CHAPTER 1
PART FOUR: The Generalized Josephus Problem
Repertoir Method
Josephus Problem Generalization

Our function \( J : \mathbb{N} - \{0\} \longrightarrow \mathbb{N} \) is defined as
\[
J(1) = 1, \quad J(2n) = 2J(n) - 1, \quad J(2n+1) = 2J(n) + 1 \quad \text{for } n > 1
\]
We generalize it to function \( f : \mathbb{N} - \{0\} \longrightarrow \mathbb{N} \) defined as follows

\[
f(1) = \alpha
\]

\[
f(2n) = 2f(n) + \beta, \quad n \geq 1
\]

\[
f(2n + 1) = 2f(n) + \gamma, \quad n \geq 1
\]

Observe that \( J = f \) for \( \alpha = 1, \beta = -1, \gamma = 1 \)

NEXT STEP: Find a Closed Formula for \( f \)
From RF to CF

**Problem:** Given RF

\[
\begin{align*}
  f(1) &= \alpha \\
  f(2n) &= 2f(n) + \beta \\
  f(2n + 1) &= 2f(n) + \gamma
\end{align*}
\]

Find a CF for it

**Step 1** Find few initial values for \( f \)

**Step 2** Find (guess) a CF formula from Step 1

**Step 3** Prove correctness of the CF formula, i.e. prove that \( RF = CF \)

This step is usually done by mathematical Induction over the domain of the function \( f \)
From RF to CF

Step 1
Evaluate few initial values for

\[ f(1) = \alpha \]
\[ f(2n) = 2f(n) + \beta \]
\[ f(2n + 1) = 2f(n) + \gamma \]
**Repertoire Method**

\[ n = 2^k + l, \quad 0 \leq l < 2^k \]

<table>
<thead>
<tr>
<th>2^0</th>
<th>1</th>
<th>( \alpha )</th>
<th>( l = 0 )</th>
<th>( f(1) = \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2^1 )</td>
<td>2</td>
<td>( 2 \alpha + 1 \beta + 0 \gamma )</td>
<td>1 = ( 2^1 - 1 - 0 ), ( l = 0 )</td>
<td>( f(2) = 2f(1) + \beta ) ( l = 0 )</td>
</tr>
<tr>
<td>( 2^1 + 1 )</td>
<td>3</td>
<td>( 2 \alpha + 0 \beta + 1 \gamma )</td>
<td>0 = ( 2^1 - 1 - 1 ), ( l = 1 )</td>
<td>( f(3) = 2f(1) + \gamma ) ( l = 1 )</td>
</tr>
<tr>
<td>( 2^2 )</td>
<td>4</td>
<td>( 4 \alpha + 3 \beta )</td>
<td>3 = ( 2^2 - 1 - 0 )</td>
<td>( f(4) = 2f(2) + \beta ) ( l = 0 )</td>
</tr>
<tr>
<td>( 2^2 + 1 )</td>
<td>5</td>
<td>( 4 \alpha + 2 \beta + \gamma )</td>
<td>2 = ( 2^2 - 1 - 1 )</td>
<td>( f(5) = 2f(2) + \gamma ) ( l = 1 )</td>
</tr>
<tr>
<td>( 2^2 + 2 )</td>
<td>6</td>
<td>( 4 \alpha + \beta + 2 \gamma )</td>
<td>2 = ( l )</td>
<td>( f(6) = 2f(3) + \beta ) ( l = 2 )</td>
</tr>
<tr>
<td>( 2^2 + 3 )</td>
<td>7</td>
<td>( 4 \alpha + 3 \gamma )</td>
<td>3 = ( l )</td>
<td>( f(7) = 2f(3) + \gamma ) ( l = 3 )</td>
</tr>
<tr>
<td>( 2^3 )</td>
<td>8</td>
<td>( 8 \alpha + 7 \beta )</td>
<td></td>
<td>( \text{F(8)} = 2f(4) + \beta ) ( l = 0 )</td>
</tr>
<tr>
<td>( 2^3 + 1 )</td>
<td>9</td>
<td>( 8 \alpha + 6 \beta + 3 \gamma )</td>
<td></td>
<td>( f(9) = 2f(4) + \gamma ) ( l = 1 )</td>
</tr>
</tbody>
</table>
Observations

\[ n = 2^k + l, \quad 0 \leq l < 2^k \]

\( \alpha \) coefficient is \( 2^k \)

\( \beta \) coefficient for the groups decreases by 1 down to 0

\( \beta \) coefficient is \( 2^k - 1 - l \)

\( \gamma \) coefficient increases by 1 up from 0

\( \gamma \) coefficient is \( l \)
General Form of CF

Given a RC function

\[ f(1) = \alpha, \quad f(2n) = 2f(n) + \beta, \quad f(2n + 1) = 2f(n) + \gamma \]

A general form of CF is

\[ f(n) = \alpha A(n) + \beta B(n) + \gamma C(n) \]

for certain \( A(n), B(n), C(n) \) to be determined

Our guess is:

\[ A(n) = 2^k, \quad B(n) = 2^k - 1 - l, \quad C(n) = l \]

for \( n = 2^k + l \)
General form of CF

RF: \( f(1) = \alpha, \quad f(2n) = 2f(n) + \beta, \quad f(2n + 1) = 2f(n) + \gamma \)

CF: \( f(n) = \alpha A(n) + \beta B(n) + \gamma C(n) \)

We prove by mathematical Induction over \( k \) that when \( n = 2^k + l, \quad 0 \leq l < 2^k \) our guess is true, i.e.

\[ A(n) = 2^k, \quad B(n) = 2^k - 1 - l, \quad C(n) = l \]

STEP 1: We consider a case: \( \alpha = 1, \beta = \gamma = 0 \) and we get

RF: \( f(1) = 1, \quad f(2n) = 2f(n), \quad f(2n + 1) = 2f(n) \) and

CF: \( f(n) = A(n) \)
Fact 1

We use $f(n) = A(n)$ and re-write RF in terms of $A(n)$ as follows

$$\begin{align*}
AR : & \quad A(1) = 1, \quad A(2n) = 2A(n), \quad A(2n + 1) = 2A(n)
\end{align*}$$

Fact 1  Closed formula CA for AR is:

$$\begin{align*}
CAR : & \quad A(n) = A(2^k + l) = 2^k, \quad 0 \leq l < 2^k
\end{align*}$$

Proof by induction on $k$

Base Case:  $k=0$, i.e. $n=2^0 + l$, $0 \leq l < 1$, and we have that $n = 1$ and evaluate

$AR$:  $A(1) = 1$,  $CAR$:  $A(1) = 2^0 = 1$, and hence $AR = CAR$
Fact 1

Inductive Assumption:
\[ A(2^{k-1} + l) = A(2^{k-1} + l) = 2^{k-1}, \quad 0 \leq l < 2^{k-1} \]

Inductive Thesis:
\[ A(2^k + l) = A(2^k + l) = 2^k, \quad 0 \leq l < 2^k \]

Two cases: \( n \in \text{even}, \quad n \in \text{odd} \)

C1: \( n \in \text{even} \)

\( n := 2n, \) and we have \( 2^k + l = 2n \) iff \( l \in \text{even} \)
Fact 1

We evaluate $n$:

$$2n = 2^k + l, \quad n = 2^{k-1} + \frac{l}{2}$$

We use $n$ in the inductive step.

Observe that the correctness of using $\frac{l}{2}$ follows from that fact that $l \in \text{even}$ so $\frac{l}{2} \in N$ and it can be proved formally like on the previous slides.

Proof

$$A(2n) = \text{reprn} \quad A(2^k + l) = \text{evaln} \quad 2A(2^{k-1} + \frac{l}{2}) = \text{ind}$$

$$2 \cdot 2^{k-1} = 2^k$$
Fact 1

C2: \( n \in odd \)

\( n := 2n+1 \), and we have \( 2^k + l = 2n + 1 \) iff \( l \in odd \)

We evaluate \( n \):

\[
2n + 1 = 2^k + l, \quad n = 2^{k-1} + \frac{l-1}{2}
\]

We use \( n \) in the inductive step. Observe that the correctness of using \( \frac{l-1}{2} \) follows from that fact that \( l \in odd \) so \( \frac{l-1}{2} \in N \)

Proof:

\[
A(2n + 1) \equiv_{\text{repr}} A(2^k + l) \equiv_{\text{eval}} 2A(2^{k-1} + \frac{l-1}{2}) \equiv_{\text{ind}} 2 \times 2^{k-1} = 2^k
\]

It ends the proof of the Fact 1: \( A(n) = 2^k \)
Repertoire Method

GENERAL PROBLEM
We have a certain recursive formula $RF$ that depends on some parameters, in our case $\alpha, \beta, \gamma$, i.e.

$$RF = RF(n, \alpha, \beta, \gamma)$$

We want to find a formula $CF$ of the form

$$CF(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$$

such that $CF = RF$

GOAL: find $A(n), B(n), C(n)$ - we have 3 unknowns so we need 3 equations to find a solution and then we have to prove

$$RF(n, \alpha, \beta, \gamma) = A(n)\alpha + B(n)\beta + C(n)\gamma$$

for all $n \in N$

In general, when there are $k$ parameters we need to develop and solve $k$ equations, and then to prove

$$RF(n, \alpha_1 \ldots \alpha_k) = A_1(n)\alpha_1 + \ldots + A_k(n)\alpha_k$$

for all $n \in N$
Repertoire Method

**METHOD:** we use a repertoire of special functions 
\[ R_1 = R_1(n), \quad R_2 = R_2(n), \quad R_3 = R_3(n) \] and form and solve a system of 6 equations:

(1) \[ RF(n, \alpha, \beta, \gamma) = R_i(n), \quad \text{for all } n \in \mathbb{N}, \quad i = 1, 2, 3 \]

(2) \[ CF(n) = A(n)\alpha + B(n)\beta + C(n)\gamma = R_i(n), \quad \text{for all } n \in \mathbb{N}, \quad i = 1, 2, 3 \]

For each repertoire function \( R_i \) we evaluate corresponding \( \alpha, \beta, \gamma \) from (1), for \( i = 1, 2, 3 \)

For each repertoire function \( R_i \), we put corresponding solutions \( \alpha, \beta, \gamma \) from (1) in (2) to get 3 equations on \( A(n), B(n), C(n) \) and solve them on \( A(n), B(n), C(n) \)

This also proves that \( RF(n) = CF(n) \), for all \( n \in \mathbb{N} \), i.e. \( RF = CF \)
Repertoire Function $R_1$

**RF:** $f(1) = \alpha$, $f(2n) = 2f(n) + \beta$, $f(2n + 1) = 2f(n) + \gamma$

**CF:** $f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$

We have already proved in **Step 1** the formula for $A(n)$, so we need only to consider **2 repertoire functions**

**Step 2:** Consider as the first repertoire function $R_1$ given by a formula

$$R_1(n) = 1 \quad \text{for all} \quad n \in \mathbb{N}$$

By (1) $f(n) = R_1(n) = 1$ for all $n \in \mathbb{N}$ i.e. we have the following condition

**C1:** $f(n) = 1$ for all $n \in \mathbb{N}$

By RF we have that $f(1) = \alpha$, and by **C1** : $f(1) = 1$, and hence $\alpha = 1$
Repertoire Function $R_1$

RF:  \( f(1) = \alpha, \quad f(2n) = 2f(n) + \beta, \quad f(2n + 1) = 2f(n) + \gamma \)

We still consider as the first repertoire function given by the formula

\[ R_1(n) = 1 \quad \text{for all} \quad n \in \mathbb{N} \]

By (1) \( f(n) = R_1(n) = 1 \) for all \( n \in \mathbb{N} \) i.e. we have the following condition

\( C1: \quad f(n) = 1 \quad \text{for all} \quad n \in \mathbb{N} \)

By RF: \( f(2n) = 2f(n) + \beta \) and by \( C1 \) we get equation:

\[ 1 = 2 + \beta, \quad \text{and hence} \quad \beta = -1 \]

By RF: \( f(2n + 1) = 2f(n) + \gamma \) and by \( C1 \) we get equation:

\[ 1 = 2 + \gamma \quad \text{and hence} \quad \gamma = -1 \]

**Solution** from first repertoire function $R_1$ is

\[ \alpha = 1 \quad \beta = -1 \quad \gamma = -1 \]
Now we use the first repertoire function $R_1$ to the closed formula

$$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$$

By (2) we get

$$f(n) = R_1 = 1,$$ for all $n \in N$

We input parameters $\alpha = 1, \beta = -1, \gamma = -1$ evaluated by $RF$ and $R_1$ in

(2) $A(n)\alpha + B(n)\beta + C(n)\gamma = R_1(n) = 1),$ for all $n \in N$

and we get the first equation

$$A(n) - B(n) - C(n) = 1, \text{ for all } n \in N$$

By the Repertoire Method we have that $CF = RF$ iff the following holds

FACT 2

$$A(n) - B(n) - C(n) = 1, \text{ for all } n \in N$$
Repertoire Function $R_2$

Step 3:
RF: $f(1) = \alpha$, $f(2n) = 2f(n) + \beta$, $f(2n + 1) = 2f(n) + \gamma$

CF: $f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$

Consider a second repertoire function $R_2$ given by the formula

$$R_2(n) = n \quad \text{for all} \quad n \in N$$

By (1) $f(n) = R_2(n) = n$ i.e. we have the following condition

C2: $f(n) = n$, for all $n \in N$

By RF we have that $f(1) = \alpha$, and by C2 : $f(1) = 1$, and hence $\alpha = 1$
Repertoire Function $R_2$

RF: $f(1) = \alpha, \ f(2n) = 2f(n) + \beta \quad f(2n + 1) = 2f(n) + \gamma$

We still consider as the second repertoire function given by the formula

$$R_2(n) = n \quad \text{for all} \quad n \in \mathbb{N}$$

By (1) $f(n) = R_2(n) = n$ i.e. we have the following condition

C2: $f(n) = n$, for all $n \in \mathbb{N}$

By RF: $f(2n) = 2f(n) + \beta$ and by C2 we get

$$2n = 2n + \beta, \quad \text{and hence} \quad \beta = 0$$

By RF: $f(2n + 1) = 2f(n) + \gamma$ and by C2 we get

$$2n + 1 = 2n + \gamma \quad \text{and hence} \quad \gamma = 1$$

Solution from second repertoire function $R_2$ is

$\alpha = 1, \quad \beta = 0, \quad \gamma = 1$
Repertoire Method

Now we use the second repertoire function $R_2$ to the closed formula

$CF : f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$

By (2) we get

$f(n) = R_2 = n$, for all $n \in N$

We input parameters $\alpha = 1$, $\beta = 0$, $\gamma = 1$ evaluated by $RF$ and $R_2$ in

(2) $A(n)\alpha + B(n)\beta + C(n)\gamma = R_2(n) = n$, for all $n \in N$

and get the second equation

$A(n) + C(n) = n$, for all $n \in N$

By the Repertoire Method we have that $CF = RF$ iff the following holds

FACT 3

$A(n) + C(n) = n$, for all $n \in N$

Remember: we have proved that $A(n) = 2^k$, for $n = 2^k + l$

so we do not need any more repertoire functions (and equations)
CF for Generalized Josephus

Step 4 A(n), B(n) and C(n) from the following equations

E1 \( A(n) = 2^k, \ n = 2^k + l, \ 0 \leq l < 2^k \)

E2 \( A(n) - B(n) - C(n) = 1, \ \text{for all} \ n \in N \)

E3 \( A(n) + C(n) = n, \ \text{for all} \ n \in N \)

E3 and E1 give us that \( 2^k + C(n) = 2^k + l \), and so

C \( C(n) = l \)

From the above and E2 we get \( 2^k - l - B(n) = 1 \) and so

B \( B(n) = 2^k - 1 - l \)
CF for Generalized Josephus

Observe that $A, B, C$ are exact formulas we have guessed and the following holds

**Fact 4**

$CF : \quad f(n) = 2^k \alpha + (2^k - 1 - l) \beta + l \gamma \quad \text{for} \quad n = 2^k + l, \quad 0 \leq l < 2^k$

is the closed formula for

$RF: \quad f(1) = \alpha, \quad f(2n) = 2f(n) + \beta \quad f(2n + 1) = 2f(n) + \gamma$

This also ends the proof that Generalized Josephus $CF$ exists and $RF = CF$
Short CF Solution

Step 2:
RF: \( f(1) = \alpha, \ f(2n) = 2f(n) + \beta \ f(2n + 1) = 2f(n) + \gamma \)

Here is a short solution as presented in our Book

You can use it for your problems solutions (also on the tests)—when you really understand what are you doing.

Consider a constant function \( f(n) = 1, \ \text{for all } n \in \mathbb{N} \) (this is our first repertoire function \( R_1 \))

We evaluate now \( \alpha, \beta, \gamma \) for it (if possible)

Solution  \( 1 = 2 + \beta, \ 1 = 2 + \gamma, \ \text{and so} \)

\[ \alpha = 1, \ \beta = -1, \ \gamma = -1 \]
Short CF Solution

\[ CF : \quad f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma \]

We evaluate \( CF \) for \( \alpha, \beta, \gamma \) being solutions for RF and \( f(n) = 1 \) and get

\( CF = RF \) iff the following holds

**Fact 2**

\[ A(n) - B(n) - C(n) = 1 \quad \text{for all} \quad n \in \mathbb{N} \]
Short CF Solution

Step 3
RF: \( f(1) = \alpha, \ f(2n) = 2f(n) + \beta \quad f(2n + 1) = 2f(n) + \gamma \)

Consider a constant function \( f(n) = n, \) for all \( n \in \mathbb{N} \)

We evaluate now \( \alpha, \beta, \gamma \) for it (if possible)

\[ 2n = 2n + \beta, \quad 2n + 1 = 2n + \gamma \]

and get

Solution: \( \alpha = 1, \beta = 0, \gamma = 1 \)
Short CF Solution

\[ CF : \quad f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma \]

Now we evaluate CF for the solutions \( \alpha = 1, \beta = 0, \gamma = 1 \) and \( f(n) = n \)
and we get

**Fact 3**

\[ A(n) + C(n) = n, \quad \text{for all} \quad n \in N \]
Final Solution for CF

Step 4
We put together Facts 1, 2, 3 to evaluate formulas for $A(n)$, $B(n)$, $C(n)$

Fact 3 and Fact 1 give that $2^k + C(n) = 2^k + l$, and so $C(n) = l$

From the above and Fact 2 we get $2^k - l - B(n) = 1$ and so $B(n) = 2^k - 1 - l$
Final Solution for CF

Given RF, CF defined as follows

RF: \( f(1) = \alpha, \ f(2n) = 2f(n) + \beta \quad f(2n + 1) = 2f(n) + \gamma \)

CF: \( f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma \)

The final form of CF is as below

Fact 4

CF: \( f(n) = 2^k\alpha + (2^k - 1 - l)\beta + l\gamma \), where
\( n = 2^k + l, \quad 0 \leq l < 2^k \)

Observe that the Book does not prove that CF = RF