_n\} \)
Sums of elements of sequences

For example we want to sum-up only each second term of \( \{a_n\} \), i.e. \( n \in \text{EVEN} \)

We write in two ways:

1. \( \sum_{1 \leq k \leq 2n, \ k \in \text{EVEN}} a_k = a_2 + a_4 + ..... + a_{2n} \)

where \( 1 \leq k \leq 2n, \ k \in \text{EVEN} \leftarrow P(k) \) summation property

2. \( \sum_{k=1}^{n} a_{2k} = a_2 + a_4 + ..... + a_{2n} \)

where \( a_{2k} \leftarrow \text{subsequence property} \)
Sums Notations

We use following notations

\[
\sum_{P(k)} a_k = \sum_{k \in K} a_k = \sum_K a_k
\]

for \( K = \{ n \in \mathbb{N} : P(n) \} \)

and \( P(n) \) is a certain formula defining our restriction on \( n \)

We assume the following

1. The set \( K \) is defined; i.e. the statement \( P(n) = True \) or \( False \) is decidable

2. The set \( K \) is finite - we consider only finite sums at this moment
Example 1

Let $P(n)$ be a property: $1 \leq n < 100$ and $n \in ODD$

$P(n)$ is a formula defining all ODD numbers between 1 and 99 (included) and hence

$$K = \{ n \in N : P(n) \} = \{ n \in ODD : 1 < n \leq 99 \} = \{1, 3, 5, \ldots, 99 \}$$

or

$$K = \{1, 3, \ldots, (2n + 1)\} \text{ for } 0 \leq n \leq 49$$
Example 1

We have that $K = \{1, 3, \ldots, (2n + 1)\}$ for $0 \leq n \leq 49$ and by definition of the sum

$$\sum_{P(n)} a_n = \sum_{K} a_k \quad \leftarrow \text{PROPERTY}$$

$$= \sum_{n=0}^{49} a_{2n+1} = a_1 + a_3 + \ldots + a_{99} \quad \leftarrow \text{subsequence}$$
Example 2

Let $P(n)$ be a property: $1 \leq n < 100$

$P(n)$ is now a formula defining natural numbers between 1 and 99 (included), i.e.

$K = \{ n \in N : P(n) \} = \{ n \in N : 1 < n \leq 99 \} = \{ 1, 2, \ldots, 99 \}$

In this case

$$\sum_{P(n)} a_n = \sum_{K} a_k = \sum_{k=1}^{99} a_k$$

$$= a_1 + a_2 + a_3 + \ldots + a_{99}$$
Example 3

Let $P(n)$ be a property: $1 \leq n < 100$ and

$$a_n = (2n + 1)^2$$

Evaluate:

$$\sum_{P(n)} a_n$$

$$K = \{P(n) : 1 \leq n < 100\} = \{1, 2, .99\}$$

and

$$\sum_{P(n)} (2n + 1)^2 = \sum_{k=1}^{99} (2n + 1)^2$$

$$= 3^2 + 5^2 + \ldots + (2 \times 99 + 1)^2$$
USEFUL NOTATION

Here is our BOOK NOTATION (from Kenneth Iverson’s programming language APL)

**Characteristic Function** of the formula $P(x)$

\[
[P(x)] = \begin{cases} 
1 & P(x) \text{ true} \\
0 & P(x) \text{ false} 
\end{cases}
\]

where $x \in X \neq \emptyset$

**Example:**
Let $P(n)$ be a property: $p$ is prime number

\[
[p \text{ prime}] = \begin{cases} 
1 & p \text{ is prime} \\
0 & p \text{ is not prime} 
\end{cases}
\]
Useful Sum Notation

We write

\[ \sum_{P(k)} a_k = \sum_k a_k[P(k)] = \sum_{k \in K} a_k \]

where

\[ K = \{ k : P(k) \} \]
Useful Sum Notation Example

Example

\[ \sum_{p} \left[ p \text{ prime} \right] \left[ p \leq n \right] \frac{1}{p} \]

Observe that now

\[ P(x) \text{ is } P_1(x) \cap P_2(x) \]

for \( P_1(x) : x \text{ is prime} \)

\( P_2(x) : x \leq n \text{ for } n \in N \)

\( P(x) \) says : \( x \) is prime \( \text{ and } x \leq n \)
Example

\[ \sum_{p \ prime} [p \leq n] \frac{1}{p} \]

\[ \sum \] means:
we sum \( \frac{1}{p} \) over all \( p \) that are PRIME and \( p \leq n \) for \( n \in \mathbb{N} \)

Case when \( n = 0 \) - as \( 0 \in \mathbb{N} \)
We have that \( P(x) \) is false as PRIMES are numbers \( \geq 2 \)
Book Notations Corrections

**Book** uses notation $p \leq N$ instead of $p \leq n$,

It is tricky!

$N$ in standard notation denotes the **set of natural numbers**

We write $n \in N$ and we **can’t write** $n \leq N$

When you read the book now and later, pay attention

**Book** also uses: $n \leq K$

This really means that $n \leq k$

In standard notation **CAPITAL LETTERS DENOTE SETS**
Book Notations Corrections

Authors never define a sequence \( \{a_n\} \) for \( \sum a_k \)

They also often state:

“\( a_k \)” is defined/not defined for all set of INTEGERS

It means they admit sequences and FINITE sequences with indices being Integers- what is OK and the set of Integers is infinitely countable
Useful Sum Notation Reminder

\[ \sum_{P(k)} a_k = \sum_{k \in K} a_k = \sum_{k} [P(k)]a_k \]

where

\[ K = \{ k \in \mathbb{Z} : P(k) \} \text{ and } K \text{ is finite} \]

or

\[ K = \{ k \in \mathbb{N} : P(k) \} \text{ and } K \text{ is finite} \leftarrow \text{This is usual case} \]

where \( \mathbb{N} \) is set of Natural numbers, \( \mathbb{Z} \) - set of Integers
Part 2: Sums and Recurrences
Some Observations

Observation 1: for any $n \in N$

$$\sum_{k=1}^{n+1} a_k = \sum_{k=1}^{n} a_k + a_{n+1}, \quad \text{and} \quad \sum_{k=1}^{1} a_k = a_1$$

Consider case $n = 0$: the sum is undefined and we put

$$\sum_{k=1}^{0} a_k = 0$$

In general we put

$$\sum_{k=a}^{b} a_k = 0 \quad \text{when} \quad b < a \quad \leftarrow \text{DEFINITION}$$
Some Observations

Observation 2: for any $n \in \mathbb{N}^+$

$$\sum_{k=0}^{n} a_k = \sum_{k=0}^{n-1} a_{k-1} + a_n$$

Now when $n = 0$ we get

$$\sum_{k=0}^{0} a_k = a_0$$

Reminder:

$$\sum_{k=0}^{-1} a_k = 0$$
Sum Recurrence

We know that for any $n \in N^+$

\[
\sum_{k=0}^{n} a_k = \sum_{k=0}^{n-1} a_{k-1} + a_n
\]

We denote

\[ S_n = \sum_{k=0}^{n} a_k \]

Observe that we have defined a function $S$

\[
S : N \rightarrow R, \quad S(n) = S_n = \sum_{k=0}^{n} a_k
\]

← SUM FUNCTION
We re-rewrite \( S(n) = S_n = \sum_{k=0}^{n} a_k \) and get a following recursive formula for \( S \):

\[
S_0 = a_0, \quad S_n = S_{n-1} + a_n \quad \text{for} \quad n > 0
\]

Sum Recurrence Formula

We will use techniques from Chapter 1 to evaluate (if possible) closed formulas for certain SUMS.
Problem

Given a sequence

\[ f : N \longrightarrow R, \text{ defined by a formula} \]

\[ f(n) = a_n \quad \text{for} \quad a_n = a + bn \]

where \( a, b \in R \) are constants

Problem

Find a closed formula \( CF \) for the following sum

\[ S(n) = \sum_{k=0}^{n} a_k = \sum_{k=0}^{n} (a + bk) \]
Sum Recurrence

The recurrence form of our sum $S_n$ is

RF: $S_0 = a$

$$S_n = S_{n-1} + (a + bn)$$

We want to find a Closed Formula CF for this recurrence formula
Generalization

Let’s generalize our formula RF to RS as follows

\[ RS : \quad R_0 = \alpha \]
\[ R_n = R_{n-1} + \beta + \gamma n \]

The previous RF is a case of RS for
\[ \alpha = a, \beta = a, \gamma = b \]
From RS to CF

RF: \( R_0 = \alpha, \; R_n = R_{n-1} + \beta + \gamma n \)

Step 1: evaluate few terms

\( R_0 = \alpha \)

\( R_1 = \alpha + \beta + \gamma \)

\( R_2 = \alpha + \beta + \gamma + \beta + 2\gamma = \alpha + 2\beta + 3\gamma \)

\( R_3 = \alpha + 2\beta + 3\gamma + \beta + 3\gamma = \alpha + 3\beta + 6\gamma \)
From RS to CF

**Step 2:** Observation - general formula for CF

\[ R_n = A(n)\alpha + B(n)\beta + C(n)\gamma \leftarrow CF \]

**GOAL:** Find \( A(n), B(n), C(n) \) and **prove** that RS = CF for RS \( R_0 = \alpha, \ R_n = R_{n-1} + \beta + \gamma n \)

**Method:** Repertoire Method
Repertoire Function 1

RS \quad R_0 = \alpha, \quad R_n = R_{n-1} + \beta + \gamma n

CF \quad R_n = A(n)\alpha + B(n)\beta + C(n)\gamma

We set the first repertoire function as

\[ R_n = 1 \quad \text{for all} \quad n \in \mathbb{N} \]

We set \( R_n = R_n \), for all \( n \in \mathbb{N} \) and \( R_0 = \alpha \), and \( R_0 = 1 \) so \( \alpha = 1 \)
Repertoire Function 1

RS: \( R_0 = \alpha, \quad R_n = R_{n-1} + \beta + \gamma n \)

Repertoire function is \( R_n = 1 \) for all \( n \in N \)

We set \( R_n = R_n \), for all \( n \in N \) and we evaluate

\[
1 = 1 + \beta + \gamma n \quad \text{for all} \quad n \in N
\]

\[
0 = \beta + \gamma n \quad \text{for all} \quad n \in N
\]

This is possible only when \( \beta = \gamma = 0 \)

Solution

\[
\alpha = 1, \quad \beta = 0, \quad \gamma = 0
\]
Equation 1

**CF:** \[ R_n = A(n)\alpha + B(n)\beta + C(n)\gamma \]

We use now the first *repertoire function*

\[ R_n = 1 \text{ for all } n \in \mathbb{N} \]

We set \( R_n = R_n \), for all \( n \in \mathbb{N} \) and use just evaluated \( \alpha = 1, \beta = 0, \gamma = 0 \) and get our **equation 1:**

\[ 1 = A(n), \text{ for all } n \in \mathbb{N} \]

**Fact 1** \( A(n) = 1, \text{ for all } n \in \mathbb{N} \)
Repertoire Function 2

RS: \( R_0 = \alpha, \quad R_n = R_{n-1} + \beta + \gamma n \)

We set the second repertoire function as

\[ R_n = n \text{ for all } n \in N \]

We set \( R_n = R_n \), for all \( n \in N \) and evaluate \( R_0 = \alpha \), and \( R_0 = 0 \) by definition, so \( \alpha = 0 \)
Repertoire Function 2

RS \( R_0 = \alpha, \quad R_n = R_{n-1} + \beta + \gamma n \)

The second repertoire function \( R_n = n \) for all \( n \in N \)

We set \( R_n = R_n \), for all \( n \in N \) and we evaluate

\( n = (n - 1) + \beta + \gamma n \), for all \( n \in N \)

\( 0 = \beta - 1 + \gamma n \), for all \( n \in N \)

\( 1 = \beta + \gamma n \), for all \( n \in N \)

This is possible only when \( \beta = 1, \gamma = 0 \)

Solution

\( \alpha = 0, \quad \beta = 1, \quad \gamma = 0 \)
Equation 2

\[ R_n = A(n)\alpha + B(n)\beta + C(n)\gamma \]

We use now the second repertoire function \( R_n = n \) for all \( n \in \mathbb{N} \)

We set \( R_n = R_n \), for all \( n \in \mathbb{N} \) and use just evaluated \( \alpha = 0, \beta = 1, \gamma = 0 \)

and get our equation 2:

\[ n = B(n), \text{ for all } n \in \mathbb{N} \]

Fact 2 \( B(n) = n, \text{ for all } n \in \mathbb{N} \)
Repertoire Function 3

RS \quad R_0 = \alpha, \quad R_n = R_{n-1} + \beta + \gamma n

We set the third \textit{repertoire function} as

\quad R_n = n^2 \quad \text{for all} \quad n \in N

We set \quad R_n = R_n, \quad \text{for all} \quad n \in N \quad \text{and evaluate}

\quad R_0 = \alpha, \quad \text{and} \quad R_0 = 0 \quad \text{so} \quad \alpha = 0
Repertoire Function 3

\[ R_0 = \alpha, \quad R_n = R_{n-1} + \beta + \gamma n \]

Third repertoire function is

\[ R_n = n^2 \quad \text{for all} \quad n \in \mathbb{N} \]

We set \( R_n = R_n \), for all \( n \in \mathbb{N} \) and evaluate

\[ n^2 = (n - 1)^2 + \beta + \gamma n, \quad \text{for all} \quad n \in \mathbb{N} \]
\[ n^2 = n^2 - 2n + 1 + \beta + \gamma n, \quad \text{for all} \quad n \in \mathbb{N} \]
\[ 0 = -2n + 1 + \beta + \gamma n, \quad \text{for all} \quad n \in \mathbb{N} \]
\[ 0 = (1 + \beta) + n(\gamma - 2), \quad \text{for all} \quad n \in \mathbb{N} \]

This is possible only when \( \beta = -1, \quad \gamma = 2 \)

Solution \( \alpha = 0, \quad \beta = -1, \quad \gamma = 2 \)
Equation 3

\[ R_n = A(n)\alpha + B(n)\beta + C(n)\gamma \]

We use now the third repertoire function 
\[ R_n = n^2 \quad \text{for all} \quad n \in \mathbb{N} \]

We set \( R_n = R_n \), for all \( n \in \mathbb{N} \) and use just evaluated \( \alpha = 0, \beta = 1, \gamma = 0 \)

and get our equation 3:

\[ 2C(n) - B(n) = n^2, \quad \text{for all} \quad n \in \mathbb{N} \]

Fact 3 \[ 2C(n) - B(n) = n^2, \quad \text{for all} \quad n \in \mathbb{N} \]
We obtained the following system of 3 equations on $A(n)$, $B(n)$, $C(n)$

1. $A(n) = 1$
2. $B(n) = n$
3. $2C(n) - B(n) = n^2$

We substitute 1. and 2. in 3. we get $n^2 = -n + 2C(n)$ and $C(n) = \frac{(n^2 + n)}{2}$

Solution

$A(n) = 1, \quad B(n) = n, \quad C(n) = \frac{(n^2 + n)}{2}$
CF Solution

We now put the solution into the general formula

\[ R_n = A(n)\alpha + B(n)\beta + C(n)\gamma \]

and get that the closed formula CF equivalent to RS:

\[ R_0 = \alpha, \quad R_n = R_{n-1} + \beta + \gamma n \]

is

\[ R_n = \alpha + n\beta + \left( \frac{n^2+n}{2} \right)\gamma \]
Let's now go back to original sum

\[ S_n = \sum_{k=0}^{n} (a + bk) \]

We have that

\[ S_n = R_n, \quad \text{for } \alpha = a, \; \beta = a, \; \gamma = b \quad \text{so} \]

\[ S_n = a + na + \left( \frac{n^2 + n}{2} \right)b = (n + 1)a + \left( \frac{n^2 + n}{2} \right)b \]

We hence evaluated

\[ S_n = \sum_{k=0}^{n} (a + bk) = (n + 1)a + \frac{n(n + 1)}{2} b \]
Simple Solution

Of course we can do it by a MUCH simpler method

$$\sum_{k=0}^{n} (a + bk) = \sum_{k=0}^{n} a + \sum_{k=0}^{n} bk$$

$$= (n + 1)a + b \sum_{k=0}^{n} k$$

$$= (n + 1)a + \frac{n(n+1)}{2}b$$

Observe that for a sequence $a_n = a$, for all $n$ we get

$$\sum_{k=0}^{n} a_n = \sum_{k=0}^{n} a = a + \ldots + a = (n + 1)a$$
Summations Laws

Distributive Law

\[ \sum_{k \in K} ca_k = c \sum_{k \in K} a_k \]

Associative Law

\[ \sum_{k \in K} (a_k + b_k) = \sum_{k \in K} a_k + \sum_{k \in K} b_k \]

Commutative Law

\[ \sum_{k \in K} a_k = \sum_{\Pi(k) \in K} a_{\Pi(k)} \]

\( \Pi(k) \) - any permutation of elements of \( K \)

\( K \) - is a finite subset of Integers
Geometric Sum

Geometric Sequence

Definition
A sequence \( f : \mathbb{N} \rightarrow \mathbb{R}, \ f(n) = a_n \) is geometric iff
\[
\frac{a_{n+1}}{a_n} = q, \quad \text{for all} \quad n \in \mathbb{N}
\]
We prove a following property of a geometric sequence \( \{a_n\} \)

\[
a_n = a_0q^n \quad \text{for all} \quad n \in \mathbb{N}
\]

Geometric Sum Formula

\[
S_n = \sum_{k=0}^{n} a_0q^k = \frac{a_0(1-q^{n+1})}{1-q}
\]
Proof of Geometric Sum Formula

\[ S_n = \sum_{k=0}^{n} a_0 q^k \]

\[ S_n = a_0 + a_0 q + \ldots + a_0 q^n \]

\[ qS_n = a_0 q + a_0 q^2 + \ldots + a_0 q^n + a_0 q^{n+1} \]

\[ S_n (1 - q) = a_0 - a_0 q^{n+1} \]

\[ S_n = \sum_{k=0}^{n} a_0 q^k = \frac{a_0 (q^{n+1} - 1)}{q - 1} \]

\[ \text{Geometric Sum} \]
Examples

Example 1

\[ S_n = \sum_{k=0}^{n} 2^{-k} = \sum_{k=0}^{n} \left( \frac{1}{2} \right)^k \]

We have \( a_0 = 1 \), \( q = \frac{1}{2} \), and

\[ S_n = \frac{\left( \frac{1}{2} \right)^{n+1} - 1}{-\frac{1}{2}} = 2 - \left( \frac{1}{2} \right)^n \]
Examples

Example 2

\[ S_n = \sum_{k=1}^{n} 2^{-k} = \sum_{k=1}^{n} \left( \frac{1}{2} \right)^k \]

We have now \( a_1 = \frac{1}{2}, \quad q = \frac{1}{2} \) and hence \( n := n - 1 \) and

\[ S_{n-1} = \frac{1}{2} \left( \left( \frac{1}{2} \right)^n - 1 \right) = 1 - \left( \frac{1}{2} \right)^n \]
From RF to Sum $S_n$ to CF

Tower of Hanoi

**RF:** $T_0 = 0, \quad T_n = 2T_{n-1} + 1$

Divide RF by $2^n$

$$\frac{T_0}{2^0} = 0, \quad \frac{T_n}{2^n} = \frac{2T_{n-1}}{2^n} + \frac{1}{2^n}$$

and we get

$$\frac{T_0}{2^0} = 0, \quad \frac{T_n}{2^n} = \frac{T_{n-1}}{2^{n-1}} + \frac{1}{2^n}$$

Denote $S_n = \frac{T_n}{2^n}$, we get a recursive sum formula **SR**

**RS:** $S_0 = 0, \quad S_n = S_{n-1} + \frac{1}{2^n}$
From RF to Sum $S_n$ to CF

SR: $S_0 = 0, \quad S_n = S_{n-1} + \frac{1}{2^n}$

It means that $S : N \to R$ and

$$S_n = \sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n} \quad \text{(as } S_n \text{ is geometric)}$$

But we have $S_n = \frac{T_n}{2^n}$ so we get

$$T_n = 2^n S_n$$

and we evaluate

$$T_n = 2^n - 1 \leftarrow \text{CF for RF}$$
Tower of Hanoi Revisited

RF:  \( T_0 = 0, \quad T_n = 2T_{n-1} + 1 \)

We have proved in Chapter 1 that

\[
T_n = 2^n - 1 \quad \leftarrow \text{Closed Formula}
\]

We now reverse the previous problem:

we will get a sum \( S_n \) and its **closed formula** from the closed formula \( \text{CF} \) for \( T_n \)

Divide \( T_n \) formula by \( 2^n \)

\[
\frac{T_0}{2^0} = 0, \quad \frac{T_n}{2^n} = \frac{2T_{n-1}}{2^n} + \frac{1}{2^n}
\]

Put \( S_n = \frac{T_n}{2^n} \) and we get

\[
\text{SR: } S_0 = 0, \quad S_n = S_{n-1} + \frac{1}{2^n}
\]

Now, \( S_n = \frac{T_n}{2^n} \) and using \( \text{CF} \) for \( T_n \) we get \( S_n = \frac{2^n-1}{2^n} \)

Thus,

\[
S_n = \sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n} \quad \leftarrow \text{SUM}
\]